Simple Groups of Finite Morley Rank

Gregory Cherlin

Luminy, September 27, 2004
Algebraicity Conjecture

An infinite simple group of finite Morley rank is an algebraic group (Chevalley group, algebraically closed field)
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Rank: $\text{rk}(S) = \text{dim}(S)$

S: Subgroup, conjugacy class, . . .

Another intuition: $|S| \approx q^d$, $\text{rk} = d$. 
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Borovik’s Program

Determine the 2-Sylow structure of a minimal counterexample … using the methods of finite group theory.

FMR=finite groups at infinity?=algebraic groups??
Morley rank

Macintyre
An infinite field of finite Morley rank is algebraically closed.
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Zilber
An $\aleph_1$-categorical structure which is not almost strongly minimal involves an infinite group of finite Morley rank.
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Hrushovski:
Non-algebraic Abelian groups of finite Morley rank are associated with abelian varieties (Mordell-Lang, Manin, Buium) and their model theory has number theoretic consequences.
(Motivation: complex analysis; Zilber-II)
Analogies with algebraic groups

Connectedness (indecomposability);
$G^0$: generic subsets, irreducibility
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Generation: $\langle X_i : i \in I \rangle = \prod X_{i_k}^{\pm1}$ definable, connected
$[G, X]$ definable, connected.
Analogies with algebraic groups

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Borel subgroups?
• Conjugacy? • Nilpotent?
Analogies with algebraic groups

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Borel subgroups?
- Conjugacy?
- Nilpotent?

**Bad group:** Minimal connected simple, all Borels nilpotent.
- No involutions (geometry of involutions)
- No involutory automorphisms
- Borels conjugate, and disjoint.
Finite Group Theory

Sylow theory
Schur-Zassenhaus
Carter subgroups
Strong embedding
Signalizer functor theory
Amalgam method

- Generalized Fitting subgroup: $F^*(G) = F(G) \ast E(G)$
  (Nilpotent*Semisimple) $\approx$ Unipotent*Reductive

Structure of $K$-groups, Generation results, etc., etc. . . .

*Coarse counting; no linear algebra, and virtually no representation theory (as yet)*
2-Sylow\textsuperscript{\textdegree} Structure

In algebraic groups

\textit{Characteristic 2:}
unipotent—[bounded exponent, definable]

\textit{Other characteristics:}
semisimple—[divisible abelian]
2-Sylow°Structure

In algebraic groups

*Characteristic 2*: unipotent—[bounded exponent, definable]

*Other characteristics*: semisimple—[divisible abelian]

In groups of FMR

\[ S^o = U \ast T: \]

2-Unipotent \times 2-torus
with finite intersection

“Prüfer rank” (dimension of the 2-torus, Lie rank)
# Types

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“Theorem” I

- Mixed type does not exist.
- Even type is algebraic.
“Theorem” I

- Mixed type does not exist.
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“Theorem” II

A minimal counterexample to the algebraicity conjecture has Prüfer rank at most two.
Mixed and Even Type

*Mixed type*: reduces to *even type.*
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*Even type:*
- Strong embedding, weak embedding
- Strongly closed abelian subgroups
- Standard components of type $SL_2$.
- Pushing-up
- $C(G, T)$
- Parabolic subgroups
- Amalgams
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3rd generation, unipotent philosophy, $p = 2$
Conjugacy Theorems

Finite
Sylow subgroups

Finite solvable
Hall and Carter subgroups
Conjugacy Theorems

Finite or Finite Morley Rank
2-Sylow subgroups
Finite or Finite Morley Rank, solvable
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Algebraic
Borel subgroups
Maximal tori
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2-Sylow subgroups

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**Finite Morley Rank**
Maximal *good* tori
Conjugacy Theorems

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2-Sylow subgroups
Finite or Finite Morley Rank, solvable
Hall and Carter subgroups

Algebraic
Borel subgroups
Maximal tori

Finite Morley Rank
Maximal $good$ tori

Good: Any definable subgroup is the definable closure of its torsion.
Odd Type

(Top Down)

- Generic identification: Berkman
- The $B$-conjecture: Burdges
- Minimal connected simple groups: Jaligot
- Solvable groups: Frécon
Tameness!

Bad field $(K; T)$; $K \rtimes T$ $T < K^\times$
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**Tame:** no bad field.
Tameness!

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**Proposition** A tame connected solvable group without involutions is nilpotent.
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**Corollary** A tame minimal connected simple group of degenerate type is a bad group, hence contains no involutions.
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Recent Theme: removal of “tameness” hypothesis.
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Recognition, Generic Case

(Berkman)

Model: $\text{SL}_n$

$S = T_2 \times W_2$:

$T = \text{diagonal}, W = \text{Sym}_n$, Coxeter group

Dynkin diagram $A_{n-1}$: structure of $W$, elementary transpositions $(i, i + 1)$

Root $\text{SL}_2$’s:

$$
\begin{pmatrix}
* & * & 0 & 0 & \ldots \\
* & * & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & \ldots \\
0 & 0 & 0 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
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$\mathcal{E}$: copies of $\text{SL}_2$ normalized by $S^\circ$.

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Recognition

\( \mathcal{E} \): copies of \( SL_2 \) normalized by \( S^o \).

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Identify \( W_0 \) geometrically (Complex Reflection Groups)

derive the Dynkin diagram, and verify "Curtis-Tits-Phan"
relations.
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**Ingredients**

- **Reductivity:** $OC(i) = 1$ (*Strong B-conjecture*)
- **Generation:** $\langle \mathcal{E} \rangle = G$. 
Recognition

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Ingredients

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**Generic Identification**
High Prüfer rank + Reductivity + Generation $\Rightarrow$ Algebraicity
The B-Conjecture

(Burdges)

Killing $OC(i)$ (or limiting its effect): Reductivity
The B-Conjecture

(Burdges)

Killing $OC(i)$ (or limiting its effect): *Reductivity*

*Signalizer Functor Method*


*Solvable* Signalizer Functor Theorem (finite groups)
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Signalizer Functor Method
Solvable Signalizer Functor Theorem (finite groups)
Nilpotent Signalizer Functor Theorem (finite Morley rank)

Need Tameness—to Apply It
The B-Conjecture

(Burdges)

Killing $OC(i)$ (or limiting its effect): Reductivity

Signalizer Functor Method


Solvable Signalizer Functor Theorem (finite groups)

Nilpotent Signalizer Functor Theorem (finite Morley rank)

Reduction: The Nilpotent Subfunctor Theorem
The Subfunctor Theorem

Idea: $\theta(i) = U(O(C(i)))$ where $U$ is the “unipotent radical”
The Subfunctor Theorem

Idea: $\theta(i) = U(O(C'(i)))$ where $U$ is the “unipotent radical”
What does unipotent mean?

At least:
The unipotent radical of a solvable group is definably characteristic and nilpotent.
If the unipotent radical of a solvable group is trivial, that group is a “torus”.

Why Not??—Because embeds into

Fact
There is a reasonable notion of unipotence, or a sliding scale of degrees of nilpotence.
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Dichotomy for minimal nonalgebraic groups

The $B$-conjecture ($\rightarrow$ Berkman),
or
minimal connected simple ($\rightarrow$ Jaligot)
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**Generation principle**

If a 4-group $V$ acts on a connected $K$-group $H$ of odd type, then $H$ is generated by $C^\circ(i)$ ($i \in V$).
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Proofette
In a counterexample, the proper subgroup of $H$ that arises is a “subsystem subgroup”, of full Lie rank, and one eliminates possibilities by inspection.
Minimal Simple Groups

(Jaligot)

“Theorem” The Prüfer rank is at most two.
Minimal Simple Groups

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*Borel subgroups*
Minimal Simple Groups

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Tame Case
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First application of tameness

Jaligot’s Lemma The intersection of Borel subgroups is disjoint from their Fitting subgroups.
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**Tame Case**
First application of tameness

*Jaligot’s Lemma* The intersection of Borel subgroups is disjoint from their Fitting subgroups.
Leads to: Standard Borel subgroups are *nilpotent*. (which is peculiar)
Minimal Simple Groups

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First application of tameness
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The torus $T$ enveloping a Sylow 2-subgroup involves almost all primes.
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(Jaligot)

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$W = N(T)/C(T)$: $W$ acts semi-regularly in each prime, regularly on the involutions, leading by *number theory* (Zsigmondy, Dirichlet) to $d \leq 2$. 
Elimination of Tameness

*Number theory* replaced by elementary “generic subsets” arguments.
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*Jaligot’s Lemma* replaced—in all probability—by close analysis of *maximal intersections* of Borel subgroups.
Carter Subgroups, Unipotence

(Frécon)

Carter subgroup: self-normalizing, nilpotent
Useful Lemma: A Carter subgroup of a standard Borel subgroup contains the Sylow 2-subgroup
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Philosophy: in a minimal simple group, the intersection of standard Borel subgroups wants to be a Carter subgroup. (e.g., SL₂)
But if the Prüfer rank is high, the intersection is unlikely to contain a Sylow 2-subgroup.
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Absolute Carter subgroups, via the unipotence theory.
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Decomposition theorem for nilpotent groups via the unipotence theory.
Geometrical themes

- The amalgam method of finite group theory (study the free product of two minimal parabolic subgroups)
- Groups generated by pseudo-reflection subgroups (even type, SCA)
- Good tori and bad fields (Wagner), Linear groups (Zilber, Poizat), Conjugacy theorems
- Complex reflection groups (generic identification, recognition of Coxeter group)
- Intersections of Borel subgroups in minimal connected simple groups and the unipotence theory.
- Geometry of involutions (bad groups)
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“Theorem” II

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Open problems

*Degenerate type (finite 2-Sylow)*

bound on 2-rank

Relation to finite group theory *(Borovik)* …