I. INTRODUCTION

We consider how material anisotropy effects the directional partition of reverberant or diffuse wave energy. Diffuse waves in solids are the long time response when multiple scattering has equilibrated the energy distribution among modes. Preferential orientation of the root mean square particle velocity does not arise in isotropic materials but is a characteristic of anisotropy. Our objective is to describe this orientation effect and to quantify it in real materials. An ability to determine, directly or by inference, the orientational distribution of kinetic energy density in a solid allows one to essentially “hear” the texture of a crystal. We will demonstrate that the key quantity that needs to be measured is the autocorrelation function, or the Green’s function evaluated at its source. By deriving an explicit formula for the autocorrelation, or the admittance matrix, we can completely describe the directional distribution of the diffuse wave energy.

We introduce two quantities for the description of reverberant energy in the presence of anisotropy: the participation tensor $G$ and the modal spectral density per unit volume, $d(\omega)$. The two are in fact intimately related as we will see. Under steady state time harmonic conditions the total energy of a body is equally divided between potential and kinetic. The latter is $1/2 \omega^2 \int dV \rho \vec{u}^2$ where $\vec{u}$ is the root mean square particle displacement, and assuming a uniform spatial distribution, the total energy is $E = V \rho \omega^2 \vec{u}^2$. This may be inverted to express the mean square displacement. Let $\vec{u}_i = \vec{u} \cdot e_i$ where $e_i$, $i = 1, 2, 3$ is an orthonormal triad. Since $\vec{u}_i^2 + \vec{u}_j^2 + \vec{u}_k^2 = |\vec{u}|^2$ we may write

$$\vec{u}_i^2 = \frac{E}{3V \rho \omega^2} \vec{G}_i, \quad \vec{G}_i = \vec{e}_i \cdot G \cdot \vec{e}_i,$$  \hspace{1cm} (1)

for $i = 1, 2, 3$ (no sum) where $G$ is a second-order symmetric tensor satisfying

$$\text{tr } G = 3. \hspace{1cm} (2)$$

For isotropic materials $G$ is simply the unit matrix or identity (second-order) tensor. Deviations from this can occur under three general situations: (i) If the field point is near a surface or boundary. This was considered in detail by Weaver, who found expressions for the components of $G$ at a free surface in terms of simple integrals, see also Egle. By analogy, $G$ will be influenced by local inhomogeneity in the material, for instance if the field point is close to a rigid inclusion, or a void. We will not discuss this further here. (ii) Material anisotropy can also influence $G$. Here we consider the simplest case of a field point in a homogeneous material of infinite extent. It is expected that $G$ displays the symmetries appropriate to the degree of anisotropy. Thus, it is characterized by a single parameter for materials with isotropic and cubic symmetries, and by two or three parameters for materials with lower symmetry.

The spectral density of modes $D$ at frequency $\omega$ in a volume $V$ is $D(\omega) = V d(\omega)$. It can be estimated as $D = \partial N/\partial \omega = V \omega^2/c^3$ by noting the total number of modes scales as $N(k) \approx V k$ where $k = \omega/c$ is typical wave number. A more precise counting yields, for isotropic bodies, the well-known result

$$d(\omega) = \frac{\omega^2}{2\pi^2} \left( \frac{2}{c_l^3} + \frac{1}{c_t^3} \right),$$ \hspace{1cm} (3)

where $c_t$ and $c_l$ are the longitudinal and transverse elastic wave speeds.

The objective is to derive analogous expressions of $d(\omega)$ and $G$ for anisotropic elastic materials. This will be achieved by explicit calculation of the admittance tensor $A$, defined in Sec. II, combined with a general relation between $d(\omega)$, $G$, and $A$. The spectral density and the participation tensor in the presence of material anisotropy do not appear to have received much attention. Some work on the related issue of admittance in bounded anisotropic thin plate systems has appeared. Weaver considered isotropic plates of finite thickness and infinite lateral extent. Tewary et al. derived an expression for the admittance at the free surface of an anisotropic half space as a double integral. Here the focus is on infinite systems, and the modal density per unit volume in this limit. Finite structures, such as plates both thin and of finite thickness, will be considered in a separate paper.
Our principal results are that the modal spectral density per unit volume and the participation tensor are given by

\[ d(\omega) = \frac{\omega^2}{2\pi^2} (\langle \mathbf{Q} \rangle^{3/2}), \]

\[ \mathbf{G} = 3 \frac{(\langle \mathbf{Q} \rangle^{3/2})}{(\langle \mathbf{Q} \rangle^{3/2})}, \]

where \( \mathbf{Q}(\mathbf{n}) \) is the acoustical or Christoffel tensor for plane waves propagating in the direction \( \mathbf{n} \), and \( \langle \cdot \rangle \) is the orientation average of a function that depends on the direction, \( \mathbf{n} \),

\[ \langle \cdot \rangle = \frac{1}{4\pi} \int_{4\pi} d\Omega(\mathbf{n})f(\mathbf{n}). \]

In an isotropic solid Eq. (4a) reduces to Eq. (3) and \( \mathbf{G} \) is simply the identity \( \mathbf{I} \). After deriving Eq. (4), the remainder of the paper will explore its implications, in particular the form of \( \mathbf{G} \) is investigated, and the parameters in Table I are deduced.

It is interesting to note that the material constant that determines the density of states of diffuse waves, \( \langle \mathbf{Q} \rangle^{3/2} \), also defines the Debye temperature \( \Theta \) of a crystal. Thus (see Chap. 9 of Ref. 17),

\[ \Theta = \frac{h}{k} \left( \frac{18\pi^2}{V_a \langle \mathbf{Q} \rangle^{3/2}} \right)^{1/3}, \]

where \( h \) is Planck’s constant, \( k \) is Boltzmann’s constant, and \( V_a \) is the volume per atom or lattice site. Fedorov \(^7\) provides a detailed discussion of \( \langle \mathbf{Q} \rangle^{3/2} \) in this context. The emphasis in this paper is on the more general tensor \( \langle \mathbf{Q} \rangle^{3/2} \) although connections with Fedorov’s analysis will be mentioned later.

The outline of the paper is as follows. The admittance tensor \( \mathbf{A} \) is defined and calculated in Sec. II, from which the main result (4) follows. Several alternative representations of the fundamental quantity \( \mathbf{Q}^{3/2} \) are developed in Sec. III. In particular it is shown that \( \mathbf{G} \) for transverse isotropy can be evaluated as a single integral. Weak anisotropy is considered in Sec. IV and numerical examples are presented in Sec. V.

II. DERIVATION OF \( d(\omega) \) AND \( \mathbf{G} \)

A. Admittance tensor

The admittance \( \mathbf{A} \) is a second-order tensor defined by the average power radiated by a time harmonic point force \( \mathbf{F} \) according to

\[ \Pi = \mathbf{F} \cdot \mathbf{A} \cdot \mathbf{F}. \]

Alternatively, \( \mathbf{A} \) is equal to the power expended at the source point—which is the more conventional definition of admittance, as the the inverse of drive point impedance. The admittance is clearly related to the autocorrelation of the Green’s function, and as such is a special case of the two-point cross correlation of the Green’s function. \(^7\) The important connection for the present purposes is the relation between the radiation from a point force and the diffuse wave density. \(^8,9\) In the present notation this becomes

\[ \mathbf{A} = \frac{\pi}{12\rho} d(\omega) \mathbf{G}. \]

A short derivation of Eq. (8) is given in Appendix A. The admittance of isotropic bodies is simply determined from Eq. (3) and \( \mathbf{G} = \mathbf{I} \). Our objective here is to calculate \( \mathbf{A} \) for anisotropic solids, and then to use the result to determine \( d(\omega) \) and \( \mathbf{G} \).

The central result for \( \mathbf{A} \) is the following: The second-order symmetric admittance tensor of Eq. (7) that determines the total power radiated to infinity from the point source averaged over a period is

\[ \mathbf{A} = \frac{\omega^2}{8\pi\rho} (\langle \mathbf{Q} \rangle^{3/2}), \]

where \( \mathbf{Q}(\mathbf{n}) \) is the acoustical tensor,

\[ Q_{ijl}(\mathbf{n}) = c_{ijkl} n_i n_l \text{ with } c_{ijkl} = \frac{1}{\rho} C_{ijkl}. \]

The elastic moduli (stiffness) \( C_{ijkl} \) have the symmetries \( C_{ijkl} = C_{klji} \) and \( C_{ijkl} = C_{jikl} \), and thus have at most 21 independent elements. Note that \( \mathbf{A} \) has dimensions of admittance (inverse impedance). We next derive Eq. (9) by explicitly calculating the admittance for a time harmonic point force.

B. Radiation from a point force

The displacement resulting from a point force \( \mathbf{F} \cos \omega t \) at the origin is \( \mathbf{u}(\mathbf{x}, t) = \text{Re} \mathbf{u}(\mathbf{x}, \omega) e^{i\omega t} \) where \( \mathbf{u} \) satisfies

\[ C_{ijkl} \tilde{u}_{kjl} + \rho \omega^2 \tilde{u}_i = -F_i \delta(x), \quad -\infty \leq x_1, x_2, x_3 \leq \infty. \]

Here \( \rho \) is the mass density and \( \delta(x) \) is the three-dimensional Dirac delta function. The equation of motion may be written

\[ \mathbf{Q} \nabla \mathbf{u} + \omega^2 \mathbf{u} = -\frac{1}{\rho} \delta(x) \mathbf{F}, \]

and the problem definition is completed by the requirement that the energy radiates away from the point source.

The solution to Eq. (11) in a solid of infinite extent is well known. For our purpose we will find the following representation from Norris [Ref. 10, Eq. (3.22)] useful for determining the admittance:
\[ \tilde{u} = \frac{1}{8 \pi^2 \rho |x|} \int d\theta(n) \sum_{j=1}^{3} \frac{q_j \otimes q_j}{\lambda_j} \mathbf{F} + \frac{1}{16 \pi^2 \rho} \int d\Omega(n) \sum_{j=1}^{3} i k_j q_j \otimes q_j e^{i k_j n \cdot x}. \] (12)

Here \( \lambda_1, \lambda_2, \lambda_3 \) are the eigenvalues and \( q_1, q_2, q_3 \) the eigenvectors of \( Q(n) \), which then has the spectral decomposition

\[ Q(n) = \lambda_1 q_1 \otimes q_1 + \lambda_2 q_2 \otimes q_2 + \lambda_3 q_3 \otimes q_3. \] (13)

Also, \( k_j = \omega / \lambda_j^{1/2} \) are the wave numbers of the three distinct branches of the slowness surface defined by the eigenvectors. The first integral in Eq. (12) is around the unit circle formed by the intersection of the plane \( n \cdot x = 0 \) with the unit \( n \) sphere. This is just the static Green’s function of elasticity.\(^{10}\)

Before considering the properties of \( Q \) and its directional density, let us evaluate its invariants.\(^{10}\)

The key quantity is the tensor \( Q^{-3/2} \) and its directional average. In practice, this may be evaluated numerically without difficulty. It is however useful to examine semieexplicit forms for the tensor, both for general anisotropy and for specific symmetries, particularly the case of transverse isotropy. We begin with two alternative and general formulations based on the spectral properties and the invariants of the acoustical tensor.

### A. General representations for arbitrary anisotropy

#### 1. A method based on invariants

Functions of a positive definite tensor can be simplified using the Cayley–Hamilton formula for the tensor, for which \( Q \) is

\[ Q^3 - I_1 Q^2 + I_2 Q - I_3 I = 0. \] (19)

The principal positive invariants of \( Q \) are

\[ I_1 = \text{tr} \, Q, \quad I_2 = \frac{1}{2} (\text{tr} \, Q)^2 - \frac{1}{2} \text{tr} \, Q^2, \quad I_3 = \det \, Q. \] (20)

Based on these fundamental properties, it can be shown that \( Q^{-3/2} \) can be expressed as functions of the invariants \( I_1, I_2, \) and \( I_3 \) which can be evaluated numerically with the invariants \( I_1, I_2, \) and \( I_3 \), see the following. Details of the derivation of Eq. (21) are given in Appendix B.

The appealing feature of Eq. (21) for \( Q^{-3/2}(n) \) is that it only involves powers of \( Q \), its three invariants, and the additional invariants \( i_1, i_2, \) and \( i_3 \). These are related to \( I_1, I_2, \) and \( I_3 \) by\(^{11,12}\)

\[ i_1^2 - 2 i_2 = I_1, \quad i_2^2 - 2 i_1 i_3 = I_2, \quad i_3^2 = I_3. \] (22)

The last implies \( i_3^2 = I_3^{-1/2} \), while expressions for \( i_1 \) and \( i_2 \) are given by Hoger and Carlson\(^{11}\) and by Norris.\(^{12}\) For instance,

\[ i_1 = \sqrt{I_1 - \beta} + 2 \sqrt{I_1 / \beta}, \] (23a)
\[ i_2 = \sqrt{I_2 - 3 I_3 / \beta} + 2 \sqrt{I_3 / \beta}, \] (23b)
\[ i_3 = \sqrt{I_3}, \] (23c)

where \( \beta \) is any eigenvalue of \( Q \), e.g.,

\[ \beta = \frac{1}{3} (I_1 + [(\xi + \sqrt{\xi^2 - (I_1^2 - 3 I_3)^2})^{1/3} + [(\xi - \sqrt{\xi^2 - (I_1^2 - 3 I_3)^2})^{1/3}], \] (24a)

\[ \xi = \frac{1}{2} (2 I_1^3 + 9 I_1 I_2 + 27 I_3). \] (24b)

Hence, the density of states per unit volume is

\[ \langle Q^{-3/2} \rangle = \frac{1}{2} (c_1^{-3} + 2 c_2^{-3}) I. \] (18)
Note that\(^{13}\) \(i_1i_2-i_3=\det(i_1I-Q^{1/2})>0\).

Taking the trace of Eq. (21) gives
\[
\text{tr } Q^{-3/2} = \frac{(I_1+i_2)I_2I_3+(I_2-2I_1I_3)i_1i_3-3I_2}{(i_1i_2-i_3)^2}.
\]

(25)

This quantity, when averaged over all orientations, gives the density of states function \(d(\omega)\) of Eq. (4a). Hence \(\lambda\) can be calculated from the invariants \(Q\) and the derived invariants \(i_1, i_2, i_3\).

2. A spectral representation

The second form for \(Q^{-3/2}\) is based on the spectral decomposition (17). The latter can be expressed in a form that does not explicitly involve the eigenvectors,
\[
Q^{-3/2} = \lambda_1^{-3/2}N_1\lambda_1^{-1/2} + \lambda_2^{-3/2}N_2\lambda_2^{-1/2} + \lambda_3^{-3/2}N_3\lambda_3^{-1/2}.
\]

(26)

The second-order tensors \(N_1, N_2, N_3\) which are alternative expressions for the dyadics formed by the eigenvectors, \(\mathbf{q}_j \otimes \mathbf{q}_j\), can be expressed in terms of \(Q\) using Sylvester’s formula
\[
N(\lambda, \mathbf{n}) = \frac{\lambda Q^2 - \lambda(I - I_1)\lambda_1^3 + I_1^3}{\lambda^3 - (I - I_1)\lambda_1^2 + I_1^2}.
\]

(27)

The identity (26) is derived in Appendix B. Calculation of Eq. (26) requires knowledge of the three eigenvalues, which are zeros of the characteristic polynomial defined by Eq. (19),
\[
p(\lambda) = \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3.
\]

(28)

The eigenvalues \(\{\lambda_1, \lambda_2, \lambda_3\}\) can be expressed in terms of the invariants as
\[
\{\beta, \frac{1}{2}(I_1 - \beta) \pm \frac{1}{4}\sqrt{(I_1 - \beta)^2 - 4I_2/\beta}\},
\]

(29)

where \(\beta\) is defined in Eq. (24a). Every\(^{14}\) derived alternate closed-form expressions based on the trigonometric solution of the characteristic cubic. The alternative version of Eq. (25) is
\[
\text{tr } Q^{-3/2} = \lambda_1^{-3/2} + \lambda_2^{-3/2} + \lambda_3^{-3/2},
\]

(30)

which is the starting point for Fedorov’s calculation\(^{17}\) of the trace.

B. Transverse isotropy

Transverse isotropy or hexagonal symmetry is an important class of anisotropy. It occurs in many practical circumstances, whether from layering in the earth to laminated composite materials, or from underlying crystal structure. It is the highest symmetry for which the participation factor tensor is not the identity, since \(G = I\) under isotropy and cubic material symmetry. We now demonstrate that the evaluation of \(d\) and \(G\) may be reduced to the evaluation of two single integrals, one for \(\text{tr } Q^{-3/2}\) and one for the parameter \(\alpha\) that defines \(G\), see Table I.

Transversely isotropic solids have five independent moduli: \(c_{11}=c_{22}, c_{23}, c_{12}, c_{13}=c_{23}, c_{44} = c_{55}\). Let \(\mathbf{e}\) be the axis of symmetry. The SH slowness decouples to give
\[
Q = \lambda_3(\mathbf{n} \cdot \mathbf{e})\mathbf{q}_3 \otimes \mathbf{q}_3 + Q_\perp,
\]

(31)

where (Ref. 15, p. 95)
\[
\lambda_3(\mathbf{n} \cdot \mathbf{e}) = c_{66} + (c_{44} - c_{66})(\mathbf{n} \cdot \mathbf{e})^2,
\]

(32)

and \(q_3 = e \otimes n / (e \cdot n)\). The two-dimensional symmetric tensor \(Q_\perp\) is\(^{15}\)
\[
Q_\perp = [c_{44} + (c_{33} - c_{44})(\mathbf{n} \cdot \mathbf{e})^2]e \otimes e + [c_{11} + (c_{44} - c_{11})(\mathbf{n} \cdot \mathbf{e})^2]d \otimes d + (c_{13} + c_{44})\mathbf{n} \cdot \mathbf{e} \mathbf{1} - (\mathbf{n} \cdot \mathbf{e})^2[d \otimes e + e \otimes d],
\]

where \(d = e \otimes q_3\). Replacing \(\mathbf{n} \cdot \mathbf{e}\) by the integration parameter \(\xi\), it follows that
\[
\langle \lambda_3^{-3/2}q_3 \otimes q_3 \rangle = \frac{1}{2} \int_0^1 d\xi \lambda_3^{-3/2}(\xi)I_\perp,
\]

(33)

where \(I_\perp\) projects onto the plane perpendicular to \(\mathbf{e}\),
\[
I_\perp = I - e \otimes e.
\]

(34)

It remains to consider the orientational average of \(Q_\perp^{-3/2}\). The tensor \(Q_\perp\) satisfies a quadratic Cayley–Hamilton equation
\[
Q_\perp^2 - J_1Q_\perp + J_2I_\perp = 0,
\]

(35)

with \(J_1 = \text{tr } Q_\perp = \lambda_1 + \lambda_2\) and \(J_2 = \det Q_\perp = \lambda_1\lambda_2\). Similarly, the Cayley–Hamilton equation for the square root is
\[
(Q_\perp^{1/2})^2 - j_1Q_\perp^{1/2} + j_2I_\perp = 0,
\]

(36)

where \(j_1 = \text{tr } Q_\perp^{1/2}\) and \(j_2 = \det Q_\perp^{1/2}\) satisfy \(J_1 = j_1^2 - 2j_2, J_2 = j_2^2\), and are therefore related to \(J_1\) and \(J_2\) by \(j_1 = \sqrt{J_1+2\sqrt{J_2}}, j_2 = \sqrt{J_2}\). Using Eqs. (35) and (36), respectively, leads to the identities
\[
Q_\perp^2 = J_2^{-2}[(J_1 - J_2)I_\perp - J_1Q_\perp],
\]

(37a)

\[
Q_\perp^{1/2} = J_1^{-1}(Q_\perp + j_2I_\perp).
\]

(37b)

Multiplication of these and further use of Eq. (35) leads to
\[
Q_\perp^{-3/2} = \frac{1}{j_1j_2J_2}[\det((J_1 - J_2)I_\perp - J_1Q_\perp) - J_2I_\perp].
\]

(38)

Again using \(\xi = n \cdot e\), we have
\[
\langle \text{tr } Q^{-3/2} \rangle = \int_0^1 d\xi \int_0^1 J_1^2[1 - \sqrt{j_2}]\sqrt{J_1 + 2\sqrt{J_2} + \lambda_3^{-3/2}(\xi)],
\]

and from Table I,
\[
\alpha = \frac{3}{3} \int_0^1 \int_0^1 \frac{(J_1 + \sqrt{j_2}j_1 - e \cdot Q_\perp \cdot e) - J_2}{J_2^{-2}\sqrt{J_1 + 2\sqrt{J_2}}}.
\]

The modal density parameter (tr \(Q^{-3/2}\)) and the scalar \(\alpha\) that defines the participation tensor can therefore be expressed as single integrals, which follow from the above-presented results and Eqs. (31)–(33), as
where
\[ a = c_{11} + c_{44}, \quad b = c_{33} - c_{11}, \]
\[ c = c_{44} - c_{11}, \quad d = c_{11}c_{44}, \]
\[ e = c_{11}c_{33} - c_{13}^2 - 2c_{44}(c_{11} + c_{13}), \]
\[ f = -c_{11}c_{33} + c_{13}^2 + c_{44}(c_{11} + c_{33} + 2c_{13}). \]

IV. WEAK ANISOTROPY

Although the general expressions for the modal density \( d \) and the participation tensor \( G \) are not difficult to compute, it is often the case that the medium is to a first approximation isotropic, and appropriate approximations can be made. The state of small or weak anisotropy is defined relative to a background isotropic medium, and it is important to select the latter properly. In this section we calculate \( d \) and \( G \) in the presence of weak anisotropy. Fedorov\textsuperscript{17} provides a detailed analysis of the expansion of \( \text{tr}(Q^{-3/2}) \) to arbitrary orders in the perturbation parameter. Our emphasis is more on obtaining estimates of the tensor \( (Q^{-3/2}) \), which is not discussed explicitly by Fedorov. We begin with a description of the comparison isotropic moduli and then proceed to calculate the first two terms in a perturbation series for \( d \) and \( G \).

A. Background isotropic moduli

Regardless of the level of the anisotropy it is always possible to define a unique set of isotropic moduli which minimize the Euclidean distance between the exact set of moduli and the equivalent isotropic moduli.\textsuperscript{16} This procedure is equivalent to requiring that the mean square Euclidean difference in the slowness surfaces is minimal.\textsuperscript{16,17} Thus, let the background isotropic moduli be
\[
C_{ijkl}^{(0)} = \frac{1}{2} \delta_{ij} \delta_{kl} + \frac{1}{2} \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - 2 \delta_{ij} \delta_{kl},
\]
where \( c_l \) and \( c_t \) are the effective longitudinal and transverse wave speeds. These are defined by simultaneously minimizing the quantity \( \langle (Q - Q_0)^2 \rangle \) with respect to both \( c_l \) and \( c_t \), where \( Q_0(n) \) is defined by the moduli \( C_{ijkl}^{(0)} \). The unique solution is
\[
c_l^2 = \frac{1}{3} \text{tr} C_r, \quad c_t^2 = \frac{1}{3} \text{tr} C_l,
\]
where the second-order tensors of reduced moduli are
\[
C_{ij} = \frac{2}{3} C_{ijkl} + \frac{1}{5} c_{ijkl}, \quad C_{ij} = \frac{3}{10} C_{ijkl} - \frac{1}{10} c_{ijkl}.
\]
The background Lamé moduli \( \lambda \) and \( \mu \) are obtained using \( c_l^2 = (\lambda + 2\mu)/\rho \) and \( c_t^2 = \mu/\rho \). The elements of \( C_l \) and \( C_t \) follow from
\[
C_{ij} = \begin{pmatrix}
    c_{11} + c_{12} + c_{13} & c_{16} + c_{26} + c_{36} & c_{15} + c_{25} + c_{35} \\
    c_{16} + c_{26} + c_{36} & c_{12} + c_{22} + c_{32} & c_{14} + c_{24} + c_{34} \\
    c_{15} + c_{25} + c_{35} & c_{14} + c_{24} + c_{34} & c_{13} + c_{23} + c_{33}
\end{pmatrix},
\]
\[
C_{ij} = \begin{pmatrix}
    c_{11} + c_{55} + c_{66} & c_{16} + c_{26} + c_{45} & c_{15} + c_{46} + c_{35} \\
    c_{16} + c_{26} + c_{45} & c_{22} + c_{44} + c_{66} & c_{24} + c_{44} + c_{56} \\
    c_{15} + c_{46} + c_{35} & c_{24} + c_{44} + c_{56} & c_{33} + c_{44} + c_{55}
\end{pmatrix}.
\]

B. Perturbation analysis

Let
\[
c_{ijkl} = c_{ijkl}^{(0)} + \epsilon c_{ijkl}^{(1)},
\]
where the nondimensional parameter \( \epsilon \) is introduced only to simplify the perturbation analysis. In practice \( \epsilon \) is set to unity on the assumption that the additional moduli \( c_{ijkl}^{(1)} \) are small in comparison with the isotropic background.

We seek expansions in powers of the small parameter \( \epsilon \). The key quantity \( Q^{-3/2} \) will be determined as the product of \( Q^{-2} \) and \( Q^{1/2} \). Based on Eq. (43), the acoustical tensor is
\[
Q = Q_0 + \epsilon Q_1,
\]
and simple perturbation gives
\[
Q^{-2} = Q_0^{-2} - \epsilon(Q_0^{-2} Q_0^{-1} + Q_0^{-1} Q_0^{-2}) + O(\epsilon^2).
\]
Let
\[
Q^{1/2} = Q_0^{1/2} + \epsilon S_1 + O(\epsilon^2),
\]
then \( S_1 \) satisfies
\[
Q_0^{1/2} S_1 + S_1 Q_0^{1/2} = Q_1.
\]
In order to calculate \( Q^{-2} \) and also the square root of \( Q \), we now use the fact that the leading order moduli \( c_{ijkl}^{(0)} \) are isotropic. The explicit form of \( Q_0^{1/2} \) follows from Eq. (17) and the identity
\[
Q_0 = c_l^2 n \otimes n + c_t^2 p \otimes p,
\]
where \( m \) is any real number and \( P = I - n \otimes n \). Equation (45) can be solved by observing that \( Q_1 \) may be partitioned as
\[
Q_1 = Q_1^{(1)} + Q_1^{(2)} + Q_1^{(3)}
\]
where \( Q_1^{(1)} = n \cdot Q_1 \cdot n \otimes n \), \( Q_1^{(2)} = P Q_1 P \), and \( Q_1^{(3)} = P Q_1 \cdot n \otimes n + n \otimes n \cdot P Q_1 \). Assuming a solution of
the form $S_1 = p_1 Q_1^{(1)} + p_2 Q_1^{(2)} + p_3 Q_1^{(3)}$, the coefficients can be determined easily from Eq. (45), i.e.,

$$S_1 = \frac{1}{2c_i} Q_1^{(1)} + \frac{1}{2c_i} Q_1^{(2)} + \frac{1}{c_i + c_i} Q_1^{(3)}.$$  

(47)

Combining the asymptotic expansions for $Q^{-2}$ and $Q^{1/2}$ gives

$$Q^{-3/2} = Q_0^{-3/2} + eV_1 + O(e^2),$$

(48)

where

$$V_1 = Q_0^{-2} S_1 - Q_0^{-2} Q_1 Q_0^{-1/2} - Q_0^{-1} Q_1 Q_0^{-3/2}$$

$$= -3 \frac{c_1}{2c_i} Q_1 - \left[ \frac{c_1^2 + c_1^2 + c_1 c_i}{c_i} \right] + 3$$

$$\times [Q_1 \cdot n \otimes n + n \otimes Q_1 \cdot n]$$

$$+ 2 \left[ \frac{c_1^2 + c_1^2 + c_1 c_i}{c_i^3} \right] - \frac{3}{2c_i^5} - \frac{3}{2c_i^5}$$

$$\times (n \cdot n) Q_1 \cdot n \otimes n.$$

The orientational average $\langle Q^{-3/2} \rangle$ can then be effected using the identities

$$\langle n \cdot n \rangle = \frac{1}{15} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) = K_{ijkl},$$

$$\langle n \cdot n \rangle = \frac{1}{15} (\delta_{ij} K_{klij} + \delta_{ik} K_{jlij} + \delta_{il} K_{klij} + \delta_{jl} K_{klji} + \delta_{kl} K_{ij} + \delta_{ij} K_{klij})$$

The resulting expressions for $\langle Q^{-3/2} \rangle$ is

$$\langle Q_{ij}^{-3/2} \rangle = \frac{1}{3} \left[ \frac{2}{c_i} + \frac{1}{c_i} \right] \right] \delta_{ij} + e^2 \left[ - \frac{1}{2c_i} c_{ijkl}^{(1)} - \frac{1}{15} \right]$$

$$\times \left[ \frac{c_1^2 + c_1^2 + c_1 c_i}{c_i} \right] - \frac{3}{2c_i^5} \left( c_{ijkl}^{(1)} + 2c_{ijkl}^{(1)} \right)$$

$$+ 1 \left[ \frac{2}{c_i^3} \right] - \frac{3}{2c_i^5} \left( c_{ijkl}^{(1)} + 2c_{ijkl}^{(1)} \right)$$

$$\times \left[ \delta_{ij} c_{ijkl}^{(1)} + 2c_{ijkl}^{(1)} + 4c_{ijkl}^{(1)} + 2c_{ijkl}^{(1)} \right] + O(e^3).$$

We note that both $c_{ijkl}^{(1)}$ and $c_{ijkl}^{(1)}$ vanish by virtue of the choice of the background isotropic moduli. This implies that the trace of $\langle Q^{-3/2} \rangle$ differs from the isotropic approximant only at the second order of anisotropic perturbation,

$$\text{tr}(Q^{-3/2}) = \frac{2}{c_i} + \frac{1}{c_i} + O(e^2).$$  

(49)

This is in agreement with Fedorov,\(^{17}\) who also provides explicit forms for the higher order terms; for instance, the expansion for cubic crystals up to fourth order in the perturbation is given by Eqs. (50.12)–(50.14) of Ref. 17. The leading order approximation of Eq. (49) when combined with the identity (4b), gives

$$G_{ij} = \delta_{ij} - e \left[ \frac{2}{c_i} + \frac{1}{c_i} \right] \left[ \frac{3}{2c_i} c_{ijkl}^{(1)} + \frac{3}{35} \left( \frac{c_1^2 + c_1^2 + c_1 c_i}{c_i} \right) \right]$$

$$+ \frac{2}{c_i^3} - \frac{3}{2c_i^5} \right] \times \left( c_{ijkl}^{(1)} + 2c_{ijkl}^{(1)} \right) + O(e^2).$$

Ignoring terms of order $e^2$ and then setting $e = 1$ yields the leading order approximation to the participation tensor as

$$G = I + a_i (I - c_i^2 C_i) + a_i (I - c_i^2 C_i),$$

(50)

where the nondimensional coefficients are

$$a_i = \frac{6}{7(2 + \kappa^3)} \left( \frac{1}{\kappa} + \frac{1}{\kappa + 1} + \frac{3}{4} \right),$$

(51a)

$$a_i = \frac{3}{2 + \kappa^3},$$

(51b)

and

$$\kappa = \frac{c_i}{c_i}.$$  

(52)

Figure 1 shows $a_i$ and $a_i$ as functions of the Poisson’s ratio $\nu$, using $\kappa^2 = 2(1 - \nu)/(1 - 2\nu)$. Note that $1.27 < a_i < 3/2$ for $0 < \nu < 1/2$ while $a_i = -5/24(1 - 2\nu)^{-1}$ as $\nu \rightarrow 1/2$.

C. Transversely isotropic materials

As an example of the general perturbation approach, we consider the particular case of TI materials. We take the axis of symmetry ($e$ in Sec. III) in the 3 direction so that

$$c_{ijkl} = \begin{pmatrix}
  c_{11} + c_{12} + c_{13} & 0 & 0 & 0 \\
  0 & c_{11} + c_{12} + c_{13} & 0 & 0 \\
  0 & 0 & c_{33} + 2c_{13} & 0 \\
  c_{11} + c_{34} + c_{66} & 0 & 0 & 0 \\
  0 & c_{11} + c_{34} + c_{66} & 0 & 0 \\
  0 & 0 & c_{33} + 2c_{44} & 0
\end{pmatrix},$$

(53a)

where $c_{66} = \frac{1}{2}(c_{11} - c_{12})$. The wave speeds in the background isotropic medium are then

$$c_i^2 = \frac{1}{15} (8c_{11} + 3c_{33} + 4c_{13} + 8c_{44}).$$

(53a)
According to Table I the participation tensor is defined by a single parameter, \( \alpha \), which to leading order is unity. Let

\[
\alpha = 1 - 2 \beta,
\]

so that

\[
G = \begin{pmatrix}
1 + \beta & 0 & 0 \\
0 & 1 + \beta & 0 \\
0 & 0 & 1 - 2 \beta
\end{pmatrix}.
\]

Applying the general perturbation theory we find that the leading order correction to the isotropic participation tensor is given by

\[
\beta = \frac{a_1}{15c_i^2}(-4c_{11} + 3c_{33} + c_{13} + 2c_{44}) \\
+ \frac{a_2}{30c_i^2}(-c_{11} + 2c_{33} - c_{13} + 3c_{44} - 5c_{66}),
\]

where \( a_1 \) and \( a_2 \) are defined in Eq. (51a).

Thomsen’s anisotropy parameters\(^\text{18} \) \( \epsilon, \gamma, \delta \) provide a means to characterize weakly anisotropic TI materials. The parameters are defined as

\[
\epsilon = (c_{11} - c_{33})/(2c_{33}), \quad \gamma = (c_{66} - c_{44})/(2c_{44}), \quad \delta = [(c_{13} + c_{44})^2 - (c_{33} - c_{44})^2]/[2c_{33}(c_{33} - c_{44})],
\]

and are commonly used in geophysical applications to describe rock properties. The correction term \( \beta \) can be expressed in terms of the Thomsen parameters as

\[
\beta = a_1 \epsilon + a_2 \delta + a_3 \gamma,
\]

where the coefficients \( a_1, a_2, \) and \( a_3 \) are (see Fig. 2)

\[
a_1 = \frac{8a_1}{15} - \frac{\kappa^2 a_i}{15}, \quad a_2 = \frac{a_i}{15}, \quad a_3 = -\frac{a_i}{3}.
\]

V. EXAMPLES AND DISCUSSION

The participation matrix was computed for many anisotropic solids. Table II summarizes the results for a selection of materials with anisotropy ranging from weak to strong. Table II provides the numerical values of diagonal elements of \( G \) (there are no off-diagonal elements for the symmetries considered). In each case the elements sum to three, \( G_{11} + G_{22} + G_{33} = 3 \), although the individual numbers can differ markedly from unity.

In order to quantify the level of anisotropy, Table II also shows the number \( \text{dist} \). This is a nondimensional positive measure of the degree of anisotropy of a set of anisotropic elastic constants. dist is chosen here as the log-Euclidean distance or length from isotropy\(^\text{19,20} \), although other measures are possible, see Norris\(^\text{19} \) for a comparative discussion. The log-Euclidean distance has the advantage that it is invariant regardless of whether the compliance or stiffness tensor is considered. We use dist as a convenient and simple measure of the degree of anisotropy. Appendix C provides a little more detail on its exact definition, including a short MATLAB script to compute dist.

Large deviations from the isotropic participation tensor are apparent. Consider the ratio \( R \) of the largest to smallest

\[
\text{TABLE II. The participation matrix } G \text{ for a variety of anisotropic materials. Sym denotes material symmetry: transversely isotropic (TI), tetragonal (Tet), or orthotropic (Orth). The Frobenius (}\sqrt{2}\text{) norm is used to compare } G \text{ with the isotropic result (I) and with the perturbation approximation } G \tilde{G} \text{ defined by Eq. (50). dist is a nondimensional and invariant measure of the anisotropy (Ref. 19), equal to zero for isotropy. dist ≈ 1 signifies considerable anisotropy.}
\]

| Material            | Sym | \( G_{11} \) | \( G_{22} \) | \( G_{33} \) | \( |G-I| \) | \( |G-\tilde{G}| \) | dist |
|---------------------|-----|-------------|-------------|-------------|---------|----------------|------|
| Beryllium\(^a\)     | TI  | 1.05        | 1.05        | 0.89        | 0.13    | 0.00           | 0.22 |
| Sulphur\(^b\)       | Ort | 0.95        | 1.32        | 0.73        | 0.42    | 0.11           | 0.95 |
| Cadmium\(^c\)       | TI  | 0.73        | 0.73        | 1.55        | 0.67    | 0.10           | 1.02 |
| Barium titanate\(^d\)| Tet | 0.81        | 0.81        | 1.39        | 0.48    | 0.01           | 1.11 |
| Rochelle salt\(^e\) | Ort | 1.38        | 0.65        | 0.97        | 0.52    | 0.09           | 1.16 |
| Zinc\(^f\)          | TI  | 0.71        | 0.71        | 1.58        | 0.71    | 0.14           | 1.17 |
| Graphite/Epoxy\(^g\)| TI  | 1.38        | 1.38        | 0.25        | 0.92    | 0.81           | 2.35 |
| Tellurium dioxide\(^h\)| Tet | 1.30 | 1.30 | 0.40 | 0.74 | 0.72 | 2.87 |
| Mercurocid iodide\(^i\)| Tet | 1.37 | 1.37 | 0.26 | 0.91 | 0.14 | 3.02 |
| Spruce\(^j\)        | Ort | 1.35        | 1.63        | 0.02        | 1.22    | 1.30           | 5.59 |

Elastic moduli from Ref. 15.
From Ref. 21.
From Ref. 22.
From Ref. 23.

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element of $G$. Even for small to moderate anisotropy, such as cadmium we see that $R = G_{33}/G_{11} > 2$. The ratio becomes much larger for the more anisotropic materials considered. Spruce is included because of its enormous ratio, $R \approx 80$. These ratios can be compared with the results for the relative partition of the diffuse wave energy at the free surface of an isotropic solid. If $\mathbf{e}_3$ is the normal to the surface, then the calculations of Weaver\(^4\) indicate that $1 \leqslant G_{33}/G_{11} \leqslant 1.25$ where the lower (upper) bound is reached as $\nu$ approaches 1/2 (0). The upper bound $\approx 1.25$ is approximate and based on Fig. 3 of Ref. 8.

The numbers in Table II indicate that the perturbation approximation is adequate for small anisotropy. This can be characterized loosely as $0 < \text{dist} \ll 1$, and strong anisotropy is $\text{dist} \geqslant 2$, roughly. The examples in Table II suggest that the weak anisotropy approximation is not useful in the presence of strong anisotropy. This is evident from the fact that the errors $|G - I|$ and $|G - \bar{G}|$ are of the same order of magnitude for the strongly anisotropic materials, whereas $|G - \bar{G}|$ is much less than $|G - I|$ for weak anisotropy.

We note that for all materials considered the numerical calculations show Eq. (49) underestimating $\text{tr}(Q^{3/2})$. However, the more refined perturbation expansion of $\text{tr}(Q^{3/2})$ by Fedorov\(^1\) suggests that this is not a universal result.

The dependence of $G$ and $d(\omega)$ on the moduli is obviously complicated by virtue of the averages required in Eq. (4). However, the formula (50) for $G$ for weak anisotropy illustrates the dependence more explicitly. The form of the matrices $C_i$ and $C_j$ imply that only 12 combinations of the 21 independent anisotropic moduli enter into the first term in the perturbation expansion. For orthotropic materials, with 9 independent moduli, this number reduces to 6, and the matrices $C_i$ and $C_j$ are then diagonal. In the case of weak TI only two combinations of moduli influence $G$, see Eq. (56).

The nondimensional tensor $G$ also has important implications for radiation from a point source. The connection follows from the relation (8) between $G$ and $A$, combined with the correspondence between the drive point admittance tensor and the radiation efficiency in Eq. (7). Thus, the direction in which a force must be applied to most efficiently radiate power is the principal direction of $G$ with the largest element. Conversely, the least amount of power is radiated if the force is directed along the principal direction with the smallest element. For instance, Table II indicates that a point force of given magnitude will radiate most power in cadmium if the force is directed along the axis of hexagonal symmetry. The situation is reversed for aligned graphite/epoxy, where forcing along the fiber direction produces the least amount of total radiated power.

The inverse problem of determining anisotropy from measurements of $G$ is clearly ill-posed. However, possible measurement could be advantageous in particular circumstances. Consider for instance, three-component measurement of the displacement downhole in a borehole environment. Assuming the frequency is such that the wavelengths are large compared with the bore radius, the three-component data are sufficient to compute the autocorrelation and hence $G$. The principal directions of $G$ and the relative magnitude of its diagonal elements provide significant information about the local geostatigraphy and formation properties.

VI. CONCLUSION

We have derived general formulas for diffuse waves in anisotropic solids. The main results are concise expressions for the modal density per unit volume and frequency, $d(\omega)$ of Eq. (4a), and the participation tensor $G$ of Eq. (4b). The latter is a material constant with one or two independent constants, and with principal axes dictated by the material symmetry. In the absence of symmetry the participation tensor defines principal axes for diffuse wave energy distribution, and for radiation efficiency. Calculation of $d(\omega)$ and $G$ requires, in general, averaging over the surface of the unit sphere. Single integrals suffice for transverse isotropy, with the important quantities given in Eq. (39). In the case of weak anisotropy, a perturbation scheme produces explicit formulas, Eqs. (49) and (50). The main quantity in all cases is the second-order averaged tensor $(Q^{3/2})$. We have illustrated the results through calculations for several materials. These display the main effects that would occur in all anisotropic solids. In particular, the deviation $G$ from the unit identity tensor can be significant. Ratios of 2 or more for the relative magnitude of diffuse wave energy in different directions in crystals can occur under moderate levels of anisotropy, with far larger ratios possible in realistic materials.

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APPENDIX A: DERIVATION OF EQ. (8)

We use an argument based on a modal representation\(^8\) for the solution to the point force problem,

$$
\left( \frac{\partial^2}{\partial t^2} - L \right) \mathbf{u} = F \delta(\mathbf{x} - \mathbf{x}_0) \cos \omega t,
$$

(A1)

where $L$ is a second-order differential operator. The resulting velocity $v = \partial \mathbf{u} / \partial t$ may be found by standard means as

$$
v = \frac{1}{\rho} \text{Re} \sum_m \frac{-i \omega F \cdot \mathbf{u}_m(\mathbf{x}_0) \mathbf{u}_m(\mathbf{x}) e^{-i\omega t}}{\omega_m^2 - \omega^2 - i\omega},
$$

where the modes $\mathbf{u}_m(\mathbf{x}) e^{-i\omega_m t}$ are solutions of the homogeneous equation (A1), with the properties

$$
\delta(\mathbf{x} - \mathbf{x}_0) I = \sum_m \mathbf{u}_m(\mathbf{x}) \mathbf{u}_m(\mathbf{x}_0),
$$

$$
\int_V d\mathbf{u}_m(\mathbf{x}) \cdot \mathbf{u}_m(\mathbf{x}) = 1.
$$

The power output averaged over a cycle is therefore
\[ \Pi(x_m, \omega) = \frac{\omega}{2 \pi} \int_0^{\frac{2\pi}{\omega}} dt \cos \omega t \cdot F \cdot v(x_0, t) \]
\[ = \frac{1}{2\rho} \sum_m |F \cdot u_m(x_0)|^2 \text{Re} \frac{-i\omega}{\omega_m^2 - \omega^2 - i\delta} \quad \text{(A2)} \]

The strict nondissipative limit of \( \text{Re}[-i\omega(\omega_m^2 - \omega^2 - i\delta)] \) is \( \pi \omega \delta(\omega_m^2 - \omega^2) = \frac{1}{2} \pi \delta(\omega_m - \omega) \) where \( \delta \) is the Dirac delta function. However, modal overlap in the presence of nonzero dissipation spreads the influence over many modes. The effect is to make \( \text{Re}[-i\omega(\omega_m^2 - \omega^2 - i\delta)] \rightarrow \frac{1}{2} \pi \delta(\omega_m - \omega) \) where \( f(\nu) \) is smooth with bounded support in \( \nu \in (-\Omega, \Omega) \), say, and unit sum:
\[ \sum_m f(\omega_m - \omega) = 1. \quad \text{(A3)} \]

Here \( \sum_m \) indicates the sum over modal frequencies \( \omega_m \in (\omega - \Omega, \omega + \Omega) \). Using the density of modes, \( V d(\omega_m) \), to replace the sum over modes in Eq. (A2) by a sum over modal frequencies, gives
\[ \Pi(x_m, \omega) = \frac{\pi V}{4\rho} \sum_m d(\omega_m) f(\omega_m - \omega) [F \cdot u_m(x_0)]^2. \quad \text{(A4)} \]

We now make the assumption that the support of \( f(\nu) \) is small enough that the modal density function, \( d(\omega_m) \), may be replaced by \( d(\omega) \). This is perfectly reasonable based on known forms for \( d(\omega) \), e.g., Eq. (3). At the same time, we assume that the support of \( f(\nu) \) is sufficiently large that we may use the equipartition of energy among modes to make the replacement [see Eq. (1)]
\[ V \sum_m f(\omega_m - \omega) u_m \otimes u_m \rightarrow V \rho a^2 \ddot{u} \otimes \ddot{u} = \frac{1}{3} G. \quad \text{(A5)} \]

Hence,
\[ \Pi(x_m, \omega) = \frac{\pi}{12\rho} d(\omega) F \cdot G \cdot F, \quad \text{(A6)} \]

and since \( F \) is arbitrary, the admittance \( A \) follows from the definition of \( \Pi \) in Eq. (7). This completes the derivation of the identity (8).

APPENDIX B: DERIVATION OF Eqs. (21) AND (26)

The Cayley–Hamilton relation for \( Q \) is \( p(Q) = 0 \), where \( p \) is the characteristic cubic polynomial defined in Eq. (28), and \( I_1(n), I_2(n), I_3(n) \) are the invariants defined in Eq. (20). Thus,
\[ I_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad I_2 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1, \]
\[ I_3 = \lambda_1 \lambda_2 \lambda_3, \]

and since \( \lambda_n = \nu^2_n \), it follows that the invariants are all positive, \( I_1 > 0, I_2 > 0 \) and \( I_3 > 0 \). Multiplying Eq. (19) by \( Q^{-1} \) and \( Q^{-2} \) yields equations for the same quantities:
\[ Q^{-1} = I_1^{-1} I_2^{-1} Q - I_1^{-1} I_2^{-1} I_3^{-1} I_1, \quad \text{(B1a)} \]
\[ Q^{-2} = I_1^{-1} Q - I_1^{-1} I_2^{-1} I_1 + I_2^{-1} I_3^{-1} Q^{-1}. \quad \text{(B1b)} \]

Eliminating \( Q^{-1} \) gives an equation for \( Q^{-2} \):
\[ Q^{-2} = I_2^{-2} (I_2^{-1} Q^2 - (I_1 I_2 - I_3) Q + I_3^{-1} I_1). \]

We next derive a similar type of equation for \( Q^{1/2} \) using a method due to Hoger and Carlson. \(^{11}\) The product of this with \( Q^{-2} \), combined with the Cayley–Hamilton equation (19) yields the desired relation (21).

First we note the general expression
\[ (Q - \lambda I)^{-1} = \frac{1}{p(\lambda)} [-Q^2 + (I_1 - \lambda)Q - (\lambda^2 - I_1 \lambda + I_2) I], \quad \text{(B2)} \]

where \( p \) is the characteristic polynomial for \( Q \), from Eq. (28). The identity (B2) may be checked by direct multiplication and use of Eq. (19). The square root tensor \( R = Q^{1/2} \) satisfies the Cayley–Hamilton equation
\[ R^3 - i_1 R^2 + i_2 R - i_3 I = 0, \quad \text{(B3)} \]

Explicit formulas for \( i_1, i_2, \) and \( i_3 \) are given in Eq. (23a). Rearranging Eq. (B3) as \( R(R^2 + i_2 I) = i_1 R^2 + i_3 I \) and using \( R^2 = Q \) gives
\[ R = (i_1 Q + i_3 I)(Q + i_2 I)^{-1}. \quad \text{(B5)} \]

Application of Eq. (B2) along with some simplifications using Eq. (B4), such as \( p(-i_2) = -(i_3 - i_1 i_2)^2 \), yields
\[ Q^{1/2} = (i_3 - i_1 i_2)^{-1} [Q^2 + (i_2 - i_3^2) Q - i_3 I]. \quad \text{(B6)} \]

Combining Eqs. (B2) and (B6) gives Eq. (21). Alternatively,
\[ Q^{-3/2} = a Q^2 + b Q + c I, \quad \text{(B7)} \]

where
\[ a = \frac{I_3 (i_2 - i_3^2) - I_2 i_1 i_3}{F_3 (i_3 - i_1 i_2)}, \]
\[ b = \frac{I_1 I_3 (i_2 - i_3^2) + (I_1 I_2 - I_3) i_1 i_3}{F_3 (i_3 - i_1 i_2)}, \]
\[ c = \frac{F_2^2 + I_2 I_3 (i_2 - i_3^2) + (I_1 I_2 - I_3) i_1 i_3}{F_3 (i_3 - i_1 i_2)}. \quad \text{(B8)} \]

The second form (26) for \( Q^{-3/2} \) is based on the identity (17). The tensor products of eigenvectors for \( \lambda_i \) satisfy
\[ \mathbf{q}_i \otimes \mathbf{q}_j = \frac{(\mathbf{Q} - \lambda_i \mathbf{I})(\mathbf{Q} - \lambda_j \mathbf{I})}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)} \quad \text{(no sum)}, \quad \text{i \neq j \neq k \neq i} \]

This follows, for example, by eliminating the other two tensor products using the spectral expressions for \( \mathbf{I}, \mathbf{Q}, \) and \( \mathbf{Q}^2 \). The dependence on \( \lambda_i \) and \( \lambda_k \) can be removed in favor of \( \lambda_j \) and the invariants \( I_1 \) and \( I_3 \), and hence Eq. (27). Note that the latter can be expressed
\[ N(\lambda, n) = \frac{1}{\lambda} p'(\lambda) \left[ \lambda Q^2 + (\lambda - I_d)\lambda Q + I_d J \right], \]

where \( p'(\lambda) \) is the derivative of the characteristic polynomial. This indicates that the general expression (27) is invalid at double roots where the slowness surface exhibits degeneracy, and proper limits are required. The possibility of such points does not present a practical impediment to numerical integration.

**APPENDIX C: THE LOG-EUCLIDEAN DISTANCE**

The procedure\(^1\) is to first calculate an effective isotropic set of moduli analogous to \( c_{ijkl}^{(0)} \) of Eq. (40) but for the matrix logarithm of the six-dimensional Voigt matrix of moduli \( C_{ij} \). Some matrix factors are required to convert from the Voigt notation. The following MATLAB lines compute dist if \( C \) is the \( 6 \times 6 \) Voigt matrix.

\[ J = 1/3 * [1 1 1 0 0 0]'; [1 1 1 0 0 0]; \]
\[ K = \text{eye}(6) - J; \]
\[ T = \text{diag}([1 1 1 \text{ sqrt}(2) * [1 1 1]]); \]
\[ L = \text{logm}(T * C * T); \]
\[ \text{dist} = \text{norm}(\text{logm}(J * \exp(\text{trace}(J * L)) \]
\[ + K * \exp(1/5 * \text{ trace}(K * L))) - L, ' f r o ' \); \]