A SIMPLE ALLOCATION PROBLEM\

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The problem studied is that of how to allocate a fixed amount of some resource among various activities where, after a period of time, unused portions of the resource lose their value. Demand for the resource at each station is considered to be random. The problem is a special case of one usually referred to as the distribution of the effort. The question of determining the optimal amount of resource is also discussed.

1. Introduction

We are concerned with the following allocation problem. Suppose at the beginning of a given time period there exists $T$ units of some product (or service) which are to be allocated among $k$ locations. The demand at each location is a random variable with a distribution which depends on the location. At the end of the time period the unused product (or service) loses its value. The problem of interest here is that of allocating the $T$ units among the $k$ locations so that the expected number of units used to satisfy demand is maximized. In the section 5 we shall also discuss the question of the optimal choice of $T$.

One situation where this problem can present itself is with a bakery chain. Each day a quantity of bread is produced. In scattered parts of a city it has $k$ shops where the bread is sold in varying quantities. The unsold bread becomes stale and has no value (this is not strictly true since day-old bread can be sold or used for other purposes). It is important for the company to know how much bread to produce and how to allocate it among its shops.

Another situation is as follows: There are $n$ repairmen in a pool and $k$ stations to be manned. In terms of hours, $T = 8n$ is the total available service time in a given day. At each station there will be varying demand for service time; idle time is lost. How many men should be stationed at each location?

The allocation problem posed is a special case of a problem sometimes referred to as the problem of the distribution of effort. This topic has been discussed by Koopman [3], Miehle [4], Vazsonyi [6], Klein [2]. See also Karush [1] and Smith, Saposnik, Lindeman [5] for specific formulations similar to that in this paper. In the next section we shall discuss this problem and prove a theorem which is useful for treating the problem under consideration.

2. Distribution of Effort Problem

Consider the function $G(a_1, \ldots, a_k)$. The problem of the distribution of effort is that of finding those values of $a_1, \ldots, a_k$ which maximizes $G$ subject to the restrictions

$$\sum_{i=1}^{k} a_i \leq T$$

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and

\[ a_i \geq 0 \quad \text{(} i = 1, \cdots, k \text{)} \]

If \( G \) is non-decreasing in each of the variables then (1) can be replaced by

\[
(1') \quad \sum_{i=1}^{k} a_i = T
\]

without changing the solution. In what follows the restriction (1') will be imposed instead of (1).

An obstacle in solving this problem, as in the linear programming problem, is the presence of the restrictions (1') and (2) on the \( a_i \)'s. Solutions can possibly occur on the boundary of the region given by (1') and (2) (in which case one or more of the \( a_i \)'s would be zero). Such a solution is not obtainable by the classical methods of analysis, i.e., finding stationary points by setting derivatives equal to zero.

Since, at times, the classical methods, when applicable, lead to easily computable solutions it is of interest to know in advance whether or not the solution lies on the boundary. If it is known that the solution is an interior point then classical methods will apply. The following theorem pertains to this question and is applicable to our specific problem.

**Theorem 1.** If \( G \) is differentiable and

\[
(3) \quad \frac{\partial G}{\partial a_i} \Big|_{a_i=0} \geq \frac{\partial G}{\partial a_j} \Big|_{0 < a_j \leq T} \quad \text{for all} \quad i, j
\]

then the maximum of \( G \) subject to (1') and (2) is achieved at an interior point.

**Proof:** Suppose that \( a_i \), say \( a_1 \), equals zero. Because of (1') there exists at least one \( a_i \), say \( a_k \), which is positive. Consider the function

\[ H(\epsilon) = G(\epsilon, a_2, \cdots, a_{k-1}, a_k - \epsilon). \]

Because of (3) it is clear that \( H'(\epsilon) \) is non-negative for \( \epsilon = 0 \) and hence \( H(\epsilon) \) is non-decreasing in a neighborhood of \( \epsilon = 0 \). Consequently \( a_1 \) can be made positive without diminishing the value of \( G \). Since this is true for every \( a_i = 0 \) it follows that there is always an interior point for which \( G \) is as large as at a boundary point. This proves the assertion.

3. **Problem of Allocation under Random Demand**

We state, more precisely, the problem given in the introduction. Let \( X_i \) be a non-negative random variable which denotes the demand for the product (or service) at location \( i \) \((i = 1, \cdots, k)\) during the time period under consideration. Let \( f_i \) denote the probability density function of \( X_i \) and \( a_i \) denote the amount of product allocated to location \( i \). The \( a_i \)'s must satisfy (1') and (2). Let \( G(a_1, \cdots, a_k) \) denote the expected amount of product supplied on demand

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\(^2\) Strictly speaking we assume the distribution function of \( X_i \) is absolutely continuous for all positive values. Thus if \( \text{Prob} \ (X_i = 0) > 0 \) then \( \int_0^\infty f_i(t) dt < 1 \).
over all $k$ locations during the time period when $a_1, \cdots, a_k$ is the allocation. We want to find that allocation which maximizes $G$. We have

Theorem 2: If $\text{Prob}(X_i = 0) = 0$ for $i = 1, \cdots, k$, then the maximum of $G$ over the region given by (1') and (2) exists and is attained at an interior point which satisfies

$$ (4) \quad \int_{a_i}^{\infty} f_i(x) \, dx = \int_{a_2}^{\infty} f_2(x) \, dx = \cdots = \int_{a_k}^{\infty} f_k(x) \, dx $$

In terms of the bakery problem, the theorem says: allocate the bread so that the probability of sell-out is equal at all locations. The probability of sell-out will be a function of $T$.

The condition that $\text{Prob}(X_i = 0) = 0$ is, to some extent, restrictive since, in most cases, this will not be strictly true. However, in many cases these probabilities will be small enough so that as an approximation, they can be considered to be zero. The remark following the proof of Theorem 2 will clarify this point further.

Proof: The function $G$ is of the form

$$ G(a_1, \cdots, a_k) = \sum_{i=1}^{k} \left\{ \int_{0}^{a_i} x f_i(x) \, dx + a_i \int_{a_i}^{\infty} f_i(x) \, dx \right\} $$

The partial derivatives are

$$ \frac{\partial G}{\partial a_i} = \int_{a_i}^{\infty} f_i(x) \, dx \quad (i = 1, \cdots, k). $$

Then because of the hypothesis of the theorem

$$ \left. \frac{\partial G}{\partial a_i} \right|_{a_i=0} - \frac{\partial G}{\partial a_j} = 1 - \int_{a_j}^{\infty} f_j(x) \, dx \geq 0 $$

for all $i$ and $j$ and $a_j > 0$. Hence the conditions of Theorem 1 hold. Equations (4) hold from the usual methods of calculus.

Remark: It can be seen from taking second derivatives that the function $G$ is concave and hence any solution to (4) will yield a maximum. Also, because of the concavity of $G$, if a solution to (4) exists which satisfies (1') and (2), then such a solution is a maximum. This assertion holds whether or not $\text{Prob}(X_i = 0) = 0$.

4. A Special Case

Assume the densities of demand are such that they can be reduced to a standard form by changing scale and location parameters. That is, suppose that there exist $\mu_i$'s and $\sigma_i$'s ($\sigma_i > 0$) such that the density function of $(X_i - \mu_i)/\sigma_i$ is $f$, independent of $i$. Let $a_i$ denote the solution to (4). We have

$$ \int_{a_i}^{\infty} f_i(x) \, dx = \int_{(a_i - \mu_i)/\sigma_i}^{\infty} f(x) \, dx \quad (i = 1, \cdots, k) $$
Then the system of equations (4) together with (1') can be written

\[ a_i - \mu_i = \frac{\sigma_i}{\sigma_k} (a_k - \mu_k) \quad (i = 1, \ldots, k - 1) \]

\[ \sum_{i=1}^{k} a_i = T, \]

which yield

(5) \[ a_i = \mu_i + \sigma_i \Delta \]

\( (i = 1, \ldots, k) \)

where

\[ \Delta = \frac{T - \sum_{i=1}^{k} \mu_i}{\sum_{i=1}^{k} \sigma_i} \]

Suppose, for example, a bakery chain has 10 stores. Past experience has shown that the number of loaves of bread sold at the shops is approximately normally distributed with means \( \mu_i \) and standard deviations \( \sigma_i \) given by the accompanying table. The bakery produces 1000 loaves every day. The third row in the table indicates the optimal allocation.

<table>
<thead>
<tr>
<th>Shops</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu )</td>
<td>100</td>
<td>110</td>
<td>105</td>
<td>90</td>
<td>95</td>
<td>100</td>
<td>103</td>
<td>95</td>
<td>90</td>
<td>102</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>10</td>
<td>11</td>
<td>11</td>
<td>9</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>9</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>( a_i )</td>
<td>101</td>
<td>111.1</td>
<td>106.1</td>
<td>19.9</td>
<td>96</td>
<td>101</td>
<td>104</td>
<td>96</td>
<td>90.9</td>
<td>103</td>
</tr>
</tbody>
</table>

distributed with means \( \mu_i \) and standard deviations \( \sigma_i \). The expected gain or profit for a given allocation is

\[ P_T(a_1, \ldots, a_k) = (S - c) G(a_1, \ldots, a_k) - c(T - G(a_1, \ldots, a_k)) \]

It follows that for a given \( T \), \( P_T \) is maximized by the optimal allocation derived in section 3. Assume that the optimum allocation is the one used. Then \( P_T \) is a function of \( T \) alone. Let

\[ g_i(a_i) = \int_{0}^{a_i} x f_i(x) \, dx + a_i \int_{a_i}^{\infty} f_i(x) \, dx \]

Also let

\[ \pi(T) = \int_{a_i}^{\infty} f_i(t) \, dt \]

5. Optimal Choice of \( T \)

Suppose \( c > 0 \) denotes the cost per unit of the product (or service) to be allocated and \( S > 0 \) denotes the return when a unit is supplied on demand.
where the $a_i$'s satisfy (4). Then
\[
\frac{dP_T}{dT} = S \frac{dG}{dT} - c
\]
\[
= S \sum_{i=1}^{k} \frac{dg_i(a_i)}{dT} - c
\]
\[
= S \sum_{i=1}^{k} \frac{dg_i(a_i)}{da_i} \frac{da_i}{dT} - c
\]
\[
= S\pi(T) \sum_{i=1}^{k} \frac{da_i}{dT} - c
\]
\[
= S\pi(T) - c.
\]
Hence setting $dP_T/dT$ equal to zero yields
\[
\pi(T) = c/S.
\]
This result can also be obtained by determining that quantity $a_i$ which maximizes the expected gain at the $i$th location for each $i$ individually. It is known, see [6] p. 257, that $a_i$ should be chosen so that
\[
\int_{a_i}^{\infty} f_i(x) = c/S.
\]
With the $a_i$'s obtained in such a manner, $\sum_{i=1}^{k} a_i$ is the optimal choice of $T$.

To carry forth the example of section 4, suppose that $c/S = \frac{1}{2}$ for a loaf of bread. It would then follow that $a_i = \mu_i$ in order for $\pi(T) = \frac{1}{2}$. Hence $T$ should equal 990.

6. The Case Where the Demand Distributions are Unknown

If the demand densities are unknown it will not be possible to derive the value of $a_i$ such that
\[
\int_{a_i}^{\infty} f_i(x) = c/S.
\]
However, a very neat statistical procedure given by Robbins and Monro [7- offers an explicit way of continually readjusting the values of $a_i$ so that the optimum value, whatever it may be, will be approached.

Let $\{\alpha_i\}$ be any sequence of positive numbers such that
\[
\sum_{i=1}^{\infty} \alpha_i = \infty \text{ and } \sum_{i=1}^{\infty} \alpha_i^2 < \infty
\]
Let $a_i^{(n)}$ be the allocation to the $i$th location at the beginning of the $n$th period; i.e. at the beginning of each period there is to be a new allocation. We can let $a_i^{(1)}$ be arbitrary. Recursively let
\[
a_i^{(n+1)} = a_i^{(n)} + \alpha_i(y_i^{(n)} - c/S)
\]
where \( y_i^{(n)} = 1 \) if the demand in the \( n \)th period at location \( i \) exceeds the supply = 0 otherwise.

It is known under fairly general conditions (see [8]) that \( a_i^{(n)} \) converges to the value \( a_i^* \) having the property that

\[
\int_{a_i^*}^{\infty} f_i(x) \, dx = c/S
\]

This procedure has the advantage that, at each period, only small adjustments need be made; thus not radically altering the status quo.

7. Reallocation During Time Period

In many situations the possibility for reallocation of the units during the time period exists. Thus if the demand at one location seems to be light, units can be shifted to another location where there is more demand. The question arises as to what is the optimal allocation among the \( k \) units, knowing that such a reallocation can take place.

A reasonable conjecture as to what the optimal allocation should be is as follows: Suppose the time period is divided into two periods \( a \) and \( b \). Let \( f_{ia} \) denote the density of demand at the \( i \)th location during the period \( a \); \( f_{ib} \) the density for period \( b \). Allocate initially so that (4) and (1') are satisfied using the densities \( \{f_{ia}\} \). Then if \( X \) denotes the total satisfied demand during period \( a \), reallocate for period \( b \) so that (4) and (1') (replacing \( T \) by \( T - X \)) are satisfied using the densities \( \{f_{ib}\} \).

Unfortunately, it turns out that the above conjecture is not true. We present a counter-example to show this. Suppose \( T = 2, k = 2 \) and

\[
\begin{align*}
  f_{1a}(t) &= \frac{1}{2} & 0 < t < 2 \\
  &= 0 & \text{otherwise} \\
  f_{2a}(t) &= 1 & 0 < t < 1 \\
  &= 0 & \text{otherwise} \\
  f_{1b}(t) &= f_{1a}(t); & f_{2b}(t) = f_{2a}(t)
\end{align*}
\]

Let \( X_a \) denote the total satisfied demand during period \( a \) and \( X_b \) denote the total satisfied demand during period \( b \). The problem is to find the allocations during periods \( a \) and \( b \) which maximize \( E(X_a + X_b) \). Let \( E(X_b \mid X_a) \) denote the conditional expectation of \( X_b \) given \( X_a \). Then

\[
E(X_a + X_b) = E(X_a + E(X_b \mid X_a)).
\]

It is clear that in order for (6) to be maximized, \( E(X_b \mid X_a) \) must be maximized. This is accomplished by allocating the \( T - X_a \) units so that they satisfy (4) and (1'). If this is done then

\[
E(X_b \mid X_a) = \frac{4}{3} - \frac{1}{2}X_a - \frac{1}{2}X_a^2
\]
Hence we want to choose the initial allocation $a_1$ and $a_2$ so that
\begin{equation}
E(X_a + \frac{4}{3} - \frac{1}{3}X_a - \frac{2}{3}X_a^2) = \frac{3}{3}EX_a - \frac{1}{3}EX_a^2 + \frac{4}{3}
\end{equation}
is maximized.

Now, for any given initial allocation satisfying $a_1 + a_2 = 2$, (also $a_2 \leq 1$) it can be shown that the distribution of $X_a$ is
\begin{align*}
f(t) &= \frac{1}{2}t & 0 \leq t < a_2 \\
&= \frac{1}{2} & a_2 \leq t < a_1 \\
&= \frac{3}{2} - \frac{t}{2} & a_1 \leq t < 2 \\
\end{align*}

\begin{equation}
P(X_2 = 2) = \frac{(1 - a_2)a_2}{2}.
\end{equation}

We have then
\begin{align*}
EX_a &= 1 + \frac{a_2}{4}; \quad EX_a^2 = \frac{4}{3} + a_2^2 - \frac{4}{3}a_2^3.
\end{align*}

Hence (7) becomes
\begin{equation}
\frac{3}{3}EX_a - \frac{1}{3}EX_a^2 + \frac{4}{3} = \frac{4}{3} - \frac{1}{3}a_2^2 + \frac{5}{2}a_2^3.
\end{equation}

It is easily seen that (8) is maximized for $a_2 = \frac{1}{5}$, $a_1 = \frac{3}{5}$. However, if the conjecture were true we would have by (4) and (1')
\begin{equation}
\frac{(2 - a_1)}{2} = 1 - a_2; \quad a_1 + a_2 = 2,
\end{equation}
i.e. $a_2 = \frac{3}{5}$, $a_1 = \frac{2}{5}$. Thus the conjectured optimal differs from the actual optimal allocation.

References