VECTORS:

Multivariate observations: \( \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} \) is a multivariate observation.

\( x_1, \ldots, x_n \) is a sample of multivariate observations.

A sample can be represented by a data matrix:

\[
\mathbf{X} = \begin{pmatrix}
  x_{11}, \ldots, x_{1p} \\
  x_{21}, \ldots, x_{2p} \\
  \vdots \\
  x_{n1}, \ldots, x_{np}
\end{pmatrix}
\]

Vector addition, transpose and multiplication:

\[
\begin{pmatrix} 1 \\
  3 \\
  4 \end{pmatrix} + \begin{pmatrix} 2 \\
  2 \\
  1 \end{pmatrix} = \begin{pmatrix} 3 \\
  5 \\
  5 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 \\
  3 \\
  4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 4 \\
  1 & 2 \end{pmatrix}
\]

\[= 1 \times 2 + 3 \times 2 + 4 \times 1 = 12\]

Norm of a vector: \( |\mathbf{v}| = (\mathbf{v}^T \mathbf{v})^{1/2} \)

Unit vector: \( |\mathbf{v}| = 1 \)

Canonical basis \( \{e_1, e_2, \ldots, e_p\} = \{ \begin{pmatrix} 1 \\
  0 \\
  \vdots \\
  0 \end{pmatrix}, \begin{pmatrix} 0 \\
  1 \\
  \vdots \\
  0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\
  \vdots \\
  \vdots \\
  1 \end{pmatrix} \} \)

Orthogonal basis, coordinate system,

Gram-Schmidt: Given a linearly independent vector system convert it to an orthogonal system.
Unary vector: \( \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \)

Matrix multiplication:

\[
\begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 \times 2 + 2 \times 2 & 1 \times 3 + 2 \times 1 \\ 3 \times 2 + 2 \times 2 & 3 \times 3 + 2 \times 1 \\ 4 \times 2 + 1 \times 2 & 4 \times 3 + 1 \times 1 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 10 & 11 \\ 10 & 13 \end{pmatrix}
\]

Determinant of a square matrix:

\[
\det \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = 3 \times 2 - 4 \times 1
\]

\[
\det \begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 4 & 1 & 2 \end{pmatrix} = 1 \det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - 2 \det \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} + 0 \det \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} = 3 - 4
\]

Covariance matrix: Symmetric positive definite

Eigenvalues and Eigenvectors of a covariance or correlation matrix

Multivariate Normal distribution has constant density on ellipsoids:

\( \Sigma \) is a \( pxp \) dimensional Covariance Matrix, Ellipsoids: \( z' \Sigma^{-1} z = c \)

Ellipsoids are characterized also by the semi-axes direction and length.

Spectral decomposition of a covariance matrix:

\[
\Sigma = \sum_{i=1}^{p} \lambda_i \mathbf{v}_i \mathbf{v}_i^T
\]

where \( \lambda_i, i=1,...,p \) is called the \( i \)-th eigenvalue and \( \mathbf{v}_i, i=1,...,p \) is called the \( i \)-th eigenvector.

Synonyms: eigenvalue == semi-axis length == variance == characteristic root == latent root, eigenvector==semi-axis direction == principal component==characteristic vector==latent vector.

In order to find the eigenvalues and eigenvectors of \( \Sigma \) we use the equation
\[ \Sigma v_i = \lambda_i v_i \]
\[ (\Sigma - \lambda_i I) v_i = 0 \]
\[ \det(\Sigma - \lambda_i I) = 0 \]

**PROPERTIES**

(i) \( \Sigma = V \Lambda V^T \), where \( V \) is the matrix of eigenvectors and \( \Lambda \) the diagonal matrix of eigenvalues of \( \Sigma \).

(ii) The eigenvectors are used to describe the orthogonal rotation from maximum variance. This type of rotation is often referred to as principal-axes rotation.

(iii) The trace of the matrix \( \Sigma \) is the sum of its eigenvalues \( \{\lambda_i\} \).

(iv) The determinant of the matrix \( \Sigma \) is the product of the \( \lambda_i \).

(v) Two eigenvectors \( v_i \) and \( v_j \) associated with two distinct eigenvalues \( \lambda_i \) and \( \lambda_j \) of a symmetric matrix are mutually orthogonal, \( v_i^T v_j = 0 \).

(vi) Square Root Matrix. There are several matrices that are the square root of \( \Sigma \).

E.g. \( \Sigma^{1/2} = V \Lambda^{1/2} V^T \), Choleski factorization: \( \Sigma = R^T R \)

(vii) Given a set of variables \( X_1, X_2, ..., X_p \), with nonsingular covariance matrix \( \Sigma \), we can always derive a set of uncorrelated variables \( Y_1, Y_2, ..., Y_p \) by a set of linear transformations corresponding to the principal-axes rotation. The covariance matrix of this new set of variables is the diagonal matrix \( \Lambda = V \Sigma V^T \)

(viii) Given a set of variables \( X_1, X_2, ..., X_p \), with nonsingular covariance matrix \( \Sigma \), a new set of variables \( Y_1, Y_2, ..., Y_p \) is defined by the transformation \( Y' = X'V \), where \( V \) is an orthogonal matrix. If the covariance matrix of the \( Y \)'s is \( \Sigma \), then the following relation holds:

\[ y^T \Sigma^{-1} y = x^T \Sigma^{-1} x \]

In other words, the quadratic form is invariant under rigid rotation.

**Singular value decomposition:**

\[ X = UDV^T \] where \( U \) is \((n \times p)\) orthogonal, \( D \) is diagonal, \( V \) is \((p \times p)\) orthogonal \( X \) is centered.