TWO PROBLEMS ON HOMOGENEOUS STRUCTURES, REVISITED

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Abstract. We take up Peter Cameron’s problem of the classification of countably infinite graphs which are homogeneous as metric spaces in the graph metric [Cam98], working toward an explicit catalog of “known” examples on the one hand, and an investigation of the cases which occur as exceptional cases from the perspective of the catalog.

We begin with a presentation of Fraïssé’s theory of amalgamation classes and the classification of homogeneous structures, with emphasis on the case of homogeneous metric spaces, from the discovery of the Urysohn space to the connection with topological dynamics developed in [KPT05]. We then turn to a discussion of the case of metrically homogeneous graphs.

We also take this opportunity to revisit another old chestnut from the theory of homogeneous structures, namely the problem of approximating the generic triangle free graph by finite graphs. Very little is known about this, but it is possible to rephrase the problem in somewhat more explicit geometric terms. And in that form one can raise questions that seem appropriate for design theorists, as well as some questions that involve structures small enough to be explored computationally.

Introduction

A metric space $M$ is said to be homogeneous if every isometry between finite subsets of $M$ is induced by an isometry taking $M$ onto itself. An interesting and early example is the Urysohn space $U$ [Ury25, Ury27] found in the summer of 1924, the last product of Urysohn’s short but intensely productive life. While the problem of Fréchet that prompted this construction concerned universality rather than homogeneity, Urysohn took particular notice of this homogeneity property in his initial letter to Hausdorff [Huš08], a point repeated in much the same terms in the posthumous announcement [Ury25]. We will discuss this in §2 below.

From the point of view of Fraïssé’s later theory of amalgamation classes [Fra54], the essential point is that finite metric spaces can be amalgamated: if $M_1, M_2$ are finite metric spaces whose metrics agree on their common part $M_0 = M_1 \cap M_2$, then there is a metric on $M_1 \cup M_2$ extending the given metrics; and more particularly, the same applies if we limit ourselves to metric spaces with a rational valued metric.

Fraïssé’s theory facilitates the construction of infinite homogeneous structures of all sorts, which are then universal in various categories, and is often used to that effect. This gives a construction of Rado’s universal graph [Rad64], as well...
as Henson’s universal $K_n$-free graphs [Hen71], and uncountably many quite similar homogeneous directed graphs [Hen72, Henson]; a variant of the same construction also yields uncountably many homogeneous nilpotent groups and commutative rings [CSW]. More subtly, the theory of amalgamation classes can also be used to classify homogeneous structures of various types: homogeneous graphs [LW80], homogeneous directed graphs [Sch79, Lac84, Che88, Che98], the finite and the imprimitive infinite graphs with two edge colors [Che99], the homogeneous partial orders with a vertex coloring by countably many colors [TT08], and even homogeneous permutations [Cam02].

There is a remarkable 3-way connection linking the theory of amalgamation classes, structural Ramsey theory, and topological dynamics, developed in [KPT05]. In this setting the Urysohn space appears as one of the natural examples, but more familiar combinatorial structures come in on an equal footing.

The Fraïssé theory, and the connection with topological dynamics, will be reviewed in §3.

In §4 we will discuss how Fraïssé’s theory has been used to obtain classifications of all the homogeneous structures in some limited classes.

Every connected graph is a metric space in the graph metric, and Peter Cameron raised the question of the classification of the graphs which are homogeneous as metric spaces [Cam98]. Such graphs are referred to as metrically homogeneous or distance homogeneous. This condition is much sharper than the condition of distance transitivity which is familiar in finite graph theory; the complete classification of the finite distance transitive graphs is much advanced and actively pursued. He raised that issue in the context of his “census” of the very rich variety of countably infinite distance transitive graphs, and our title echoes his. We aim for a more complete catalog of the known graphs meeting this very strong condition, as well as a demonstration that at the margins this catalog is reasonably complete. This remains a work in progress: the catalog is still not in satisfactory form, as we shall see.

We also take this occasion to look at another problem connected with homogeneous structures, namely the question of the finite model property for the universal homogeneous triangle free graph [Che93], one of Henson’s homogeneous graphs [Hen71]. The problem is the construction, for each $k$, of a finite triangle free graph with the properties that (i) any maximal independent set of vertices has order at least $k$; and (ii) for any set $A$ of $k$ independent vertices, and any subset $B$ of $A$, there is a vertex adjacent to all vertices of $B$ and no vertices of $A \setminus B$—or, of course, a proof that for some $k$ there is no such finite graph. Some examples are known for the case $k = 3$, notably the Higman-Sims graph (as observed by Simon Thomas), as well as an infinite family constructed by Michael Albert, again with $k = 3$. We still have no example with $k = 4$, and we will suggest that the lack of variety among the known examples for $k = 3$ is worth further attention in its own right, and raises some problems that seem relatively accessible.

This lack of variety in the case $k = 3$ leads to some problems that seem relatively accessible. We will review the situation in §4. One arrives at a class of combinatorial geometries whose constraints do not appear to be overly restrictive, but for which we have remarkably few examples. Some of the best examples to date come from strongly regular triangle free graphs, but there are also more elementary examples, in infinite families. The supply is rather limited, and to get an example with $k = 4$
would require, in particular, having much more satisfactory examples for \( k = 3 \). So we will make this point explicit. What we have in mind is somewhat in the spirit of [Bon09]. We will also show that the case \( k = 4 \) cannot be handled using strongly regular graphs. Peter Cameron mentioned to me many years ago that bounds in the theory of strongly regular graphs should limit the possibilities for some such value of \( k \). My impression is that the case \( k = 4 \) is something of a squeaker, and I found the explicit formulas in [Big09] helpful in dealing with this case. I don’t suppose, in the present state of knowledge, one can expect to bound the size of a strongly regular graph satisfying our conditions for \( k = 3 \), as this seems to be bound up with the central problems of the field. But perhaps the experts can do something with that as well.

The theory of homogeneous structures has many other aspects that we will not touch upon, many connected with the study of the automorphism groups of homogeneous structures, e.g. the small index property and reconstruction of structures from their automorphism groups [DNT86, HHLS93, KT01, Rub94, Tr09], group theoretic issues [Tru85, Tru03, Tr09], and the classification of reducts of homogeneous structures [Tho91, Tho91], which is tied up with structural Ramsey theory. There is also an elaborate theory due to Lachlan treating stable homogeneous relational structures systematically as limits of finite structures, and by the same token giving a very general analysis of the finite case.

The subject of homogeneity falls under the much broader heading of “oligomorphic permutation groups”, that is the study of infinite permutation groups having only finitely many orbits on \( n \)-tuples for each \( n \). As the underlying set is infinite, this property has the flavor of a very strong transitivity condition, and leads to a very rich theory [Cam09, Cam96].
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Part 1. Homogeneous Structures and Amalgamation Classes

1. URYSOHN’S SPACE

1.1. A little history. A special number of Topology and its Applications (vol. 155) contains the proceedings of a conference on the Urysohn space (Beer-Sheva, 2006). A detailed account of the circumstances surrounding the discovery of that space, shortly before a swimming accident took Urysohn’s life off the coast of Brittany, can be found in the first section of [Hu808], which we largely follow here. There is also an account of Urysohn’s last days in [GK09], which provides additional context.

Urysohn completed his habilitation in 1921, and his well known contributions to topology were carried out in the brief interval between that habilitation and his fatal accident on August 17th, 1924. Fréchet raised the question of the existence of a universal complete separable metric space (one into which any other should embed isometrically) in an article published in the American Journal of Mathematics in 1925 with a date of submission given as August 21, 1924. Fréchet had communicated his question to Aleksandrov and Urysohn some time before that, and an announcement of a solution is contained in a letter from Aleksandrov and Urysohn to Hausdorff dated August 3, 1924, a letter to which Hausdorff replied in detail on August 11. The letter from Aleksandrov and Urysohn is quoted in the original German in [Hu808]. In that letter, the announcement of the construction of a universal complete separable metric space is followed immediately by the remark: “...and in addition [it] satisfies a quite powerful condition of homogeneity: the latter being, that it is possible to map the whole space onto itself (isometrically) so as to carry an arbitrary finite set \( M \) into an equally arbitrary set \( M_1 \), congruent to the set \( M \).” The letter goes on to note that this pair of conditions, universality together with homogeneity, actually characterizes the space constructed up to isometry. This comment on the property of homogeneity is highlighted in very similar terms in the published announcement [Ury25].

Urysohn’s construction proceeds in two steps. He first constructs a space \( U_0 \), now called the rational Urysohn space, which is universal in the category of countable metric spaces with rational-valued metric. This space is constructed as a limit of finite rational-valued metric spaces, and Urysohn takes its completion \( \hat{U} \) as the solution to Fréchet’s problem.

It is the rational Urysohn space which fits neatly into the general framework later devised by Fraïssé [Fra54]. A countable structure is called homogeneous if any isomorphism between finitely generated substructures is induced by an automorphism of \( M \). If we construe metric spaces as structures in which the metric defines a weighted complete graph, with the metric giving the weights, then finitely generated substructures are just finite subsets with the inherited metric, and isomorphism is isometry. Other examples of homogeneity arise naturally in algebra, such as vector spaces (which may carry forms—symplectic, orthogonal, or unitary), or algebraically closed fields. We will mainly be interested in relational systems, that is combinatorial structures in which “f.g. substructure” simply means “finite subset, with the induced structure.” But Fraïssé’s general theory, to which we turn in the next section, does allow for the presence of functions.

1.2. Topological dynamics and the Urysohn space. The Urysohn space, or rather its group of isometries, turns up in topological dynamics as an example of the phenomenon of extreme amenability. A topological group is said to be extremely
amenable if any continuous action on a compact space has a fixed point. The isometry group of Urysohn space is shown to be extremely amenable in [Pes02], and subsequently the general theory of [KPT05] showed that the isometry group of the ordered rational Urysohn space (defined in the next section) is also extremely amenable. The general theory of [KPT05] requires Fraïssé’s setup, but we quote one of the main results in advance:

**Theorem 1.** [KPT05, Theorem 2] *The extremely amenable closed subgroups of the infinite symmetric group $\text{Sym}_\infty$ are exactly the groups of the form $\text{Aut}(F)$, where $F$ is the Fraïssé limit of a Fraïssé order class with the Ramsey property.*

We turn now to the Fraïssé theory.

2. Fraïssé Classes and the Ramsey Property

2.1. Amalgamation Classes. It is not hard to show that any two countable homogeneous structures of a given type will be isomorphic if and only if they have the same isomorphism types of f.g. substructures. This uses a “back-and-forth” constructions, as in the usual proof of that any two countable dense linear orders are isomorphic, which is indeed a particular instance. In view of this uniqueness, it is natural to look for a characterization of countable homogeneous structures directly in terms of the associated class $\text{Sub}(M)$ of f.g. structures embedding into $M$. Fraïssé identified the relevant properties:

1. $\text{Sub}(M)$ is hereditary (closed downward, and under isomorphism): in other words, if $A$ is in the class, then any f.g. structure isomorphic with a substructure of $A$ is in the class;
2. There are only countably many isomorphism types represented in $\text{Sub}(M)$;
3. $\text{Sub}(M)$ has the joint embedding and amalgamation properties: if $A_1, A_2$ are f.g. substructures in the class, and $A_0$ embeds into $A_1$, $A_2$ isomorphically via embeddings $f_1, f_2$, then there is a structure $\hat{A}$ in the class with embeddings $A_1, A_2 \to \hat{A}$, completing the diagram

```
  A0
 / \   /
|   |  /
|   | /\A1
|   |
|   |
A2
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The joint embedding property is the case in which $M_0$ is empty, which should be treated as a distinct condition if one does not allow empty structures.

A key point is that amalgamation follows from homogeneity: taking $A_0$ to be a subset of $A_1$, and embedded into $A_2$, apply an automorphism of the ambient structure to move the image of $A_0$ in $A_2$ back to $A_0$, and $A_2$ to some isomorphic structure $A_2'$ containing $A_0$; then the structure generated by $A_1 \cup A_2$ will serve as an amalgam.

Conversely, if $\mathcal{A}$ is a class of structures with properties (I – III), then there is a countable homogeneous structure $M$, unique up to isomorphism, for which $\text{Sub}(M) = \mathcal{A}$. This homogeneous structure $M$ is called the *Fraïssé limit* of the class $\mathcal{A}$. Thus with $\mathcal{A}$ taken as the class of finite linear orders, the Fraïssé limit is
isomorphic to the rational order; with \( A \) the class of all finite graphs, the Fraïssé limit is Rado’s universal graph [Rad64]; and with \( A \) the class of finite rational-valued metric spaces, after checking amalgamation, we take the Fraïssé limit to get the rational Urysohn space. The computation that checks amalgamation can be found in Urysohn’s own construction, though not phrased as such.

So we see that Fraïssé’s theory is at least a ready source of “new” homogeneous structures, and we now give a few more examples in the same vein. Starting with the class of all partial orders, we obtain the “generic” countable partial order (to call it merely “dense,” as in the linear case, would be to understate its properties). Or starting with the class of finite triangle free graphs, we get the “generic” triangle free graph, and similarly for any \( n \) the generic \( K_n \)-free graph will be obtained [Hen71]. The amalgamation procedure here is simply graph theoretical union, and the special role of the complete graphs here is due to their indecomposability with respect to this amalgamation procedure: a complete graph which embeds into the graph theoretical union of two graphs (with no additional edges permitted) must embed into one of the two. The generalization to the case of directed graphs is immediate: amalgamating via the graph theoretical union, the indecomposable directed graphs are the tournaments. So we associate with any set of tournaments \( T \) the Fraïssé class of finite directed graphs omitting \( T \)—i.e., with no directed subgraph isomorphic to one in \( T \). The corresponding Fraïssé limits are the generic \( T \)-free graphs considered by Henson [Hen72]. The richness of the construction is confirmed by showing that \( 2^{\aleph_0} \) directed graphs arise in this way, because of the existence of an infinite antichain in the class of finite tournaments, that is an infinite set \( \mathcal{X} \) of finite tournaments, which are pairwise incomparable under embedding, so that each subset of \( \mathcal{X} \) gives rise to a different Fraïssé limit. A suitable construction of such an antichain is given by Henson [Hen72]. A parallel construction in the category of commutative rings provides, correspondingly, uncountably many homogeneous commutative rings [CSW].

The structure of the infinite antichains of finite tournaments has been investigated further, but has not been fully elucidated. Any such antichain lies over one which is minimal in an appropriate sense, and after some close analysis by Latka [Lat94, La03, La02] a general finiteness theorem emerged [CL00] to the effect that for any fixed \( k \), there is a finite set of minimal antichains which will serve as universal witnesses for any collection of finite tournaments determined by \( k \) constraints (forbidden tournaments) which allows an infinite antichain. This means that whenever such an antichain is present, one of the given antichains is also present, up to a finite difference. But there is still no known a priori bound for the number of antichains required, as a function of \( k \). Even the question as to whether the number of antichains needed is bounded by a computable function of \( k \) remains open.

In the terminology of [KPT05], the notion of Fraïssé class is taken to incorporate a further condition of local finiteness, meaning that all f.g. structures are finite. This may be viewed as a strengthening of condition (II). This convention is in force, in particular, in the statement of Theorem 1, which we now elucidate further.

2.2. Order Classes. The Fraïssé classes that occur in the theorem of Kechris-Pestov-Todorcevic above are order classes: this means that the structures considered are equipped with a distinguished relation \(<\) representing a linear order. Thus in this theorem nothing is said about graphs, directed graphs, or metric spaces, but rather their ordered counterparts: ordered graphs, ordered directed graphs,
ordered metric spaces. In particular the ordered rational Urysohn space is, by definition, the homogeneous ordered rational valued metric space delivered by Fraissé’s theory. As there is no connection between the order and the metric the necessary amalgamation may be carried out separately in both categories.

In the most straightforward, and most common, applications of the Fraissé theory there is often some notion of “free amalgamation” in use. In the case of order classes amalgamation cannot be entirely canonical, as some “symmetry breaking” is inevitable. But there are also finite homogeneous structures—such as the pentagon graph, or 5-cycle—for which the theory of amalgamation classes is not illuminating, and the amalgamation procedure consists largely of the forced identification of points.

When one passes from the construction of examples to their systematic classification, there is typically some separation between the determination of more or less sporadic examples, and the remaining cases described naturally by the Fraissé theory. Such a classification has only been carried out in a few cases, and perhaps a more nuanced picture will appear eventually.

We will say something more about how the theory of [KPT05] applies in the absence of order, but first we complete the interpretation of Theorem 1 by discussing the second key property required.

2.3. The Ramsey Property. The ordinary Ramsey theorem is expressed in Hungarian notation by the symbolism:

$$\forall k, m, n \exists N : N \rightarrow (n)^m_k$$

meaning that for given $k, m, n$, there is $N$ so that: for any coloring of increasing $m$-tuples from $A = \{1, \ldots, N\}$ by $k$ colors, there is a subset $B$ of cardinality $n$ which is monochromatic with respect to the coloring.

Structural Ramsey theory deals with a locally finite hereditary class $\mathcal{A}$ of finite structures of fixed type, which on specialization to the case of the class $\mathcal{L}$ of finite linear orders will degenerate to the usual Ramsey theory. In general, given two structures $A, B$ in $\mathcal{A}$, write $(B)_A$ for the class of induced substructures of $B$ isomorphic to $A$. This gives $i^N_n$ an appropriate meaning if $\mathcal{A} = \mathcal{L}$, namely increasing sequences of length $n$ from an ordered set of size $N$.

We may then use the Hungarian notation

$$M \rightarrow (B)_A^k$$

to mean that whenever we have a coloring of of $\binom{M}{A}$ by $k$ colors, there is a copy of $B$ inside $M$ which is monochromatic with respect to the induced coloring of $\binom{B}{A}$. And the Ramsey property will be:

$$\forall A, B \in \mathcal{A} \forall k \exists M \in \mathcal{A} : M \rightarrow (B)_A^k$$

So the Ramsey property for $\mathcal{L}$ is the usual finite Ramsey theorem.

Ramsey theory for Fraissé classes is a subtle matter, but a highly developed one. In [HN05] it is shown that the Ramsey property implies the amalgamation property, by a direct argument. What one would really like to classify are the Fraissé classes with the Ramsey property, but according to [HN05] the most promising route toward that is via classification of amalgamation classes first, and then the identification of the Ramsey classes.
For unordered graphs, the only instances of the Ramsey property that hold are those for which the subgraphs being colored are complete graphs $K_n$, or their complements [NR75b]. But the collection of finite ordered graphs does have the Ramsey property [NR77a, NR77b, AH78].

To illustrate the need for an ordering, consider colorings of the graph $A = K_1 + K_2$, a graph on 3 vertices with one edge, and let $B = 2 \cdot K_2$ be the disjoint sum of two complete graphs of order 2. If we order $B$ in any way, we may color the copies of $A$ in $B$ by three colors according to the relative position of the isolated vertex of $A$, with respect to the other two vertices, namely before, after, or between them. Then $B$, with this coloring, cannot be monochromatic. Thus we can never have a graph $G$ satisfying $G \rightarrow (B)_3^A$, since given such a graph $G$ we would first order $G$, then define a coloring of copies of $A$ in $G$ as above, using this order, and there could be no monochromatic copy of $B$.

The significance of the ordered rational Urysohn space only emerged in [KPT05], and the appropriate structural Ramsey theorem was proved “on demand” by Nešetřil [Nes07].

At this point, we have collected all the notions needed for Theorem 1. Before we leave this subject, we note that the theory of [KPT05] also exhibits a direct connection between the more classical examples of the Fraïssé theory (lacking a built-in order). The following is a fragment of Theorem 5 of [KPT05].

**Theorem 2.** Let $G$ be the automorphism group of one of the following countable structures $M$:

1. The random graph;
2. The generic $K_n$-free graph, $n \geq 2$;
3. The rational Urysohn space.

Let $L$ be the space of all linear orderings of $M$, with its compact topology as a closed subset of $2^{M \times M}$. Then under the natural action of $G$ on $L$, $L$ is the universal minimal compact flow for $G$.

The **minimality** here means that there is no proper closed invariant subspace; and the **universality** means that this is the largest such minimal flow (projecting on to any other). Again, Theorem 2 has an abstract formulation in terms of Fraïssé theory [KPT05]. The following is a special case.

**Theorem 3** ([KPT05, Theorem 4]). Let $A$ be a (locally finite) Fraïssé class and let $A^+$ be the class of ordered structures $(K, <)$ with $K \in A$. Suppose that $A^+$ is a Fraïssé class with the order property and the Ramsey property. Let $M$ be the Fraïssé limit of $A$, $G = \text{Aut}(M)$, and $\mathcal{L}$ the space of linear orderings of $M$, equipped with the compact topology inherited by inclusion into $2^{M \times M}$. Then under the natural action of $G$ on $\mathcal{L}$, the space $\mathcal{L}$ is the universal minimal compact flow for $G$.

Here one has in mind the case in which amalgamation in $A$ does not require any identification of vertices (strong amalgamation); then $A^+$ is certainly an amalgamation class. The **order property** is the following additional condition: given $A \in A$, there is $B \in A$ such that under any ordering on $A$, and any ordering on $B$, there is some order preserving isomorphic embedding of $A$ into $B$. This is again a property which must be verified when needed, and is known in the cases cited. To see an example where the order property does not hold, consider the class $A$ of finite equivalence relations. Any equivalence relation $B$ may be ordered so that its classes are intervals; thus the order property fails.
We turn next to the problem of classifying homogeneous structures of particular
types. Here again the Fraïssé theory can be helpful.

3. Classification

The homogeneous structures of certain types have been completely classified,
notably homogeneous graphs [LW80], homogeneous tournaments [Lac84], homo-
geneous tournaments with a coloring by finitely many colors and homogeneous
directed graphs [Che88], homogeneous partial orders with a coloring by countably
many colors [TT08], and homogeneous permutations [Cam02]: this last is a less
familiar notion, that we will enlarge upon. There is also work on the classifica-
tion of homogeneous 3-hypergraphs [LT95, AL95], and on graphs with two colors of
edges [Lac86, Che99], the latter covering only the finite and imprimitive cases: this
uncovers some sporadic examples, but the main problem remain untouched in this
class.

3.1. Homogeneous Permutations. Cameron observed that permutations have
a natural interpretation as structures, and that when one adopts that point of
view the model theoretic notion of embedding is the appropriate one. A finite
permutation may naturally be viewed as a finite structure consisting of two linear
orderings. This is equivalent to a pair of bijections between the structure and the
set \{1, \ldots, n\}, \(n\) being the cardinality, and thus to a permutation. In this setting,
an embedding of one permutation into another is an occurrence in the second of a
\textit{permutation pattern} corresponding to the first, so that this formalism meshes nicely
with the subject of permutations omitting specified patterns.

By a very direct analysis, Cameron showed that there are just 6 homogeneous
permutations, in this sense, up to isomorphism: the trivial permutation of order
1, the identify permutation of \(\mathbb{Q}\) or its reversal, the class corresponding to the lex-
ographic order on \(\mathbb{Q} \times \mathbb{Q}\), where the second order agrees with the first in one
coordinate and reverses the first in the other coordinate, and the generic permu-
tation (corresponding to the class of all finite permutations). The existence of
the generic permutation is immediate by the Fraïssé theory.

As amalgamations of linear orders are tightly constrained, the classification of
the amalgamation classes of permutations is quite direct. Cameron also observes
that it would be natural also to generalize from structures with two linear orders to
an arbitrary finite number, but I do know of any further progress on this question.

3.2. Homogeneous Graphs. This is the case that really launched the classifica-
tion project. The classification of homogeneous graphs by Lachlan and Woodrow
involves an ingenious inductive setup couched directly in terms of amalgamation
classes of finite graphs. We will need the results of that classification later, when
discussing distance homogeneous graphs. Indeed, homogeneous graphs are just the
diameter two (or less) case of distance homogeneous graphs. Furthermore, in any
distance homogeneous graph, the graph induced on the neighbors of a point is a
homogeneous graph, and one may consider each of the possibilities individually.

We begin with a catalog of the homogeneous graphs. We use the notation \(K_n\)
for a complete graph of order \(n\), allowing \(n = \infty\), meaning \(\aleph_0\) in this context. We
write \(I_n\) for the complement of \(K_n\), that is (an independent set of vertices of order
\(n\), and \(m \cdot K_n\) for the disjoint sum of \(m\) copies of \(K_n\), again allowing \(m\) and \(n\) to
become infinite. Bearing in mind that the complement of a homogeneous graph is again a homogeneous graph, we arrange the list as follows.

I. **Degenerate cases**, $K_n$ or $I_n$; here, one of the available 2-types is not actually realized, so these are actually structures for a simpler language.

II. **Imprimitive homogeneous graphs**, $m \cdot K_n$ and their complements, where $m, n \geq 2$. The complement of $m \cdot K_n$ is complete $n$-partite with parts of constant size.

III. **Primitive, nondegenerate, homogeneous finite graphs** (highly exceptional): the pentagon or 5-cycle $C_5$, and a graph on 9-points which may be described as the line graph of the complete bipartite graph $K_{3,3}$, or the graph theoretic square of $K_3$.

IV. **Primitive, nondegenerate, infinite homogeneous graphs**, with which the classification is primarily concerned: Henson’s generic graphs omitting $K_n$, and their complements, generic omitting $I_n$, as well as the generic or random graph $\Gamma_\infty$ (Rado’s graph) corresponding to the class of all finite graphs.

In this setting there is no difficulty identifying the degenerate and imprimitive examples, and little difficulty in identifying the remaining finite ones by an inductive analysis. Since the class of homogeneous graphs is closed under complementation, the whole classification comes down to the following result.

**Theorem 4** ([LW80, Theorem 2', paraphrased]). Let $\Gamma$ be a homogeneous non-degenerate primitive graph containing an infinite independent set, as well as the complete graph $K_n$. Then $\Gamma$ contains every finite graph not containing a copy of $K_{n+1}$.

Let us see that this completes the classification in the infinite, primitive, nondegenerate case. As the graph $\Gamma$ under consideration is infinite, by Ramsey’s theorem it contains either $K_\infty$ or $I_\infty$, and passing to the complement if necessary, we may suppose the latter. So if $K_n$ embeds in $\Gamma$ and $K_{n+1}$ does not, then Theorem 4 says that $\Gamma$ is the corresponding Henson graph, while if $K_n$ embeds in $\Gamma$ for all $n$, Theorem 4 then says that it is the Rado graph.

The method of proof is by induction on the order $N$ of the finite graph $G$ which we wish to embed in $\Gamma$. The difficulty is that on cursory inspection, Theorem 4 does not at all lend itself to such an inductive proof. Lachlan and Woodrow show that as sometimes happens in such cases, a stronger statement may be proved by induction. Their strengthening is on the extravagant side, and involves some additional technicalities, but it arises naturally from the failure of the the first try at an inductive argument. So let us first see what difficulties appear in a direct attack.

Let $G$ be a graph of order $N$, not containing $K_{n+1}$, and let $\Gamma$ be the homogeneous graph under consideration. We aim to show that $G$ embeds in $\Gamma$, proceeding by induction on $N$. Pick a vertex $v$ of $G$. If $v$ is isolated, or if $v$ is adjacent to the remaining vertices of $G$, we will need some special argument (even more so later, once we strengthen our inductive claim). For example, if $v$ is adjacent to the remaining vertices of $G$, then we have an easy case: we identify $v$ with any vertex of $\Gamma$, we consider the graph $\Gamma_1$ on the vertices adjacent to $v$ in $\Gamma$, and after verifying that $\Gamma_1$ inherits all hypotheses on $\Gamma$ (with $n$ replaced by $n - 1$) we can conclude directly by induction on $n$. If the vertex $v$ is isolated, the argument will be less immediate, but still quite manageable.
Turning now to the main case, when the vertex \( v \) has both a neighbor and a non-neighbor in \( G \), matters are considerably less simple. Let \( G_0 \) be the graph induced on the other vertices of \( G \), and let \( a, b \) be vertices of \( G_0 \) chosen with \( a \) adjacent to \( v \), and \( b \) not. At this point, we must build an amalgamation diagram which forces a copy of \( G \) into \( \Gamma \), and we hope to get the factors of the diagram by induction on \( N \), which of course does not quite work. It goes like this.

Let \( A \) be the graph obtained from \( G \) by deleting \( v \) and \( a \), and let \( B \) be the graph obtained from \( G \) by deleting \( v \) and \( b \). Let \( H \) be the disjoint sum \( A + B \) of \( A \) and \( B \), and form two graphs \( H_1 = H_0 \cup \{ u \} \) and \( H_2 \cup \{ c \} \), with edge relations as follows.

The vertex \( u \) plays the role of \( v \), and is therefore related to \( A \) and \( B \) as \( v \) is in \( G \). The vertex \( c \) plays a more ambiguous role, as \( a \) or \( b \), and is related to \( A \) as \( a \) is, and to \( B \) as \( b \) is.

Suppose for the moment that copies of \( H_1, H_2 \) occur in \( \Gamma \). Then so does an amalgam \( H_1 \cup H_2 \) over \( H_0 \), and in that amalgam either \( u \) is joined to \( c \), which may then play the role of \( a \), with the help of \( A \), or else \( u \) is not joined to \( c \), and then \( c \) may play the role of \( b \), with the help of \( B \). In either case, a copy of \( G \) is forced into \( \Gamma \).

What may be said about the structure of \( H_1 \) and \( H_2 \)? These are certainly too large to be embedded into \( \Gamma \) by induction, but they have a simple structure: \( H_1 \) is the free amalgam of \( A \cup \{ a \} \) with \( B \cup \{ b \} \) under the identification of \( a \) with \( b \), and \( H_2 \) is constructed similarly, over \( u \). Here each factor (e.g., \( A \cup \{ a \}, A \cup \{ b \} \) in the case of \( H_1 \)) embeds into \( \Gamma \) by induction, but we need also the sum of the two factors over a common vertex. This leads to the following definitions.

**Definition 3.1.**

1. A pointed graph \((G, v)\) is a graph \( G \) with a distinguished vertex \( v \).
2. The pointed sum of two pointed graphs \((G, v)\) and \((H, w)\) is the graph obtained from the disjoint sum \( G + H \) by identifying the base points.
3. Let \( A(n) \) be the set of finite graphs belonging to every amalgamation class which contains \( K_n, I_\infty \), the path of order \( 3 \), and its complement (the last two eliminate imprimitive cases).
4. Let \( A^*(n) \) be the set of finite graphs \( G \) such that for any vertex \( v \) of \( G \), and any pointed graph \((H, w)\) with \( H \in A(n) \), the pointed sum \((G, v) + (H, w)\) belongs to \( A(n) \).

Notice that \( A^*(n) \) is contained in \( A \) for trivial reasons, simply taking for \((H, w)\) the pointed graph of order 1. Now we can state the desired strengthening of Theorem 2'.

**Theorem 5** ([LW80, Lemma 6]). For any \( n \), if \( G \) is a finite graph omitting \( K_{n+1} \), then \( G \) belongs to \( A^*(n) \).

With this definition, the desired inductive proof actually goes through. Admittedly the special cases encountered in our first run above become more substantial the second time around. As a result, this version of the main theorem will be preceded by 5 other preparatory lemmas required to support the final induction. However the process of chasing one’s tail comes to an end at this point.

### 3.3. Homogeneous Tournaments

In Lachlan’s classification of the homogeneous tournaments [Lac84] two new ideas occur, which later turned out to be sufficient to carry out the full classification of the homogeneous directed graphs
[Che98], with suitable orchestration. A byproduct of that later work was a more efficient organization of the case of tournaments, given in [Che88]. The main idea introduced at this stage was a certain use of Ramsey’s theorem that we will describe in full. The second idea arises naturally at a later stage as one works through the implications of the first; it involves an enlargement of the setting beyond tournaments, where much as in the case of the Lachlan/Woodrow argument, the point is to find an inductive framework large enough to carry through an argument that leads somewhat beyond the initial context of homogeneous tournaments.

It turns out that there are only 5 homogeneous tournaments, four of them of a special type which are easily classified, and the last one fully generic. The whole difficulty comes in the characterization of this last tournament as the only homogeneous tournament of general type, in fact the only one containing a specific tournament of order 4 called \([T_1, C_3]\). In this notation, \(T_1\) is the tournament of order 1, \(C_3\) is a 3-cycle, and \([T_1, C_3]\) is the tournament consisting of a vertex \((T_1)\) dominating a copy of \(C_3\). So the analog of Theorem 2′ of [LW80] is the following.

**Theorem 6** ([Lac84]). Let \(T\) be a countable homogeneous tournament containing a tournament isomorphic with \([T_1, C_3]\). Then every finite tournament embeds into \(T\).

We now give the classification of the homogeneous tournaments explicitly, and indicate the reduction of that classification to Theorem 6.

A local order is a tournament with the property that for any vertex \(v\), the tournaments induced on the sets \(v^+ = \{ u : v \rightarrow u \}\) and \(v^- = \{ u : u \rightarrow v \}\) are both transitive (i.e., given by linear orders). Equivalently, these are the tournaments not embedding \([T_1, C_3]\) or its dual \([C_3, T_1]\). There is a simple structure theory for the local orders, which we will not go into here. But the result is that there are exactly four homogeneous local orders, two of them finite: the trivial one of order 1, and the 3-cycle \(C_3\). The infinite homogeneous local orders are the rational order \((\mathbb{Q}, <)\) and a very similar generic local order, which can be realized equally concretely.

Now a tournament \(T\) which does not contain a copy of \([T_1, C_3]\) can easily be shown to be of the form \([S, L]\) where \(S\) is a local order whose vertices all dominate a linear order \(L\); here \(S\) or \(L\) may be empty. Indeed, if \(T\) is homogeneous, one of the two must be empty, and in particular \(T\) is a local order. Thus if the homogeneous tournament \(T\) contains a copy of \([T_1, C_3]\) then it contains a copy of \([C_3, T_1]\) as well, and it remains only to prove Theorem 6.

At this point, the following interesting technical notion comes into the picture. If \(\mathcal{A}\) is an amalgamation class, let \(\mathcal{A}^*\) be the set of finite tournaments \(T\) such that every tournament \(T^*\) of the following form belongs to \(\mathcal{A}\): \(T^* = T \cup L\), \(L\) is linear, and any pattern of edges between \(T\) and \(L\) is permitted. Theorem 6 is equivalent to the following rococo variation.

**Theorem 7.** If \(\mathcal{A}\) is an amalgamation class containing \([T_1, C_3]\), then \(\mathcal{A}^*\) is an amalgamation class containing \([T_1, C_3]\).

That \(\mathcal{A}^*\) is an amalgamation class is simply an exercise in the definitions, but worth working through to see why the definition of \(\mathcal{A}^*\) takes the particular form that it does (because linear orders have strong amalgamation). The deduction of Theorem 6 from Theorem 7 is also immediate. Assuming Theorem 7, we argue by induction on \(N = |T|\) that \(T\) belongs to any amalgamation class \(\mathcal{A}\) containing
[\text{T}_1, \text{C}_3]. \) Take any vertex \( v \) of \( T \) and let \( T_0 \) be the tournament induced on the remaining vertices. By induction, \( T_0 \) belongs to every amalgamation class containing \([\text{T}_1, \text{C}_3]\), in particular \( T_0 \in \mathcal{A}^* \). Since \( T \) is the extension of \( T_0 \) by a single vertex, and since a single vertex constitutes a linear tournament, then from \( T_0 \in \mathcal{A}^* \) we derive \( T \in \mathcal{A} \), and we are done.

Note the progress which has been made. In Theorem 6 we consider arbitrary tournaments; in Theorem 7 we consider only linear extensions of \([\text{T}_1, \text{C}_3]\). Now a further reduction comes in, and eventually the statement to be proved reduces to a finite number of specific instances of Theorem 6 which can be proved individually. But we have not yet encountered the leading idea of the argument, which comes in at the next step.

### 3.4. The Ramsey Argument.

We introduce another class closely connected with \( \mathcal{A}^* \).

#### Definition 3.2.

1. For tournaments \( A, B \) we define the composition \( A[B] \) as usual as the tournament derived from \( A \) by replacing each vertex of \( A \) by a copy of \( B \), with edges determined within each copy of \( B \) as in \( B \), and between each copy of \( B \) as in \( A \). The composition of two tournaments is a tournament.

2. If \( T \) is any tournament, a stack of copies of \( T \) is a composition \( L[T] \).

3. If \( \mathcal{A} \) is an amalgamation class of finite tournaments, let \( \mathcal{A}^{**} \) be the set of tournaments \( T^* \) such that every tournament \( T^* \) of the following form belongs to \( \mathcal{A} \):
   \[ T^* = L[T] \cup \{v\} \]
   is the extension of some stack of copies of \( T \) by one more vertex.

The crucial point here is the following.

#### Fact 3.3. Let \( \mathcal{A} \) be an amalgamation class of finite tournaments, and \( T \) a finite tournament in \( \mathcal{A}^{**} \). Then \( T \) belongs to \( \mathcal{A}^* \).

We will not give the argument in detail here. It is a direct application of the Ramsey theorem, given explicitly in [Lac84] and again in [Che88, Che98]. The idea is that one may amalgamate a large number of one point extensions of a long stack of copies of \( T \) so that in any amalgam, the additional points contain a long linear tournament, and one of the copies of \( T \) occurring in the stack will hook up with that linear tournament in any previously prescribed fashion desired.

This leads to our third, and nearly final, version of the main theorem.

#### Theorem 8. Let \( \mathcal{A} \) be an amalgamation class of finite tournaments. Then \( \text{C}_3 \) belongs to \( \mathcal{A}^{**} \).

Notice that a stack of copies of \([\text{T}_1, \text{C}_3]\) embeds in a longer stack of copies of \( \text{C}_3 \), so that Theorem 8 immediately implies that the same result for \([\text{T}_1, \text{C}_3]\). Since we already saw that \( \mathcal{A}^{**} \subseteq \mathcal{A}^* \), Theorem 8 implies Theorem 7. In view of the very simple structure of a stack of copies of \( T \), we are almost ready to prove this last statement by induction on the length of the stack. Unfortunately the additional vertex \( v \) occurring in \( T^* = L[T] \cup \{v\} \) complicates matters, and leads to a further reformulation of the statement.

At this point, it is convenient to return from the language of amalgamation classes to the language of structures. So let the given amalgamation class correspond to the homogeneous tournament \( T \), and let \( T = L[\text{C}_3] \cup \{v\} \) be a 1-point extension of a finite stack of 3-cycles. Theorem 8 says that \( T \) embeds into \( T \). It will be simpler to strengthen the statement slightly, as follows.
Let $a$ be an arbitrary vertex in $T$, and consider $T = a^+$ and $T_2 = a^-$ separately. We claim that $T$ embeds into $T$ with $L[C_3]$ embedding into $T_1$, and with the vertex $v$ going into $T_2$. This now sets us up for an inductive argument in which we consider a single 3-cycle $C$ in $T_1$, and the parts $T'_1$ and $T'_2$ defined relative to $C$ as follows: $T'_1$ consists of the vertices of $T_1$ dominated by the three vertices of $C$, and $T'_2$ consists of the vertices $v'$ which relate to $C$ as the specified vertex $v$ does. What remains at this point is to clarify what we know, initially, about $T_1$ and $T_2$, and to show that these properties are inherited by $T'_1$ and $T'_2$ (in particular, $T'_2$ should be nonempty!). This then allows an inductive argument to run smoothly.

At this point we have traded in the tournament $T$ for a richer structure $(T_1, T_2)$ consisting of a tournament with a distinguished partition into two sets. The homogeneity of $T$ will give us the homogeneity of $(T_1, T_2)$ in its expanded language. Such structures will be called 2-tournaments, and the particular class of 2-tournaments arising here will be called ample tournaments. The main inductive step in the proof of Theorem 8 will be the claim that an ample tournament $(T_1, T_2)$ gives rise to an ample tournament $(T'_1, T'_2)$ if we fix a 3-cycle in $T_1$ and pass to the subsets considered above.

We do not wish at all to dwell on this last part. The main steps in the proof are the reduction to Theorem 8, and then the realization that we should step beyond the class of homogeneous tournaments to the class of homogeneous 2-tournaments, to find a setting which is appropriately closed under the construction corresponding to the inductive step of the argument. This then leaves us concerned only about the base of the induction, which reduces to a small number of specific claims about tournaments of order not exceeding 6. Once the problem is finitized, it can be settled by explicit amalgamation arguments.

Lachlan’s Ramsey theoretic argument functions much the same way in the context of directed graphs as it does for tournaments, and comes more into its own there, as it is not a foregone conclusion that Ramsey’s theorem will necessarily produce a linear order; but it will produce something, and modifying the definition of $A^+$ to allow for this additional element of vagueness, things proceed much as they did before.

In [Che98] there is also a treatment of the case of homogeneous graphs using the ideas of [Lac84] in place of the methods of [LW80]. This cannot be said to be a simplification, having roughly the complexity of the original proof, but it is a viable alternative, and the proof of the classification of homogeneous directed graphs is more or less a combination of the ideas of the tournament classification with the ideas which appear in a treatment of homogeneous graphs by this second method.

This ends our general survey of the general theory of amalgamation classes its application to classification results. It is not clear how much further those ideas can be taken. The proofs are long and ultimately computational even when the final classifications have a reasonably simple form, and the subject has not gone much beyond this point. There may be some scope for further applications of these ideas in the case of metrically homogeneous graphs, but this still remains to be seen. So we have looked more toward the construction of generic types of metrically homogeneous graphs on the one hand, and the classification of the more exceptional types, on the other, for which such techniques are not needed. So as far as the present exposition is concerned, we will not be returning to these methods. But their availability should be borne in mind.
In the remaining sections we will focus on the two open problems mentioned already in the introduction.

Part 2. Metrically Homogeneous Graphs

4. Metrically homogeneous graphs: Toward a catalog

Any connected graph may be considered as a metric space under the graph metric, and if the associated metric space is homogeneous then the graph is said to be metrically homogeneous [Cam98]), or (by analogy with distance transitivity) distance homogeneous. Cameron asked whether this class of graphs can be completely classified, and gave some examples of constructions via the Fraïssé theory of amalgamation classes in [Cam98]. He also noted that for graphs of diameter at most 2, the metric structure is the same as the graph structure, so that the problem becomes the classification of homogeneous graphs, whose solution by Lachlan and Woodrow was discussed in §3. Also noteworthy in this regard is the classification by Macpherson of the infinite, locally finite distance transitive graphs [Mph82], and the treatment of the finite case in [Cam80]. Further examples are found in tables given at the end of [Che98]. These tables present all the primitive metrically homogeneous graphs of diameter 3 or 4 which can be defined by forbidding a set of triangles, excluding those of diameter 3 in which none of the forbidden triangles involve the distance 2 (in which case one automatically has an amalgamation class). Those tables require some decoding for our purposes: they make no explicit mention of metric spaces, but simply deal with homogeneous structures determined by constraints on triples—many, but not all, of these can be construed as metric spaces. So we will translate this information into our present context. While preparing this report, I learned of ongoing work in [AMP10], which sheds considerable light on the case of diameter 3.

For any homogeneous metric space \( \Gamma \), and for any \( i \) up to the diameter of \( \Gamma \), the set \( \Gamma_i \) of points at distance \( i \) from a fixed vertex of \( \Gamma \) will again be a homogeneous metric space. But if \( \Gamma \) is a metrically homogeneous graph, the distance 1 may not occur in the metric space \( \Gamma_i \), in which case the induced metric will not come from the induced graph. This is a major complication for any inductive analysis of the metrically homogeneous graphs, and suggests that in order to treat this problem it may eventually be necessary to embed it in an even larger classification problem.

On the other hand, when distance 1 does occur in \( \Gamma_i \), then \( \Gamma_i \) can be viewed as a metrically homogeneous graph (not necessarily connected). There is a small point to be checked here, which is covered in [Cam98]. In particular for small values of \( i \), \( \Gamma_i \) will have smaller diameter than \( \Gamma \), and if the metric is in fact a graph metric then that graph can be supposed known, inductively. For larger values of \( i \) the induced graph will be no more complicated than \( \Gamma \), a point which may sometimes be exploited, in a more delicate way.

4.1. Exceptional and imprimitive metrically homogeneous graphs. In setting out our catalog, we begin by noticing two classes of metrically homogeneous graphs which deserve special attention. On the one hand, we have the imprimitive ones: these have a nontrivial equivalence relation definable without parameters (equivalently, invariant under the full automorphism group). In the finite distance transitive case, these are described by Smith’s Theorem [Smi71]; this theorem is an
effective tool for reduction to the primitive case when the graph is finite, and also
to a significant degree when the graph is infinite.

An explicit version of Smith’s theorem, laying out the various subcases which
naturally arise, is given in [AH06]. Leaving aside the case of $n$-cycles, which are
imprimitive whenever $n$ is composite, there are only two ways in which a connected
distance transitive graph can be imprimitive: it may be bipartite, or it may be
“antipodal”. This means that the diameter $\delta$ is finite and the relation “$d(x, y) \in
\{0, \delta\}$ is an equivalence relation. Of course, the graph may be both bipartite
and antipodal. Smith’s theorem says a good deal more than this, and we will elaborate
when we explore the imprimitive metrically homogeneous case.

The other special class of metrically homogeneous graphs worthy of particular
attention arises from the Lachlan/Woodrow classification. If $\Gamma$ is a metrically ho-
homegeneous graph, $\Gamma_1$ is a metrically homogeneous graph with respect to the edge
relation “$d(x, y) = 1$” (though $\Gamma_1$ may be an independent set). The cases which
arise naturally in connection with Fraïssé style constructions are the following: $\Gamma_i$
is an independent set; $\Gamma_1$ is generic omitting $K_n$ for some $n$; $\Gamma_1$ is the Rado graph
(fully generic). Any other possibility will be considered exceptional: that is $\Gamma_1$
finite, or infinite imprimitive, or generic omitting an independent set $I_n$ for some
$n$.

These exceptional graphs may be explicitly classified, and this classification will
give the point of departure for our catalog. The following result, to be proved in
§6, refers to some specific constructions that will be described subsequently.

**Theorem 9.** Let $\Gamma$ be a connected metrically homogeneous graph. Then one of the
following occurs.

1. $\Gamma$ has diameter at most 2, and is homogeneous as a graph (cf. §3):
2. $\Gamma$ has degree 2, and is an $n$-cycle for some $n$.
3. $\Gamma_1$ is finite or imprimitive, and $\Gamma$ is one of the following.
   (a) Antipodal of diameter 3, and the result of doubling $C_5$, $E(K_{3,3})$, or a
   finite independent set (Theorem 15);
   (b) A tree-like graph $T_{r,s}$ as described by Macpherson [Mph82].
4. $\Gamma$ is of generic type, meaning that $\Gamma_1$ is one of the following three types:
   (a) An infinite independent set; and furthermore, for $u \in \Gamma_2$, the set of
   neighbors of $u$ in $\Gamma_1$ is infinite;
   (b) Generic omitting a complete graph $K_n$, for some $n \geq 3$ finite;
   (c) Homogeneous universal (Rado’s graph).

Given the Lachlan/Woodrow theorem, observe that the locally finite case is
covered by the result of Cameron in the finite case, and the result of Macpherson in
the infinite case. We will indicate direct proofs below, but it should be noted that
the analyses of Cameron and Macpherson use less than metric homogeneity. The
proof of Macpherson’s theorem in full generality is far more subtle than the special
case that concerns us here. We claim in addition that no new cases arise with $\Gamma_1$
finite and imprimitive, or generic omitting $I_n$ for some $n$.

Next in order, we consider the infinite bipartite case. If $\Gamma$ is a connected bipar-
tite metrically homogeneous graph, then one denotes by $B\Gamma$ the metric structure
induced on either “half” of $\Gamma$, rescaled by a factor of $1/2$; in particular $B\Gamma$ becomes
a graph with edge relation “$d(x, y) = 2$”. This gives us another metrically homo-
ogeneous graph, of half the diameter (rounded down) of $\Gamma$. In particular up through
diameter 5, the associated graph $B\Gamma$ is an ordinary homogeneous graph, found in the Lachlan/Woodrow classification.

By definition any infinite bipartite metrically homogeneous graph falls on the “generic” side of our initial classification, but the same need not apply to $B\Gamma$. The next result says that apart from some explicitly classifiable examples, which are either trees or have diameter at most 5, our graph $B\Gamma$ is of generic type, and even has $B\Gamma_1$ homogeneous universal (i.e., $B\Gamma_1$ is Rado’s graph). In particular, $B\Gamma$ is not itself bipartite.

**Theorem 10.** Let $\Gamma$ be a connected, bipartite, and metrically homogeneous graph, of diameter at least 3, and degree at least 3. Then one of the following occurs, writing $B\Gamma_1$ for $(B\Gamma)_1$.

1. $\Gamma$ is a tree;
2. $\Gamma$ has diameter 3, and $\Gamma$ is either the complement of a perfect matching, or a generic bipartite graph;
3. $\Gamma$ has diameter 4, $B\Gamma \cong K_\infty[I_2]$, and $\Gamma$ is a double cover of a generic bipartite graph, described in Lemma 7.2 below.
4. $\Gamma$ has diameter 4, and $B\Gamma_1$ is generic omitting an independent set of order $n+1$, for some $n \geq 2$. With $n$ fixed, $\Gamma$ is determined up to isomorphism.
5. $\Gamma$ has diameter 5 and is antipodal; $B\Gamma_1$ is generic omitting an independent set of order 3. $\Gamma$ is determined up to isomorphism.
6. $B\Gamma_1$ is a homogeneous universal graph.

Each of these possibilities occurs.

In the antipodal case, our analysis deviates considerably from the finite case. In the finite case, one works with the broader class of distance transitive graphs (or even more broadly than that). Then there is a natural quotient operation which gives a distance transitive graph structure on the equivalence classes for the antipodality relation: we join two classes by an edge if there are representatives joined by an edge. Unfortunately, this does not work well in the metrically homogeneous setting, as we shall see. On the other hand, we will derive the following strong constraint.

**Proposition 4.1.** Let $\Gamma$ be a connected metrically homogeneous and antipodal graph, of diameter $\delta \geq 3$. Then for each vertex $u \in \Gamma$, there is a unique vertex $u' \in \Gamma$ at distance $\delta$ from $u$, and we have the law

$$d(u, v) = \delta - d(u', v)$$

for $u, v \in \Gamma$. In particular, the map $u \mapsto u'$ is an automorphism of $\Gamma$.

This does not give us a direct reduction of the classification to the primitive case, but it does suggest that the antipodal, and not bipartite, case may be broadly similar to the primitive case.

We have still to consider graphs which are both antipodal and bipartite. In the case of odd diameter, that is covered by the following.

**Theorem 11.** Let $\Gamma$ be a metrically homogeneous graph of odd diameter $2d + 1$ which is both antipodal and bipartite. Then $B\Gamma$ is a primitive infinite metrically homogeneous graph with the following properties:

1. $B\Gamma$ has diameter $d$;
2. No triangle in $B\Gamma$ has perimeter greater than $2d + 1$;
For any metrically homogeneous graph $G$ with these three properties, there is an antipodal bipartite graph of diameter $2d + 1$ with $B \Gamma \cong G$, and $G$ determines $\Gamma$ up to isomorphism.

We do not know anything similar in the case of even diameter. In this case, $B \Gamma$ will again be antipodal, but not bipartite. We do not have a general description of the class of graphs occurring as $B \Gamma$, or information about the extent to which $B \Gamma$ determines $\Gamma$.

At this point, we have looked at the various special cases which naturally arise, and we have seen a number of issues that need further exploration. We turn now to a discussion of the known constructions that produce graphs of generic type, including some which are bipartite, antipodal, or both. Two of our constructions are already familiar from the study of universal graphs with forbidden subgraphs, see [KMP88, Kom99].

5. CONSTRUCTIONS OF METRICALLY HOMOGENEOUS GRAPHS

We are concerned here with the use of Fraissé constructions to produce metrically homogeneous graphs of generic type, including some bipartite and antipodal cases.

5.1. Constructions. We begin by setting out three examples of amalgamation classes of finite integral metric spaces, with our usual proviso that the distances occurring exhaust an interval $[0, \delta]$ in $\mathbb{N}$, so that the corresponding homogeneous structure can be construed as a graph with the graph metric. In general, we expect the typical amalgamation class to be an intersection of these three types.

Fix $\delta$ with $3 \leq \delta \leq \infty$, and let $\mathcal{M}^{\delta}$ be the collection of all finite integral-valued metric spaces of diameter at most $\delta$. With $\delta$ fixed, we will consider forbidden structures of the following two types:

- A triangle is a metric space with 3 points; its perimeter is the sum of the three distances involved. The type of a triangle is the triple $(i, j, k)$ of distances between pairs of points, taken in any order.
- A $(1, \delta)$-space is a metric space in which all distances are equal to 1 or $\delta$, where if $\delta = \infty$ this means that all distances are equal to 1.

We then define the following classes of finite metric spaces contained in $\mathcal{M}^{\delta}$.

1. For $1 \leq K_{1} \leq K_{2} \leq \delta$ or $K_{1} = \infty$, $K_{2} = 0$, let $\mathcal{A}^{\delta}_{K_{1}, K_{2}}$ be the class of $X \in \mathcal{M}^{\delta}$ such that for any triangle of type $(i, j, k)$ embedding in $X$, if the perimeter $P = i + j + k$ is odd then it satisfies:

\[ P \geq 2K_{1} + 1 \text{ and } 2P \leq 2K_{2} + \min(i, j, k) \]

2. For any $C_{0}, C_{1} \geq 2\delta + 1$ with $C_{0}$ odd and $C_{1}$ even, let $\mathcal{B}^{\delta}_{C_{0}, C_{1}}$ be the class of $X \in \mathcal{M}^{\delta}$ such that for any triangle of type $(i, j, k)$ embedding in $X$, if the perimeter $P = i + j + k$ has parity $\epsilon$ (i.e. $P \equiv \epsilon \mod 2$ and $\epsilon = 0$ or 1) then

\[ P < C_{\epsilon} \]

3. For any set $S$ of $(1, \delta)$-spaces, let $\mathcal{C}^{\delta}_{S}$ be the class of $X \in \mathcal{M}^{\delta}$ such that no space in $S$ embeds isometrically into $X$. 


(4) With $\delta, K_1, K_2, C_0, C_1, S$ as above let $A^\delta_{K_1, K_2; C_0, C_1; S}$ be the intersection

$$A^\delta_{K_1, K_2} \cap B^\delta_{C_0, C_1} \cap C^\delta_S$$

Certain values of the $K_i$ and $C_i$ will result in the exclusion of various $(1, \delta)$-spaces and as a rule we would include in $S$ only the minimal constraints not already excluded on that basis, but there is no real harm in allowing some redundancy. Many but not all of the classes $A^\delta_{K_1, K_2; C_0, C_1; S}$ will be amalgamation classes. To describe the precise conditions needed on the parameters, it is convenient to take the pair $C_0, C_1$ in increasing order, so we set

$$C = \min(C_0, C_1), \quad C' = \max(C_0, C_1)$$

In laying out the precise conditions needed we take the case $K_1 = \infty$ as a special case, and for $K_1 < \infty$ we consider further whether or not $C \leq 2\delta + K_1$; and some further conditions are necessary when $C' > C + 1$. So we distinguish three cases, with some further subdivision.

**Theorem 12.** Let $\delta \geq 3$, $1 \leq K_1 \leq K_2 \leq \delta$, $2\delta + 1 \leq C_0, C_1 \leq 3\delta + 2$ with $C_0$ even and $C_1$ odd, and $S$ a set of $(1, \delta)$-spaces occurring in $A^\delta_{K_1, K_2; C_0, C_1}$. Let $C = \min(C_0, C_1)$ and $C' = \max(C_0, C_1)$. Then the class

$$A^\delta_{K_1, K_2; C_0, C_1; S}$$

is an amalgamation class if and only if one of the following sets of conditions is satisfied.

1. $K_1 = \infty$:
   $$K_2 = 0, \quad C_1 = 2\delta + 1, \quad \text{and } S \text{ is a set of } \delta\text{-cliques.}$$

2. $K_1 < \infty$ and $C \leq 2\delta + K_1$:
   $$C = 2K_1 + 2K_2 + 1, \quad K_1 + K_2 \geq \delta, \quad \text{and } K_1 + 2K_2 \leq 2\delta - 1$$
   If $C' > C + 1$ then $K_1 = K_2$ and $3K_2 = 2\delta - 1$.
   If $K_1 = 1$ then $S$ is empty.

3. $K_1 < \infty$, and $C > 2\delta + K_1$:
   $$K_1 + 2K_2 \geq 2\delta - 1 \quad \text{and } 3K_2 \geq 2\delta$$
   If $K_1 + 2K_2 = 2\delta - 1$ then $C \geq 2\delta + K_1 + 2$.
   If $C' > C + 1$ then $C \geq 2\delta + K_2$.
   If $K_1 = 1$, $K_2 = \delta$, and $C = 2\delta + 2$, then $S$ is empty.

The Fraïssé limits of these classes will give us a wide variety of metrically homogeneous graphs, taking the edge relation given by $d(x, y) = 1$.

We will not give the proof of the theorem here, but we will say something about how one verifies the amalgamation property when it holds. The converse direction, showing that all of these conditions are necessary, requires many explicit auxiliary constructions.

For the proof of amalgamation, it suffices to consider amalgamation diagrams of the special form $A_1 = A_0 \cup \{a_1\}$, $A_2 = A_0 \cup \{a_2\}$, that is with only one distance $d(a_1, a_2)$ needing to be determined; we call these 2-point amalgamations. Furthermore, any distance lying between

$$d^- (a_1, a_2) = \max_{x \in A_0} (|d(a_1, x) - d(a_2, x)|)$$
and
\[ d^+(a_1, a_2) = \min_{x \in A_0} (d(a_1, x) + d(a_2, x)) \]
will give at least a pseudometric (and if \( d^-(a_1, a_2) = 0 \), we might as well identify \( a_1 \) and \( a_2 \)).

When \( C = C' \) we have the requirement that all triangles have perimeter at most \( C - 1 \) and therefore we consider a third value
\[ \tilde{d}(a_1, a_2) = \min_{x \in A_0} (C - 1 - [d(a_1, x) + d(a_2, x)]) \]
In this case the distance \( i = d(a_1, a_2) \) must satisfy:
\[ d^-(a_1, a_2) \leq i \leq \min(d^+(a_1, a_2), \tilde{d}(a_1, a_2)) \]
Similarly when \( K_1 = \infty \) then as there are no odd triangles we modify the definition of \( \tilde{d} \) as follows:
\[ \tilde{d}(a_1, a_2) = \min_{x \in A_0} (C_0 - 2 - [d(a_1, x) + d(a_2, x)]) \]

The amalgamation procedure is then given by the following rules for completing a 2-point amalgamation diagram \( A_i = A_0 \cup \{a_i\} \) \((i = 1, 2)\). We will assume throughout that \( d^-(a_1, a_2) > 0 \), as otherwise we may simply identify \( a_1 \) with \( a_2 \).

We will also write:
\[ i^− = d^−(a_1, a_2), i^+ = d^+(a_1, a_2), i = \tilde{d}(a_1, a_2) \]
and we seek a suitable value for \( i = d(a_1, a_2) \).

(1) If \( K_1 = \infty \):
Then the parity of \( d(a_1, x) + d(a_2, x) \) is independent of the choice of \( x \in A_0 \).
If \( S \) is empty then any value \( i \) with \( i^− \leq i \leq i^+ \) and of the correct parity will do.
If \( S \) is nonempty (and irredundant) then \( S \) consists of a \( \delta \)-clique, and \( \delta \) is even. In particular \( \delta \geq 4 \). In this case take \( d(a_1, a_2) = i \) with \( 1 < i < \delta \) and with \( i \) of the correct parity. As \( i^− < \delta, i^+ > 1, \) and \( \delta \geq 4 \), there is at least one such value of \( i \).

(2) If \( K_1 < \infty \) and \( C \leq 2\delta + K_1 \):
(a) If \( C' = C + 1 \) then:
\[(i) \text{ If } \min(i^+, i) \leq K_2 \text{ let } d(a_1, a_2) = \min(i^+, i) \text{. Otherwise:} \]
\[(ii) \text{ If } i^− \geq K_1 \text{ let } d(a_1, a_2) = i^− \text{. Otherwise:} \]
\[(iii) \text{ Let } d(a_1, a_2) = K_2 \text{.} \]
(b) If \( C' > C + 1 \) then:
\[(i) \text{ If } i^+ < K_2 \text{ let } d(a_1, a_2) = i^+ \text{. Otherwise:} \]
\[(ii) \text{ If } d^− > K_2 \text{ let } d(a_1, a_2) = i^− \text{. Otherwise:} \]
\[(iii) \text{ Take } d(a_1, a_2) = K_2 \text{ unless there is } x \in A_0 \text{ with } d(a_1, x) = d(a_2, x) = \delta, \text{ in which case take } d(a_1, a_2) = K_2 - 1 \text{.} \]
(3) If \( K_1 < \infty \) and \( C > 2\delta + K_1 \):
   (a) If \( i^- \geq K_1 \), let \( d(a_1, a_2) = i^- \).
   (b) Otherwise:
      (i) If \( C' = C + 1 \):
          (A) If \( i^+ \leq K_1 \) let \( d(a_1, a_2) = \min(i^+, i^-) \).
          Otherwise:
              Let \( d(a_1, a_2) = K_1 \) unless we have one of the following:
              There is \( x \in A_0 \) with \( d(a_1, x) = d(a_2, x) \),
              and \( K_1 + 2K_2 = 2\delta - 1 \); or \( K_1 = 1 \).
              In these cases, take \( d(a_1, a_2) = K_1 + 1 \).
      (ii) If \( C' > C + 1 \):
          If \( i^+ < K_2 \) let \( d(a_1, a_2) = i^+ \).
          Otherwise, let \( d(a_1, a_2) = \min(K_2, C - 2\delta - 1) \).

We note some extreme cases. With \( K_1 = \infty \) we are dealing with bipartite graphs;
with \( C = 2\delta + 1 \) we are dealing with antipodal graphs. With \( K_1 > 1 \) we have the case \( \Gamma_1 \cong I_\infty \); with \( K_1 = 1 \) and \( S = \{K_n\} \) (a clique), we have the case of \( \Gamma_1 \) the
generic \( K_n \)-free graph.

Lemma 5.1. Let \( \Gamma \) be a metrically homogeneous graph of diameter \( \delta \). Then \( \Gamma \) is
antipodal if and only if no triangle has perimeter greater than \( 2\delta \).

Proof. Bearing in mind that geodesics are triangles, the bound on perimeter implies
antipodality.

For the converse, let \( (a, b, c) \) be a triangle with \( d(a, b) = i, d(a, c) = j, d(b, c) = k, \)
and let \( a' \) be the antipodal point to \( a \). Then the triangle \( (a', b, c) \) has distances \( \delta - i, \)
\( \delta - j, \) and \( k, \) and the triangle inequality yields \( i + j + k \leq 2\delta, \) as claimed. \( \square \)

5.2. Antipodal Variations. We consider modifications of our definitions which
allow us to include arbitrary \((1, \delta)\)-constraints when the associated graph is antipo-
dal and \( K_1 = 1 \). In this case the associated amalgamation class may have additional
constraints which are neither triangles nor \((1, \delta)\)-spaces.

Definition 5.2. Let \( \delta \geq 4 \) be finite and \( 2 \leq n \leq \infty \). Then

(1) \( A_0^\delta = A_0^{4, \delta; 2\delta + 2, \delta+1, \infty} \) is the set of finite metric spaces in which no triangle
has perimeter greater than \( 2\delta \).

(2) \( A_{a,n}^\delta \) is the subset of \( A_0^\delta \) containing no subspace of the form \( K_k \cup K_\ell \) with
\( K_k, K_\ell \) cliques (at distance 1), \( k + \ell > n \), and \( d(x, y) = \delta - 1 \) for \( x \in K_k, \)
\( y \in K_\ell \). In particular, \( K_{n+1} \) does not occur.

(3) \( A_{ab,n}^\delta \) is the subset of \( A_{a,n}^\delta \) in which no triangles of odd perimeter occur
(“ab” stands for “antipodal bipartite”).

Theorem 13. If \( \delta \geq 4 \) is finite and \( 2 \leq n \leq \infty \), then \( A_{a,n}^\delta \) and \( A_{ab,n}^\delta \) are amal-
gamation classes. If \( n \geq 3 \) then the associated Fraïssé limit is a connected an-
tipodal metrically homogeneous graph which is said to be generic for the specified
constraints.

Here the parameter \( n \) stands in place of the set \( S \); since there are no triangles
of perimeter greater than \( 2\delta \), the only relevant \((1, \delta)\)-spaces are 1-cliques. In these
graphs \( \Gamma_1 \) is the generic graph omitting \( K_n \).

As the proof of amalgamation for these particular classes is not very elaborate,
we will give it here. The following lemma is helpful.
Lemma 5.3. Let $\delta$ be fixed, and let $A$ be a finite metric space with no triangle of perimeter greater than $\delta$. Then there is a unique “antipodal” extension $\hat{A}$ of $A$, up to isometry, to a metric space satisfying the same condition, in which every vertex is paired with an antipodal vertex at distance $\delta$, and every vertex not in $A$ is antipodal to one in $A$.

If $A$ is in $A_{\delta,n}^a$ or $A_{\delta,b,n}^b$, then $\hat{A}$ is in the same class.

Proof. The uniqueness is clear: let $B = \{a \in A: \text{There is no } a' \in A \text{ with } d(a, a') = \delta\}$ and introduce a set of new vertices $B' = \{b' : b \in B\}$. Let $\hat{A} = A \cup B'$ as a set. Then there is a unique symmetric function on $\hat{A}$ extending the metric on $A$, with $d(x, b') = d(x, b) - d(x, b)$ for $x \in A, b \in B$.

So the issue is one of existence, and for that we may consider the problem of extending $A$ one vertex at a time, that is to $A \cup \{b'\}$ with $b \in B$, as the rest follows by induction.

We need to show that the canonical extension of the metric on $A$ to a function $d$ on $A \cup \{b'\}$ is in fact a metric, satisfies the antipodal law for $\delta$, and also satisfies the constraints corresponding to $n$, and is bipartite if that is required.

The triangle inequality for triples $(b', a, c)$ or $(a, b', c)$ corresponds to the ordinary triangle inequality for $(a, b, c)$ or the bound on perimeter for $(a, b, c)$ respectively, and the bound on perimeter for triangles $(a, b', c)$ follows from the triangle inequality for $(a, b, c)$.

Now suppose $n < \infty$ and $b'$ belongs to a configuration $K_k \cup K_\ell$ with $k + \ell > n$ and $d(x, y) = \delta - 1$ for $x \in K_k$, $y \in K_\ell$. We may suppose that $b' \in K_k$; then $K_k \setminus \{b'\} \cup (K_\ell \cup \{b'\})$ provides a copy of $K_{k-1} \cup K_{\ell+1}$ of forbidden type.

Finally, triangles $(a, b', c)$ and $(a, b, c)$ with $a, c \in A$, $b \in B$ have perimeters of the same parity. \hfill $\Box$

Lemma 5.4. If $\delta \geq 4$ is finite, $2 \leq n \leq \infty$, then $A_{\delta,n}^a$ and $A_{\delta,b,n}^b$ are amalgamation classes.

Proof. We consider a two-point amalgam with $A_i = A_0 \cup \{a_i\}$, $i = 1, 2$. If $d(a_2, x) = \delta$ for some $x \in A_0$ then there is a canonical amalgam $A_1 \cup A_2$ embedded in $A_1$. So we will suppose $d(a_i, x) < \delta$ for $i = 1, 2$ and $x \in A_0$.

We claim that any metric $d$ on $A_1 \cup A_2$ extending the given metrics $d^i$ on $A_i$ will satisfy the antipodal law for $\delta$. So with $d$ such a metric, consider a triangle of the form $(a_0, a_1, a_1$ with $a_i \in A_0$, $i = 0, 1, 2$. By the triangle law for $(a_1, a_0, a_2)$ we have

$$d(a_1, a_2) \leq 2\delta - [d(a_1, a_0) + d(a_2, a_0)]$$

and this is the desired bound on perimeter.

We know by our general analysis that any value $r$ for $d(a_1, a_2)$ with

$$d^-(a_1, a_2) \leq r \leq d^+(a_1, a_2)$$

will give us a metric, and in the bipartite case we will want $r$ to have the same parity as $d^- (a_1, a_2)$ (or equivalently, $d^+ (a_1, a_2)$).

To deal with the the constraints involving the parameter $n$, it is sufficient to avoid the values $r = 1$ and $r = \delta - 1$. But $d^+ (a_1, a_2) > 1$, and $d^- (a_1, a_2) < \delta - 1$, so we may take $r$ equal to one of these two values unless we have

$$d^- (a_1, a_2) = 1, d^+ (a_1, a_2) = \delta - 1.$$

In this case, we take \( r \) to be some intermediate value of the same parity, and as \( \delta > 3 \), there is such a value.

5.3. Smith’s Theorem. We now turn to Smith’s Theorem, a general description of the imprimitive case, following [AH06] (cf. [BCN89, Smi71]). This result applies to imprimitive distance transitive graphs (that is, the homogeneity condition is assumed to hold for pairs of vertices), and even more generally in the finite case. There are three points to this theory: (1) the imprimitive graphs are of two extreme types, bipartite or antipodal; (2) associated with each type there is a reduction (folding or halving) to a potentially simpler graph; (3) with few exceptions, the reduced graph is primitive. Among the exceptions that need to be examined are the graphs which are both antipodal and bipartite. As our hypothesis of metric homogeneity is not preserved by the folding operation in general, we lose a good deal of the force of (2) and a corresponding part of (3). On the other hand, metric homogeneity implies that with trivial exceptions, in antipodal graphs the antipodal equivalence classes have order two, and one may hope to classify these directly without passing through the primitive case.

We will first take up the explicit form of Smith’s Theorem given in [AH06], restricting ourselves to the distance transitive case. If \( \Gamma \) is a distance transitive graph, then any binary relation \( R \) invariant under \( \text{Aut}(\Gamma) \) is a union of relations \( R_i \) defined by \( d(x, y) = i \), \( R = \bigcup_{i \in I} R_i < d \) for some set \( I \), with \( \delta \) the diameter (possibly infinite). We denote by \( (t) \) the union \( \bigcup_{t_1} R_i \) taken over the multiples of \( t \). The first point is the following.

\textbf{Fact 5.5} (cf. [AH06, Theorem 2.2]). Let \( \Gamma \) be a connected distance transitive graph of diameter \( \delta \), and let \( E \) be a congruence of \( \Gamma \).

1. \( E = (t) \) for some \( t \).
2. If \( 2 < t < \delta \), then \( \Gamma \) has degree 2.
3. If \( t = 2 \) then either \( \Gamma \) is bipartite, or \( \Gamma \) is a complete regular multipartite graph, of diameter 2.

In particular, if the degree of \( \Gamma \) is at least 3, then \( \Gamma \) is either bipartite or antipodal (and possibly both).

Of course, the exceptional case of diameter 2 has already been noticed within the Lachlan/Woodrow classification, where it occurs as the complement of \( m \cdot K_n \), with \( m, n \leq \infty \).

If \( \Gamma \) is a connected distance transitive bipartite graph, we write \( B\Gamma \) for the graph induced on either of the two equivalence classes for the congruence \( (2) \); these are isomorphic, with respect to the edge relation \( R_2 \); \( d(x, y) = 2 \). This is called a \textit{halved graph} for \( \Gamma \), and \( \Gamma \) is a \textit{doubling} of \( B\Gamma \). If \( \Gamma \) is a connected distance transitive antipodal graph of diameter \( \delta \) (necessarily finite), then \( A\Gamma \) denotes the graph induced on the quotient \( \Gamma / R_\delta \) by the edge relation: \( C_1 \) is adjacent to \( C_2 \) if there are \( u_i \in C_i \) with \( (u_1, u_2) \) an edge of \( \Gamma \). This is called a \textit{folding} of \( \Gamma \), and \( \Gamma \) is called an \textit{antipodal cover} of \( A\Gamma \). In our context, as mentioned, the halving construction is more useful than the folding construction.

\textbf{Fact 5.6} (cf. [AH06, Theorem 2.3]). Let \( \Gamma \) be a connected metrically homogeneous bipartite graph. Then \( B\Gamma \) is metrically homogeneous.

\textit{Proof.} Since \( \text{Aut}(\Gamma) \) preserves the equivalence relation whose classes are the two halves of \( \Gamma \), the homogeneity condition is inherited by each half. \( \square \)
Some insight into the folding construction is afforded by the following.

**Lemma 5.7.** Let $\Gamma$ be a connected distance transitive antipodal graph of diameter $\delta$, and let $C_1, C_2$ be two equivalence classes for the antipodality relation $R_\delta$. Then the set of distances $d(u, v)$ for $u \in C_1, v \in C_2$ is a pair of the form $\{i, \delta - i\}$ (which is actually a singleton if $i = \delta/2$ with $\delta$ even).

**Proof.** Since there are geodesics $(u, v, u')$ with $d(u, v) = i$, $d(v, u') = \delta - i$, and $d(u, u') = \delta$, whenever we have $d(u, v) = i$ we also have $d(u', v) = \delta - i$ for some $u'$ antipodal to $u$, by distance transitivity.

We claim that for $u \in C_1, v, w \in C_2$, with $d(u, v) = i$, $d(u, w) = j$ and $i, j \leq \delta/2$, we have $i = j$. If $i < j$, take $w' \in C_2$ with $d(u, w') = \delta - j$. Then $d(v, w') \leq i + (\delta - j) < \delta$, so $v = w'$ and $i = \delta - j$, in which case $i = j = \delta/2$.

For $u \in C_1$ the set of distance $d(u, v)$ with $v$ in $C_2$ has the form $\{i, \delta - i\}$, and for $v$ in $C_2$ the same applies with respect to $C_1$, with the same pairs of values. This implies our claim. $\square$

**Corollary 5.8 (cf. [AH06, Proposition 2.4]).** Let $\Gamma$ be a connected distance transitive antipodal graph of diameter $\delta$, and consider the graph $A\Gamma$ as a metric space. If $u, v \in \Gamma$ with $d(u, v) = i$, then in $A\Gamma$ the corresponding points $\bar{u}, \bar{v}$ lie at distance $\min(i, \delta - i)$.

**Proof.** Replacing $v$ by $v'$ with $d(u, v) = \delta - i$, we may suppose $i = \min(i, \delta - i)$. We have $d(\bar{u}, \bar{v}) \leq d(u, v)$.

Let $j = d(\bar{u}, \bar{v})$ and lift a path of length $j$ from $\bar{u}$ to $\bar{v}$ to a walk $(u, \ldots, v^*)$ in $\Gamma$. If $v^* = v$ then $d(u, v) \leq j$ and we are done. Otherwise, $\delta = d(v, v^*) \leq d(v^*, u) + d(u, v) \leq j + i$, and as $i, j \leq \delta/2$, we find $i = j = \delta/2$. $\square$

This implies in particular that the folding of an antipodal metrically homogeneous graph of diameter at most 3 is complete, and that the folding of any connected distance transitive antipodal graph is distance transitive. However, as we will see later, there is a “generic” antipodal graph $\Gamma$ of any diameter, with equivalence classes of size 2, such that for any pair $u, u'$ at distance $\delta$, $(u, v, u')$ is a geodesic for all $v$ (i.e., $d(u, v) + d(v, u') = \delta$). Supposing that the diameter $\delta$ is at least 4, consider an induced path $P = (u_0, \ldots, u_\delta)$ of order $\delta + 1$ in $\Gamma$, and an induced cycle $C = (v_0, \ldots, v_\delta, v_0)$ of order $\delta$. On $P$ we have $d(u, u_j) = |i - j|$ and on $C$ we have $d(v_i, v_j) = \min(|i - j|, \delta - |i - j|)$, so the images of these graphs in $A\Gamma$ are isometric. We claim however that there is no automorphism of $A\Gamma$ taking one to the other.

Let $\delta_1 = \lfloor \delta/2 \rfloor$ and $\delta' = \lceil \delta_1/2 \rceil$. Then there is a vertex $w_C$ in $\Gamma$ at distance precisely $\delta'$ from every vertex of $C$. Hence the same applies in $A\Gamma$ to the image of $C$. We claim that this does not hold for the image of $P$. Supposing the contrary, there would be a vertex $w_P$ in $\Gamma$ whose distance from each vertex of $P$ is either $\delta'$ or $\delta - \delta'$. On the other hand, the distances $d(w, u_i), d(w, u_{i+1})$ can differ by at most 1, and $\delta - \delta' > \delta' + 1$ since $\delta \geq 4$, so the distance $d(w, u_i)$ must be independent of $i$. However $d(u_0, w_w) = \delta - d(u_\delta, w_w)$, which would mean $\delta = 2\delta'$, while in fact $\delta > 2\delta'$.

Still, we can get a decent grasp of the antipodal case in another way.

5.4. **The Antipodal Case.** All graphs considered under this heading are connected and of finite diameter.
Proposition 5.9. Let $\Gamma$ be a connected metrically homogeneous and antipodal graph, of diameter $\delta \geq 3$. Then for each vertex $u \in \Gamma$, there is a unique vertex $u' \in \Gamma$ at distance $\delta$ from $u$, and we have the law
\[ d(u, v) = \delta - d(u', v) \]
for $u, v \in \Gamma$. In particular, the map $u \mapsto u'$ is an automorphism of $\Gamma$.

For $v \in \Gamma$, $\Gamma_i(v)$ denotes the graph induced on the vertices at distance $i$ from $v$, and since the isomorphism type is independent of $v$, this will sometimes be denoted simply by $\Gamma_i$, when the choice of $v$ is immaterial. In particular, $\Gamma_{\delta} \cong K_n^{(\delta)}$, the complete graph with edge relation given by $R_{\delta}$, for some $n$, with $1 \leq n \leq \infty$. Our claim is that $n = 1$.

We begin with a variation of Lemma 5.7.

Lemma 5.10. Let $\Gamma$ be a metrically homogeneous and antipodal, of diameter $\delta$. Suppose $u, u' \in \Gamma$, and $d(u, u') = \delta$. Then for $i < \delta/2$, the relation $R_{\delta}$ defines a bijection between $\Gamma_i(u)$ and $\Gamma_i(u')$, while $\Gamma_{\delta/2}(u) = \Gamma_{\delta/2}(u')$.

Proof. Fix $i < \delta/2$, and $v \in \Gamma_i(u)$. We work with the equivalence classes $C_1, C_2$ of $u$ and $v$ respectively, with respect to the relation $R_{\delta}$. As $d(u, v) = i$, $d(u, u') = \delta$, $i \leq \delta/2$, and $d(v, u') \in \{i, \delta - i\}$, we have $d(v, u') = \delta - i$.

Now $(v, u')$ extends to a geodesic $(v, u', v')$ with $d(v, v') = \delta$, $d(u', v') = i$, and we claim that $v'$ is unique. If $(v, u', v'')$ is a second such geodesic then we have $d(v, v') = d(v, v'') = \delta$ and $d(v', v'') \leq 2(\delta - i) < \delta$, so $v' = v''$.

Thus we have a well-defined function from $\Gamma_i(u)$ to $\Gamma_i(u')$, and interchanging $u, u'$ we see that this is a bijection.

For the final point, apply Lemma 5.7: with $i = \delta/2$, the set $\{i, \delta - i\}$ is a singleton.

Taking $n > 1$, we will first eliminate some small values of $\delta$.

Lemma 5.11. Let $\Gamma$ be a metrically homogeneous and antipodal graph, of diameter $\delta \geq 3$, and let $\Gamma_{\delta} \cong K_n^{(\delta)}$ with $1 < n \leq \infty$. Then $\delta \geq 5$.

Proof. Take $a_1, a_2, a_3$ at mutual distance $\delta$, and take $u_1, v_1 \in \Gamma_1(a)$ with $d(u_1, v_1) = 2$.

Suppose the diameter is 3. Using the previous lemma, take vertices $u_2, v_2 \in \Gamma_1(a_2)$, $u_3, v_3 \in \Gamma_1(a_3)$, with $u_1, u_2, u_3$ and $v_1, v_2, v_3$ triples with pairwise distances all equal to $\delta$.

Then $\delta(u_1, v_3) = \delta - d(u_1, v_1) = \delta - 2$ and similarly $\delta(u_2, v_3) = \delta - 2$, so $\delta = d(u_1, u_2) \leq 2(\delta - 2)$, a contradiction.

So now suppose the diameter is 4. Consider $v_2 \in \Gamma_1(a_2)$ with $d(v_1, v_2) = \delta$. Observe that the triangles $(a_3, u_1, v_1)$ and $(a_3, u_1, v_2)$ are isometric. Now apply metric homogeneity to find an isometry carrying $(a_3, u_1, v_1, a_1)$ to $(a_3, u_1, v_2, b_1)$ for some $b_1$. Then $u_1, v_2 \in \Gamma_1(b_1)$ and $d(b_1, a_3) = 4$. But then $d(a_1, b_1), d(a_2, b_1) \leq 2$ while $a_1, a_2, a_3, b_1$ are all in the same antipodality class, forcing $a_1 = b_1 = a_2$, a contradiction.

Now we need to extend Lemma 5.7 to some cases involving distances which may be greater than $\delta/2$.

Lemma 5.12. Let $\Gamma$ be a metrically homogeneous and antipodal, of diameter $\delta \geq 3$, and let $\Gamma_{\delta} \cong K_n^{(\delta)}$ with $1 < n \leq \infty$. Suppose $d(a, a') = \delta$ and $i < \delta/2$. Suppose
u ∈ Γ_i(a), u' ∈ Γ_i(a'), with d(u, u') = δ. If v ∈ Γ_i(a) and d(u, v) = 2i, then d(u', v) = δ - 2i.

Proof. We have d(a, u') = δ - i. Take v_0 ∈ Γ_i(a) so that (a, v_0, u') is a geodesic, that is d(v_0, u') = δ - 2i. As u, v_0 ∈ Γ_i(a) we have d(u, v_0) ≤ 2i. On the other hand d(u, u') = δ and d(v_0, u') = δ - 2i, so d(u, v_0) ≥ 2i. Thus d(u, v_0) = 2i.

So we have at least one triple (a, u, v_0) with v_0 ∈ Γ_i(a), d(u, v_0) = 2i, and with d(v_0, u') = δ - 2i. Let (a, u, v) be any triple isometric to (a, u, v_0). Then the quadruples (a, u, v, a') and (a, u, v_0, a') are also isometric since u, v, v_0 ∈ Γ_i(a) with i < δ/2. But as a, u together determine u', we then have (a, u, v, a', u') and (a, u, v_0, a', u') isometric, and in particular d(v, u') = δ - 2i.

After these preliminaries we can prove the proposition.

Proof. We show that n = 1, after which the rest follows directly since if u determines u', then d(u, v) must determine d(u', v).

We have δ ≥ 5. We fix a_1, a_2, a_3 at mutual distance δ, and fix i < δ/2, to be determined more precisely later.

Take u_1, v_1 ∈ Γ_i(a_1) with d(u_1, v_1) = 2i, and then correspondingly u_2, v_2 ∈ Γ_i(a_2), u_3, v_3 ∈ Γ_i(a_3), with u_1, u_2, u_3 and v_1, v_2, v_3 triples of vertices at mutual distance δ.

Now d(u_1, v_3) = δ - 2i, and d(u_1, v_3) = δ, so as usual d(u_3, v_3) = 2i. We now consider the following property of the triple (u_1, v_1, a_3): For v_3 ∈ Γ_i(a_3) with d(v_3, v_3) = δ, we have d(u_1, v_3) = δ - 2i. The triple (u_1, v_1, a_3) is isometric with (v_3, u_3, a_2). It follows that d(v_1, u_2) = δ - 2i. SO

δ = d(u_1, u_2) ≤ 2(δ - 2i)

This shows that i ≤ δ/4, so for a contradiction we require

δ/4 < i < δ/2

and for δ > 4 this is possible.

We can now give the classification of distance homogeneous antipodal graphs of diameter 3.

**Theorem 14.** Let G be one of the following graphs: the pentagon (5-cycle), the line graph E(K_{3,3}) for the complete bipartite graph K_{3,3}, an independent set I_n (n ≤ ∞), or the random graph Γ_∞. Let G* be the graph obtained from G by adjoining an additional vertex adjacent to all vertices of G, and let Γ be the graph obtained by taking two copies H_1, H_2 of G*, with a fixed isomorphism u → u' between them, and with additional edges (u, v') or (v', u), for u, v ∈ H_1, just when (u, v) is not an edge of H_1. Then Γ is a homogeneous antipodal graph of diameter 3 with pairing the given isomorphism u → u'. Conversely, any connected metrically homogeneous antipodal graph of diameter 3 is of this form.

Proof. Let Γ be connected, metrically homogeneous, and antipodal, of diameter 3. Fix a basepoint * ∈ Γ and let G = Γ_1(*). Then G is a homogeneous graph, which may be found in the Lachlan/Woodrow catalog given in §3.

Let H_1 = G ∪ {*}. Then the pairing u → u' on Γ gives an isomorphism of H_1 with H_2 = Γ_1(*') ∪ {*'} . Furthermore, for u, v ∈ Γ_1(*), we have d(u, v') = δ - d(u, v), so the edge rule in Γ is the one we have described. It remains to identify
the set of homogeneous graphs $G^*$ for which the associated graph $\Gamma$ is metrically homogeneous.

We claim that for any homogeneous graph $G^*$, the associated graph $\Gamma$ has the following homogeneity property: if $A, B$ are finite subgraphs of $\Gamma$ both containing the point $\ast$, then any isometry $A \to B$ fixing $\ast$ extends to an automorphism of $\Gamma$. Given such $A, B$, we first extend to $\hat{A}, \hat{B}$ by closing under the pairing $u \leftrightarrow u'$, then reduce to $G$ by taking $\tilde{A} = \hat{A} \cap G, \tilde{B} = \hat{B} \cap G$. Then apply the homogeneity of $G$ to get an isometry extending the given one on $\tilde{A}$ to all of $H_1$, fixing $\ast$, which then extends canonically to $\Gamma$. It is easy to see that this agrees with the given isometry on $A$.

This homogeneity condition implies that for such graphs $\Gamma$, the graph will be metrically homogeneous if and only if $\text{Aut}(\Gamma)$ is transitive on vertices. For any of these graphs $\Gamma$, whether metrically homogeneous or not, we have the pairing $u \leftrightarrow u'$. Furthermore, we can reconstruct $\Gamma$ from $\Gamma_1(v)$ for any $v \in \Gamma$. So the homogeneity reduces to this: $\Gamma_1(v) \cong G$ for $v \in \tilde{G} = \Gamma_1(\ast)$; here we use the pairing to reduce to the case $v \in G$.

Now $\Gamma_1(v)$ is the graph obtained from the vertex $\ast$, the graph $G_1(v)$ induced on the neighbors of $v$ in $G$, and the graph $G_2(v)$ induced on the non-neighbors of $v$ in $G$, by taking the neighbors of $\ast$ to be $G_1(v)$, and switching the edges and nonedges between $G_1(v)$ and $G_2(v)$. Another way to view this would be to replace $v$ by $\ast$, and then perform the switching between $G_1(v)$ and $G_2(v)$. So it is really only the latter that concerns us.

We go through the catalog. In the degenerate cases, with $G$ complete or independent, there is no switching, so the corresponding graph $\Gamma$ is homogeneous. But when $G$ is complete this graph is not connected, so we set that case aside.

When $G$ is imprimitive, we switch edges and non-edges between the equivalence classes not containing the fixed vertex $v$, and the vertices in the equivalence class of $v$ other than $v$ itself. As a result, the new graph becomes connected with respect to the equivalence relation on $G$, so this certainly does not work.

When $G$ is primitive, nondegenerate, and finite, we have just the two examples mentioned above for which the construction does work, by inspection.

Lastly, we consider the Henson graphs $G = \Gamma_n$, generic omitting $K_n$, their complements, and the random graph $\Gamma_\infty$. The Henson graphs will not work here. For example, if $G = \Gamma_n$, then $G_2(v)$ contains $K_{n-1}$, and switching edges and nonedges with $G_1(v)$ will extend this to $K_n$. The complementary case is the same. So we are left with the case of the Rado graph. This is characterized by extension properties, and it suffices to check that these still hold after performing the indicated switch; and using the vertex $v$ as an additional parameter, this is clear. $\square$

There is a good deal more to the general analysis of [AH06], Proposition 2.5 through Corollary 2.10, all with some parallels in our case, but the main examples in the finite case do not satisfy our conditions, while the main examples in our case have no finite analogs, so the statements gradually diverge, and it is better for us to turn to the consideration of graphs which are exceptional in another sense, and only then come back to the bipartite case.

In particular the main result of [AH06] is the following.

**Fact 5.13 ([AH06, Theorem 3.3]).** An antipodal and bipartite finite distance transitive graph of diameter 6 and degree at least 3 isomorphic to the 6-cube.
We have infinite connected metrically homogeneous graphs of any diameter $\delta \geq 3$ which are both bipartite and antipodal. In the case $\delta = 6$, the associated graph $\Gamma_2$ is the generic bipartite graph of diameter 4, and the associated graph $\Gamma_3$ is isomorphic to $\Gamma$ itself. So even in the context of Smith’s Theorem, the two pictures eventually diverge. But graphs of odd diameter which are both bipartite and antipodal are quite special.

Recall the statement of Theorem 11.

**Theorem (11).** Let $\Gamma$ be a metrically homogeneous graph of odd diameter $2d + 1$ which is both antipodal and bipartite. Then $B\Gamma$ is a primitive infinite metrically homogeneous graph with the following properties:

1. $B\Gamma$ has diameter $d$;
2. No triangle in $B\Gamma$ has perimeter greater than $2d + 1$;
3. $B\Gamma$ is not antipodal.

For any metrically homogeneous graph $G$ with these three properties, there is an antipodal bipartite graph of diameter $2d + 1$ with $B\Gamma \cong G$, and $G$ determines $\Gamma$ up to isomorphism.

We remark that conditions (1-3) on $B\Gamma$ imply that $B\Gamma_d$ is a clique of order at least 2. One exceptional case included under this theorem is that of the $(4d + 2)$-gon, of diameter $2d + 1$, associated with the $(2d + 1)$-gon, of diameter $d$.

**Proof.** Let $A, B$ be the two halves of $\Gamma$. Then the metric on $\Gamma$ is determined by the metrics on $A$ and $B$ and the pairing

$$a \leftrightarrow a'$$

between $A$ and $B$ determined by $d(a, a') = 2d + 1$, since

$$d(a_1, a'_2) = 2d + 1 - d(a_1, a_2)$$

So the uniqueness is clear.

Let us next check that the conditions on $B\Gamma$ are satisfied. The first is clear. For the second, suppose we have vertices $(a_1, a_2, a_3)$ in $B\Gamma$ forming a triangle of perimeter at least $2d + 2$; we may construe these as vertices of $A$ forming a triangle of perimeter $P \geq 4d + 4$. Then looking at the triangle $(a_1, a_2, a'_3)$, we have

$$d(a_1, a'_3) + d(a_2, a'_3) = (4d + 2) - [d(a_1, a_3) + d(a_2, a_3)]$$

$$= (4d + 2) - P + d(a_1, a_2)$$

$$< d(a_1, a_2),$$

contradicting the triangle inequality. Finally, consider an edge $(a_1, a_2)$ of $B\Gamma$, which we construe as a pair of vertices of $A$ at distance 2. Then there must be a vertex $a$ such that $a'$ is adjacent to both, and this means that $a_1, a_2 \in \Gamma_{2d}(a)$, that is $a_1, a_2 \in B\Gamma_d(a)$.

Conversely, suppose $G$ is a metrically homogeneous graph of diameter $d$, and $\Gamma$ is the metric space on $G \times \{0, 1\}$ formed by doubling the metric of $G$ on $A = G \times \{0\}$ and on $B = G \times \{1\}$, pairing $A$ and $B$ by $(a, \epsilon)' = (a, 1 - \epsilon)$, and defining

$$d(a_1, a'_2) = 2d + 1 - d(a_1, a_2)$$

for $a_1, a_2$ in $A$

The triangle inequality follows directly from the bound on the perimeters of triangles. We claim that $\Gamma$ is a homogeneous metric space. Furthermore, the pairing
$a \leftrightarrow a'$ is an isometry of $\Gamma$, and is recoverable from the metric: $y = x'$ if and only if $d(x, y) = 2d + 1$.

Let $X, Y$ be finite subspaces of $\Gamma$, and $f$ an isometry between them. Then $f$ extends canonically to their closures under the antipodal pairing. So we may suppose $X$ and $Y$ are closed under the antipodal pairing; and composing $f$ with the antipodal pairing if necessary, we may suppose $f$ preserves the partition of $\Gamma$ into $A, B$. Then restrict $f$ to $A \cap X$, extend to $A$ by homogeneity, and then extend back to $\Gamma$. Thus $\Gamma$ is a homogeneous metric space.

Finally, we claim that the metric on $\Gamma$ is the graph metric, and for this it suffices to show that vertices at distance 2 in the metric have a common neighbor in $\Gamma$. So let $a_1, a_2$ be two such vertices, taken for definiteness in $A$; write $a_i = (v_i, 0)$. Taking $v \in G$ with $v_1, v_2 \in G_d(v)$, and $a = (v, 1)$, we find that $a$ is adjacent to $a_1$ and $a_2$. $\square$

The case of diameter 5 is of particular interest as it turns up naturally when cataloguing exceptional cases.

**Corollary 5.14.** Let $\Gamma$ be an antipodal bipartite graph of diameter 5. Then $B\Gamma$ is either a pentagon, or the generic homogeneous graph omitting $I_3$.

**Proof.** Let $G = B\Gamma$. Then $G$ has diameter 2, so it is a homogeneous graph, on the list of Lachlan and Woodrow. Furthermore, by the theorem, $G$ contains $I_2$ but not $I_3$. As $d = 2$ here, $G$ contains a path of length 2 as well as a vertex at distance 2 from both vertices of an edge, and thus is primitive. By the classification, in the finite case $G$ is a 5-cycle (with $\Gamma$ a 10-cycle) and in the infinite case it must be the generic graph omitting $I_3$. $\square$

### 6. Exceptional Metrically Homogeneous Graphs

In the classification of metrically homogeneous graphs, imprimitivity is one kind of exception meriting special analysis. Another equally important kind of exception is associated with the Lachlan/Woodrow classification of the homogeneous graphs.

If $\Gamma$ is a metrically homogeneous graph, then $\Gamma_1$ is a homogeneous graph, and must occur in the short list of such graphs described in §3. There are three possibilities for $\Gamma_1$ which are compatible with the Fraïssé constructions we have described, and any classification of the corresponding possibilities for $\Gamma$ would be expected to be a large undertaking. In the other cases, we have an explicit classification of the possibilities for $\Gamma$. Accordingly we make the following definition.

**Definition 6.1.** A metrically homogeneous graph $\Gamma$ is of generic type if the graph $\Gamma_1$ is of one of the following three types: an infinite independent set $I_\infty$; generic omitting $K_{n+1}$ for some $n$; or fully generic (the random graph). If $\Gamma$ is not of generic type, then we say it is of exceptional type.

Note that all the bipartite graphs (and quite a bit more!) are included under “generic type.” The classification of primitive triangle free metrically homogeneous graphs is likely to pose particular difficulties. But it is still very useful to have a classification of the exceptional metrically homogeneous graphs in this sense, and this we now turn to.

**Definition 6.2.** For \(2 \leq r, s \leq \infty\), we may construct an \(r\)-tree of \(s\)-cliques \(T_{r,s}\) as follows. Take a tree \(T(r,s)\) partitioned into two sets of vertices \(A, B\), so that each vertex of \(A\) has \(r\) neighbors, all in \(B\), and each vertex of \(B\) has \(s\) neighbors, all in \(A\). Consider the graph induced on \(A\) with edge relation given by \(d(u,v) = 2\). This is \(T_{r,s}\) (and the corresponding graph on \(B\) is \(T_{s,r}\)).

**Lemma 6.3.** For any \(r, s\) the tree \(T(r,s)\) is homogeneous as a metric space with a fixed partition into two sets, and the graph \(T_{r,s}\) is metrically homogeneous.

**Proof.** For any finite subset \(A\) of a tree \(T\), one can see that the metric structure on \(A\) induced by \(T\) determines the structure of the convex closure of \(A\), the smallest subtree of \(T\) containing \(A\). Given that, a map between two finite subsets of \(T(r,s)\) that respects the partition will extend first to the convex closures and then to the whole of \(T(r,s)\).

This applies in particular to the two halves of \(T(r,s)\).

With \(r, s < \infty\) these graphs are locally finite (that is, the vertex degrees are finite). Conversely:

**Fact 6.4** (Macpherson, [Mph82]). Let \(G\) be an infinite locally finite distance transitive graph. Then \(G\) is \(T_{r,s}\) for some finite \(r, s \geq 2\).

The proof uses a result of Dunwoody on graphs with nontrivial cuts given in [Dun82].

From the point of view of the Lachlan/Woodrow classification, these graphs fall mainly under the case \(\Gamma_1\) imprimitive. If we require metric homogeneity rather than distance transitivity then a similar result applies to this broader class.

We will prove the following.

**Theorem 15.** Let \(\Gamma\) be a connected metrically homogeneous graph of diameter \(\delta\), and suppose \(\Gamma_1\) is finite or imprimitive. Then one of the following occurs.

1. \(\delta \leq 2\), \(\Gamma\) is found under the Lachlan/Woodrow classification.
2. \(\Gamma\) has degree 2, a cycle.
3. \(\delta = 3\), \(\Gamma\) is obtained by doubling \(C_5\), \(E(K_{3,3})\), or an independent set.
4. \(\delta = \infty\), \(\Gamma = T_{r,s}\) for some \(r, s \geq 2\).

**Lemma 6.5.** Let \(\Gamma\) be a connected metrically homogeneous graph of diameter at least 3 and degree at least 3, and suppose that \(\Gamma_1\) is one of the primitive finite homogeneous graphs containing both edges and nonedges, that is \(C_5\) or \(E(K_{3,3})\). Then \(\Gamma\) is the antipodal graph of diameter 3 obtained from \(\Gamma_1\) in the manner of Theorem 14.

**Proof.** We fix a basepoint \(*\) in \(\Gamma\) so that \(\Gamma_i\) is viewed as a specific subgraph of \(\Gamma\) for each \(i\). The proof proceeds in two steps.

1. There is a \(*\)-definable function from \(\Gamma_1\) to \(\Gamma_2\).

   We will show that for \(v \in \Gamma_1\), the vertices of \(\Gamma_1\) not adjacent to \(v\) have a unique common neighbor \(v'\) in \(\Gamma_2\).

   If \(\Gamma_1\) is a 5-cycle then this amounts to the claim that every edge of \(\Gamma\) lies in two triangles, and this is clear by inspection of an edge \((*, v)\) with \(*\) the basepoint and \(v \in \Gamma_1\).
Now suppose $\Gamma_1$ is $E(K_{3,3})$. We claim that every induced 4-cycle $C \cong C_4$ in $\Gamma$ has exactly two common neighbors.

Consider $u, v \in \Gamma_1$ lying at distance 2, and let $G_{u,v}$ be the metric space induced on their common neighbors. This is a homogeneous metric space. Since these common neighbors consist of the basepoint $*$, the two common neighbors $a, b$ of $u, v$ in $\Gamma_1$, and whatever common neighbors $u, v$ may have in $\Gamma_2$, we see that pairs at distance 1 occur, and the corresponding graph has degree 2 (looking at $*$) and is connected (looking at $(a, *, b)$). So $G_{u,v}$ is a connected metrically homogeneous graph of degree 2, and furthermore embeds in $\Gamma_1(u) \cong \Gamma_1$. So $G_{u,v}$ is a 4-cycle, and therefore $(u, a, v, b)$ has exactly 2 neighbors, as claimed. This proves (1).

Now let $f : \Gamma_1 \to \Gamma_2$ be $*$-definable. By homogeneity $f$ is surjective, and as $\Gamma_1$ is primitive, it is bijective. It also follows from homogeneity that for $u, v \in \Gamma_1$, $d(u, v)$ determines $d(f(u), f(v))$, so $f$ is either an isomorphism or an anti-isomorphism. Since $\Gamma_1$ is isomorphic to its complement, $\Gamma_1 \cong \Gamma_2$ in any case. Hence the vertices of $\Gamma_2$ have a common neighbor $v$, and $v \in \Gamma_3$. We claim that $|\Gamma_3| = 1$.

By homogeneity all pairs $(u, v)$ in $\Gamma_2 \times \Gamma_3$ are adjacent. In particular for $v_1, v_2 \in \Gamma_3$ we have $\Gamma_1(v_1) = \Gamma_1(v_2)$ and $d(v_1, v_2) \leq 2$. Since we have pairs of vertices $u_1, u_2$ in $\Gamma_1$ at distance 1 or 2 for which $\Gamma_1(u_1) \neq \Gamma_1(u_2)$, we find $|\Gamma_3| = 1$.

It now follows that $\Gamma$ is antipodal of diameter 3 and the previous analysis applies. \hfill \Box

**Lemma 6.6.** Let $\Gamma$ be a metrically homogeneous graph of diameter at least 3 and degree at least 3, and suppose that $\Gamma_1$ is a complete multipartite graph of the form $K_m[I_n]$ (the complement of $m \cdot K_n$). Then $m = 1$.

**Proof.** Fix an induced path $(*, u, v)$ of length 2 and let $\Gamma_1 = \Gamma_i(*)$. Let $A$ be the set of neighbors of $u$ in $\Gamma_1$; if $m > 1$, then $A$ is nonempty. Then $A \cong K_{m-1}[I_n]$, and the neighbors of $u$ in $\Gamma$ include $*, v$, and $A$. Now “$d(x, y) > 1$” is an equivalence relation on $\Gamma_1(u)$, and $*$ is adjacent to $A$, so $v$ is adjacent to $A$. Now if we replace $u$ by $u' \in A$ and argue similarly with respect to $(*, u', v)$, we see that the rest of $\Gamma_1$ is also adjacent to $v$, that is $\Gamma_1 \subseteq \Gamma_1(v)$. Now switching $* \text{ and } v$, by homogeneity $\Gamma_1(v) \subseteq \Gamma_1$. But then the diameter of $\Gamma$ is 2, a contradiction. Thus $m = 1$. \hfill \Box

It remains to consider the case $\Gamma_1 \cong m \cdot K_n$, or rather the two cases arising under the assumption either that $n \geq 2$ or that $m < \infty$.

We first deal with the case $n$ which $\Gamma_1$ is an independent set of vertices, under a milder hypothesis than the finiteness of $m$.

**Lemma 6.7.** Let $\Gamma$ be a metrically homogeneous graph of diameter at least 2, with $\Gamma_1 \cong I_m$, $3 \leq m \leq \infty$. Suppose that for $u, v \in \Gamma$ at distance two, the number $k$ of common neighbors of $u, v$ is finite. Then either $k = 1$, or $m = k + 1$.

**Proof.** We consider $\Gamma_1$ and $\Gamma_2$ with respect to a fixed basepoint $* \in \Gamma$. For $u \in \Gamma_2$, let $I_u$ be the $k$-set consisting of its neighbors in $\Gamma_1$. Any $k$-subset of $\Gamma_1$ occurs as $I_u$ for some $u$. For $u, v \in \Gamma_2$ let $u \cdot v = |I_u \cap I_v|$. If $u \cdot v \geq 1$ then $d(u, v) = 2$. Now $\text{Aut}(\Gamma_1)_*$ has a single orbit on pairs in $\Gamma_2$ at distance 2, while every value $i$ in the range $\max(1, 2k - m) \leq i \leq k - 1$, will occur as $u \cdot v$ for some such $u, v$. Therefore $k \leq 2$ or $k = m - 1$, with $m$ finite in the latter case.

The case $k = 2 < m - 1$ is eliminated by a characteristic application of homogeneity. A set of three pairs in $\Gamma_1$ which intersect pairwise may or may not have a
common element (once \( m \geq 4 \)), so if we choose \( u_1, u_2, u_3 \) and \( v_1, v_2, v_3 \) in \( \Gamma_2 \) corresponding to these two possibilities for the associated \( I_{u_i} \) and \( I_{v_j} \), we get isometric configurations \((*, u_1, u_2, u_3)\) and \((*, v_1, v_2, v_3)\) which lie in distinct orbits of \( \text{Aut} \Gamma \). So we have \( k = 1 \) or \( m = k + 1 \).

**Lemma 6.8.** Let \( \Gamma \) be a connected metrically homogeneous graph of diameter at least 3, with \( \Gamma_1 \cong I_m \), and \( 3 \leq m \leq \infty \). Suppose any pair of vertices at distance 2 have \( k \) common neighbors, with \( k < \infty \). Then one of the following occurs.

1. \( m = k + 1 \), and \( \Gamma \) is the complement of a perfect matching, in other words the antipodal graph of diameter 3 obtained by doubling \( \Gamma_1 \).
2. \( k = 1 \), and \( \Gamma \) is a \( k \)-regular tree.

**Proof.** We fix a basepoint \( * \) and write \( \Gamma_i \) for \( \Gamma_i(*) \).

Suppose first that \( m = k + 1 \). Then any two vertices of \( \Gamma_2 \) lie at distance 2, and there is a \(*\)-definable function \( f : \Gamma_2 \to \Gamma_1 \) given by the nonadjacency relation. By homogeneity \( f \) is surjective, and as \( \Gamma_2 \) is, primitive, \( f \) is bijective. In particular \( \Gamma_2 \cong \Gamma_1 \) and there is a vertex \( v \in \Gamma_3 \) adjacent to all vertices of \( \Gamma_2 \), hence \( \Gamma_1(v) = \Gamma_2 \).

It follows readily that \( |\Gamma_3| = 1 \) and \( \Gamma \) is antipodal of diameter 3. The rest follows by our previous analysis.

Now suppose that \( k = 1 \). It suffices to show that \( \Gamma \) is a tree.

Suppose on the contrary that there is a cycle \( C \) in \( \Gamma \), which we take to be of minimal diameter \( d \). Then the order of \( C \) is \( 2d \) or \( 2d + 1 \).

Suppose the order of \( C \) is \( 2d \). Then for \( v \in \Gamma_d \), \( v \) has at least two neighbors \( u_1, u_2 \) in \( \Gamma_{d-1} \), whose distance is therefore 2. Furthermore, in \( \Gamma_{d-2} \) there are no edges, and each vertex of \( \Gamma_{d-2} \) has a unique neighbor in \( \Gamma_{d-3} \), so each vertex of \( \Gamma_{d-2} \) has at least two neighbors \( u_1, u_2 \) in \( \Gamma_{d-1} \), whose distance is therefore 2.

So for \( u_1, u_2 \in \Gamma_{d-1} \), there is a common neighbor in \( \Gamma_{d} \), and also in \( \Gamma_{d-2} \). This gives a 4-cycle in \( \Gamma \), contradicting \( k = 1 \).

So the order of \( C \) is \( 2d + 1 \). In particular, each \( v \in \Gamma_d \) has a unique neighbor in \( \Gamma_{d-1} \), and \( \Gamma_{d-1} \) contains edges.

Let \( G \) be a connected component of \( \Gamma_d \). Suppose \( u, v \) in \( G \) are at distance 2 in \( G \). Then \( u, v \) must have a common neighbor in \( \Gamma_{d-1} \) as well as in \( G \), and this contradicts the hypothesis \( k = 1 \). So the connected components of \( \Gamma_d \) are simply edges.

Take \( a \in \Gamma_{d-1} \), \( u_1, v_1 \in \Gamma_d \) adjacent to \( a \), and take \( u_2, v_2 \in \Gamma_d \) adjacent to \( u_1, v_1 \) respectively. By homogeneity there is an automorphism fixing the basepoint \( * \) and interchanging \( u_1 \) with \( v_1 \) and also interchanges \( u_2 \) and \( v_1 \). Hence \( d(v_1, v_2) = 2 \). This follows that \( v_1, v_2 \) have a common neighbor \( b \) in \( \Gamma_{d-1} \). Now \( (a, u_1, v_1, b, v_2, u_2, a) \) is a 6-cycle. Since the minimal cycle length is odd, we have \( |C| = 5 \) and \( d = 2 \).

Furthermore the element \( b \) is determined by \( a \in \Gamma_1 \) and the basepoint \( * \): we take \( u \in \Gamma_2 \) adjacent to \( a \), \( v \in \Gamma_2 \) adjacent to \( u \), and \( b \in \Gamma_1 \) adjacent to \( v \). So the function \( a \mapsto b \) is \(*\)-definable. However \( \Gamma_1 \) is an independent set of order at least 3, so this violates homogeneity.

The remaining case is the main one: \( \Gamma_1 = m \cdot K_n, n \geq 2 \).

6.2. **The case** \( \Gamma_1 = m \cdot K_n, n \geq 2 \). Our main goal is to show that any two vertices at distance two have a unique common neighbor. We divide up the analysis into three cases. Observe that in a metrically homogeneous graph with \( \Gamma_1 \cong m \cdot K_n \), the common neighbors of a pair \( a, b \in \Gamma \) at distance 2 will be an independent set,
since for \( u_1, u_2 \) adjacent to \( a, b \), and each other, we would have the path \((a, u_1, b)\) inside \( \Gamma_1(u_2) \).

**Lemma 6.9.** Let \( \Gamma \) be a metrically homogeneous graph of diameter at least 3 and suppose that \( \Gamma \cong n \cdot K_n \) with \( n \geq 3 \). Then for \( u, v \in \Gamma \) at distance 2, there are at most two vertices adjacent to both.

**Proof.** Supposing the contrary, every induced path of length 2 is contained in two distinct 4-cycles. Fix a basepoint \( * \in \Gamma \) and let \( \Gamma_i = \Gamma_i(*) \).

Fix \( v_1, v_2 \in \Gamma_1 \) adjacent. For \( i = 1, 2 \), let \( H_i \) be the set of neighbors of \( v_i \) in \( \Gamma_2 \).

We claim

\[
H_1 \cap H_2 = \emptyset
\]

Otherwise, consider \( v \in H_1 \cap H_2 \) and the path \((*, v_1, v)\) contained in \( \Gamma_1(v_2) \).

Next, we will find \( u_1 \in H_1 \), and \( u_2, u_2' \in H_2 \) distinct, so that

\[
d(u_1, u_2) = d(u_1, u_2') = 1
\]

Extend the edge \((v_1, v_2)\) to a 4-cycle \((v_1, v_2, u_2, u_1)\). Then \( u_1, u_2 \notin \Gamma_1 \cup \{\ast\} \), so \( u_1 \in H_1 \) and \( u_2 \in H_2 \). By our hypothesis there is a second choice of \( u_2 \) with the same properties.

With the vertices \( u_1, u_2, u_2' \) fixed, let \( A, B, B' \) denote the components of \( H_1 \) and \( H_2 \), respectively, containing the specified vertices. Observe that \( B \) and \( B' \) are distinct: otherwise, the path \((u_1, u_2, v_2)\) would lie in \( \Gamma_1(u_2') \).

\[
(2) \quad \text{The relation } d(x, y) = 1 \text{ defines a bijection between } A \text{ and } B
\]

With \( u \in A \) fixed, it suffices to show the existence and uniqueness of the corresponding element of \( B \). The uniqueness amounts to the point just made for \( u_1 \), namely that \( B \neq B' \).

For the existence, we may suppose \( u \neq u_1 \). Then \( d(u, v_2) = d(u, u_2) = 2 \). We have an isometry

\[
(*) \quad \text{and hence the triple } (u, u_1, u_2) \text{ with } u_1 \in H_1 \text{ corresponds to an isometric triple } (u_2', u_1, u) \text{ with } u_1 \in H_2.
\]

Thus we have a bijection between \( A \) and \( B \) definable from \((*, v_1, v_2, u_1, u_2)\) and hence derive a bijection between \( B \) and \( B' \) definable from \((*, v_1, v_2, u_1, u_2, u_2')\). Using this, we show

\[
n = 2
\]

The graph induced on \( B \cup B' \) is \( 2 \cdot K_{n-1} \) and any isometry between finite subsets of \( B \cup B' \) containing \( u_2, u_2' \) will be induced by \( \text{Aut}(\Gamma) \). So if there is a bijection between \( B \) and \( B' \) invariant under the corresponding automorphism group, we find \( n = 2 \).

**Lemma 6.10.** Let \( \Gamma \) be a metrically homogeneous graph of diameter at least 3 and suppose that \( \Gamma \cong m \cdot K_n \) with \( n \geq 2 \). Let \( u, v \in \Gamma \) lie at distance 2, and suppose \( u, v \) have finitely many common neighbors. Then they have a unique common neighbor.

**Proof.** We fix a basepoint \( * \), and for \( u \in \Gamma_2 \) we let \( I_u \) be the set of neighbors of \( u \) in \( \Gamma_1 \). Our assumption is that \( k = |I_u| \) is finite. Then any independent subset of \( \Gamma_1 \) of cardinality \( k \) occurs as \( I_u \) for some \( u \in \Gamma_2 \).
We consider the $k - 2$ possibilities:

$$|I_u \cap I_v| = i \text{ with } 1 \leq i \leq k - 1$$

As $n \geq 2$, all possibilities are realized, whatever the value of $m$. However in all such cases, $d(u, v) \leq 2$, so we find $k - 1 \leq 2$, and $k \leq 3$.

We claim that for $u, v \in I_2$ adjacent, we have $|I_u \cap I_v| = 1$.

There is a clique $v, u_1, u_2$ with $v \in I_1$ and $u_1, u_2 \in I_2$. As $u_1, u_2$ are adjacent their common neighbors form a complete graph. On the other hand $I_{u_1}$ and $I_{u_2}$ are independent sets, so their intersection reduces to a single vertex. By homogeneity the same applies whenever $u_1, u_2 \in I_2$ are adjacent, proving our claim.

Now suppose $k = 3$. Then $|I_u \cap I_v|$ can have cardinality 1 or 2, and the case $|I_u \cap I_v| = 2$ must then correspond to $d(u, v) = 2$.

Now $k \leq m$, so we may take pairs $(a_i, b_i)$ for $i = 1, 2, 3$ lying in distinct components of $I_1$. Of the eight triples $t$ formed by choosing one of the vertices of each of these pairs, there are four in which the vertex $a_i$ is selected an even number of times. Let a vertex $v_t \in I_2$ be taken for each such triple, adjacent to its vertices. Then the four vertices $v_t$ form a complete graph $K_4$. It follows that $K_3$ embeds in $I_1$, that is $n \geq 3$. So we may find independent triples $I_1, I_2, I_3$ such that $|I_1 \cap I_2| = |I_2 \cap I_3| = 1$ while $I_1 \cap I_3 = \emptyset$. Take $u_1, u_2, u_3 \in I_2$ with $I_i = I_{u_i}$ and with $d(u_1, u_2) = d(u_2, u_3) = 1$. Then $I_{u_1} \cap I_{u_3} = \emptyset$, while $d(u_1, u_2) \leq 2$, a contradiction. Thus $k = 2$.

With $k = 2$, suppose $m > 2$. For $u \in I_2$, let $I_u$ be the set of components of $I_1$ meeting $I_u$. We consider the following two properties of a pair $u, v \in I_2$:

$$|I_u \cap I_v| = 1; |I_u \cap I_v| = i \text{ (} i = 1 \text{ or } 2 \text{)}$$

These both occur, and must correspond in some order with the conditions $d(u, v) = 1$ or 2. But just as above we find $u_1, u_2, u_3$ with $d(u_1, u_2) = d(u_2, u_3) = 1$ and $I_{u_1} \cap I_{u_3} = \emptyset$, and as $d(u_1, u_3) \leq 2$ this is a contradiction.

So we come down to the case $m = k = 2$. But then for $v \in I_2$, all components of $I_1(v)$ are represented in $I_1$, and hence $v$ has no neighbors in $I_3$, a contradiction. □

**Lemma 6.11.** Let $\Gamma$ be a metrically homogeneous graph of diameter at least 3 and suppose that $\Gamma_1 \cong m \cdot K_n$ with $n \geq 2$. Let $u, v \in \Gamma$ lie at distance 2, and suppose $u, v$ have infinitely many common neighbors. Then $n = \infty$.

**Proof.** We fix a basepoint $*$, and for $u \in I_2$ we let $I_u$ be the set of neighbors of $u$ in $\Gamma_1$. Our assumption is that $I_u$ is infinite. Then any finite independent subset of $\Gamma_1$ is contained in $I_u$ for some $u \in I_2$.

For $u \in I_2$, let $I_u$ be the set of components of $\Gamma_1$ which meet $I_u$, and let $J_u$ be the set of components of $\Gamma_1$ which do not meet $I_u$. We show first that $J_u$ is infinite.

Supposing the contrary, let $k = |J_u| < \infty$ for $u \in I_2$. Any set of $k$ components of $\Gamma_1$ will be $J_u$ for some $u \in I_2$, and the $k + 1$ relations on $\Gamma_2$ defined by

$$|J_u \cap J_v| = i$$

for $i = 0, 1, \ldots, k$ will be nontrivial and distinct. Furthermore, for any pair pre-assigned $k$ components $J$, and any vertex $a \in \Gamma_1$ not in the union of $J$, there is a vertex $u$ with $J_u = J$ and $a \in I_u$, so our $(k + 1)$ relations are realized by pairs $u, v \in I_2$ with $I_u \cap I_v \neq \emptyset$, and hence $d(u, v) \leq 2$. Hence $k + 1 \leq 2$, $k \leq 1$. 


Suppose $k = 1$ and fix a vertex $v_0 \in \Gamma_1$. Then for $u, v \in \Gamma_2$ adjacent to $v_0$, the two relations $J_u = J_v$, $J_u \neq J_v$ correspond in some order to the relations $d(u, v) = 1$, $d(u, v) = 2$, and since the first relation is an equivalence relation, they correspond in order.

With $u \in \Gamma_2$, $v_0, v_1 \in I_u$ distinct, there are $u_0, u_1$ in $\Gamma_2$ with $u_0$ adjacent to $u$ and $v_0$, and with $u_1$ adjacent to $u$ and $v_1$. The neighbors of $u$ form a graph of type $\infty \cdot K_n$, so $d(u_0, u_1) = 2$. However $d(u, u_0) = d(u, u_1) = 1$ and hence $J_{u_0} = J_{u_1}$, a contradiction.

So $k = 0$ and for $u \in \Gamma_2$, the set $I_u$ meets every component of $\Gamma_1$. That is, $\Gamma_1(u)$ meets every component of $\Gamma_1$, and after switching the roles of $u$ and the basepoint $v$, we conclude $\Gamma_1$ meets every component of $\Gamma_1(u)$, which is incompatible with the condition $\delta \geq 3$. So $\hat{J}_u$ is infinite for $u \in \Gamma_2$.

Now we claim

For $u, v \in \Gamma_2$ adjacent, $\hat{I}_u \setminus \hat{I}_v$ is infinite

Supposing the contrary, for all adjacent pairs $u, v \in \Gamma_2$, the sets $\hat{I}_u$ and $\hat{I}_v$ coincide up to a finite difference.

Take $u \in \Gamma_2$, $v_0, v_1 \in \Gamma_1$ adjacent to $u$, and $u_0, u_1$ adjacent to $u$, $v_0$ or $u, v_1$ respectively. Then, as above, $d(u_0, u_1) = 2$, while $J_{u_0} = J_u = J_{u_1}$. Thus for $u, v \in \Gamma_2$ with $d(u, v) \leq 2$ we have $\hat{J}_u = \hat{J}_v$. Furthermore, the size of the difference $|J_u \setminus J_v|$ is bounded, say by $\ell$. But we can fix $u \in \Gamma_2$ and then find $v \in \Gamma_2$ so that $I_u$ meets $I_v$ but $I_u$ picks up more than $\ell$ components of $\hat{J}_u$. Since $I_u$ meets $I_v$, we have $d(u, v) \leq 2$, and thus a contradiction. This proves our claim.

We make a third and last claim of this sort.

For $u, v \in \Gamma_2$ adjacent, $\hat{I}_u \cap \hat{I}_v$ is infinite

Supposing the contrary, let $k'$ be $|\hat{I}_u \cap I_u|$ for $u, v \in \Gamma_2$ adjacent, fix $u, v \in \Gamma_2$ adjacent, and let $I$ be a finite independent subset consisting of representatives for more than $k'$ components in $\hat{I}_v \setminus \hat{I}_u$.

Take $a \in I_u \cap I_v$ and take $v' \in \Gamma_2$ adjacent to $a$ and to $u$, with $J \subseteq \hat{I}_{v'}$ and $I_v \neq I_{v'}$.

Now $u, v, v'$ are adjacent to $a$, and $v, v'$ are adjacent to $u$, so $v$ and $v'$ are adjacent. But by construction $|I_{v'} \setminus I_v| > k'$. This proves the third claim.

Now, finally, suppose $n$ is finite. Take $u \in \Gamma_2$, $v \in I_u$, and let $A, B$ be components of $\Gamma_1$ which are disjoint from $I_u$ but meet $I_{v'}$ for some $v'$ adjacent to $u, v$. Then we can then find neighbors $u_{a,b}$ of $u, v$ in $\Gamma_2$ for which the intersection of $I_v$ with $A$ and $B$ respectively is an arbitrary pair of representatives $a, b$. But this requires $n - 1 \geq n^2$, and gives a contradiction.

\[ \square \]

**Corollary 6.12.** Let $\Gamma$ be a metrically homogeneous graph of diameter at least 3 and suppose that $\Gamma_1 \equiv m \cdot K_n$ with $n \geq 2$. Then for $u, v \in \Gamma$ with $d(u, v) = 2$, there is a unique vertex adjacent to both.

**Proof.** Apply the last three lemmas. If $u, v$ have infinitely many common neighbors, then $n$ is infinite. In particular, $n \geq 3$. But then they have at most two common neighbors. So in fact $u, v$ have finitely many neighbors, and we apply Lemma 6.10.

After this somewhat laborious reduction, we can complete the classification in this case.
Proposition 6.13. Let $\Gamma$ be a connected distance homogeneous graph with $\Gamma_1 \cong m \cdot K_n$, with $n \geq 2$ and $\delta \geq 3$. Then $m \geq 2$, and $\Gamma_1 \cong T_{m,n+1}$.

Proof. If $m = 1$ then evidently $\Gamma$ is complete, contradicting the hypothesis on $\delta$. So $m \geq 2$.

By definition, the blocks of $\Gamma$ are the maximal 2-connected subgraphs. Any edge of $\Gamma$ is contained in a unique clique of order $n + 1$. It suffices to show that these cliques form the blocks of $\Gamma$, or in other words that any cycle in $\Gamma$ is contained in a clique.

Supposing the contrary, let $C$ be a cycle of minimal order in $\Gamma$, not contained in a clique. The cycle $C$ carries two metrics: its metric $d_C$ as a cycle, and the metric $d_\Gamma$ induced by $\Gamma$. We claim these metrics coincide. In any case, $d_\Gamma \leq d_C$.

If the metrics disagree, let $u, v \in C$ be chosen at minimal distance such that $d_\Gamma(u, v) < d_C(u, v)$, and let $P = (u, \ldots, v)$ be a geodesic in $\Gamma$, and let $Q = (u, \ldots, v)$ be a geodesic in $C$. Let $v'$ be the first point of intersection of $P$ with $Q$, after $u$. Then $P \cup Q$ contains a cycle $C'$ smaller than $C$, and containing $u, v'$. Hence by hypothesis the vertices of $C'$ form a clique in $\Gamma$, and in particular $u, v'$ are adjacent in $\Gamma$.

If $u, v'$ are adjacent in $C$, then $d_C(v, v') = d_C(u, v) - 1$, $d_\Gamma(v, v') = d_\Gamma(u, v) - 1$, so $d_C(v, v') = d_\Gamma(v, v')$, and this contradicts the choice of $u, v$. So they are not adjacent, and the edge $(u, v')$ gives belongs to two cycles $C_1, C_2$ whose union contains $C$, both smaller than $C$. So the vertices of $C_1$ and $C_2$ are cliques. Since $C$ is not a clique, there are $u_1 \in C_1$ and $u_2 \in C_2$ nonadjacent in $\Gamma$. Then $d(u_1, u_2) = 2$, and there is a unique vertex adjacent to $u_1$ and $u_2$; but $u, v'$ are two such, a contradiction.

Thus the embedding of $C$ into $\Gamma$ respects the metric. In particular, $C$ is an induced subgraph of $\Gamma$.

Let $d$ be the diameter of $C$, so that the order of $C$ is either $2d$ or $2d + 1$. Fix a basepoint $\ast \in C$, and let $\Gamma_i = \Gamma_i(\ast)$.

For $v \in \Gamma_i$ with $i < d$, we claim that there is a unique geodesic $[\ast, v]$ in $\Gamma$. Otherwise, take $i < d$ minimal such that $\Gamma$ contains vertices $u, v$ at distance $i$ with two distinct geodesics $P, Q$ from $u$ to $v$. By the minimality, these geodesics are disjoint. Their union forms a cycle smaller than $C$, hence they form a clique. As they are geodesics, $i = 1$ and then in any case the geodesic is unique.

Let $v \in \Gamma_{d-1}$, and let $H = \Gamma_1(v)$, a copy of $m \cdot K_n$. Then $H$ contains a unique component meeting $\Gamma_{d-2}$. We claim that no other component of $H$ meets $\Gamma_{d-1}$.

Suppose on the contrary that $u_1, u_2$ are adjacent to $v$ with $u_1 \in \Gamma_{d-1}, u_2 \in \Gamma_{d-2}$, and $d(u_1, u_2) = 2$. Then taking $P_1, P_2$ to be the unique geodesics from $u_1$ or $u_2$ to $v$, $P_1 \cup P_2$ contains a cycle smaller than $C$, and containing the path $(u_1, v, u_2)$, hence not a clique. This is a contradiction.

Since $\Gamma_d$ contains at least one component of $H$, in particular there is an edge in $\Gamma_d$ whose vertices have a common neighbor in $\Gamma_{d-1}$. Using this, we eliminate the case $|C| = 2d + 1$ as follows.

If $|C| = 2d + 1$ then $C \cap \Gamma_d$ consists of two adjacent vertices $v_1, v_2$, whose other neighbors $u_1, u_2$ in $C$ lie in $\Gamma_{d-1}$. Furthermore $v_1, v_2$ have a common neighbor $u$ in $\Gamma_{d-1}$, and $u \neq u_1, u_2$. Let $P, P_1$ be the unique geodesics in $\Gamma$ connecting $\ast$ with $u, u_1$ respectively. Then $P \cup P_1 \cup \{v_1\}$ contains a cycle shorter than $C$, which contains the path $(u_1, v, u_1)$, and we have a contradiction.
Thus $|C| = 2d$, in other words the vertices $v \in \Gamma_d$ are connected to the basepoint $*$ by at least two distinct geodesics, and any two such geodesics will be disjoint.

Take $u \in \Gamma_{d-1}$, and $u' \in \Gamma_1$, with $d(u, u') = d - 2$. Take $v' \in \Gamma_1$ with $d(u', v') = 2$. We claim

$$d(u, v') = d$$

Otherwise, with $P, Q$ the geodesics from $u$ to $u'$ and $v'$ respectively, we have $|P \cup Q| \leq 2d - 1 < |C|$, and hence $(u', *, v')$ is contained in a cycle smaller than $C$, a contradiction since $u', v'$ are nonadjacent.

Thus $d(u, v') = d$, and the extension of the geodesic from $u$ to $v$ by the path $(u', *, v')$ gives a geodesic $Q$ from $u$ to $v'$. There is a second geodesic $Q'$ from $u$ to $v'$, disjoint from $Q$. Let $u_1$ be the unique neighbor of $u$ in $\Gamma_{d-1}$; this lies on $Q$. Let $u_1'$ be the neighbor of $u$ in $Q'$. As the cycle $Q \cup Q'$ satisfies the same condition as the cycle $C$, the metric on this cycle agrees with the metric in $\Gamma$, and in particular $d(u_1, u_1') = 2$. Hence $u_1' \in \Gamma_d$. Let $u_2'$ be the following neighbor of $u_1$ in $Q'$. Then $d(u, u_2') = 2$, with $u, u_2' \in \Gamma_{d-1}$.

Suppose $m \geq 3$. Then there is $u^* \in \Gamma_{d-2}$ adjacent to both $u$ and $u_2'$. But the only neighbor of $u$ in $\Gamma_{d-2}$ is $u_1$, so $u_1$ is adjacent to $u_2'$. In this case, $u$ and $u_2'$ have two distinct common neighbors, a contradiction. We conclude

$$m = 2$$

Fix an edge $u_1, u_2$ in $\Gamma_{d-1}$. For $i, j = 1, 2$ in some order, set

$$H_{ij} = \{v \in \Gamma : d(v, u_i) = 1, d(v, u_j) = 21\}$$

We claim $H_{ij} \subseteq \Gamma_d$.

As $u_1, u_2$ have the same unique neighbor in $\Gamma_{d-2}$, we have

$$(H_{12} \cup H_{21}) \cap \Gamma_{d-2} = \emptyset$$

Now suppose $v \in H_{12} \cap \Gamma_{d-1}$. Then $d(v, u_2) = 2$ and $v, u_2 \in \Gamma_{d-1}$. Now since $|C| = 2d$, there is $w \in \Gamma_d$ adjacent to $v$ and $u_2$. Then $u_2, v$ have the two common neighbors $w$ and $u_1$, a contradiction. So $H_{12}, H_{21} \subseteq \Gamma_d$.

Let $v_1 \in H_{12}, v_2 \in H_{21}$. We claim that

$$d(v_1, v_2) = 3$$

Otherwise, there is a cycle of length at most 5, not contained in a clique. As $|C|$ is even, it follows that $C$ is a 4-cycle, in other words vertices at distance 2 have at least two common neighbors, a contradiction. Thus for any other choice $v_1' \in H_{12}, v_2' \in H_{21}$, we have $(*, u_1, u_2, v_1, v_2)$ isometric with $(*, u_1, u_2, v_1', v_2')$.

There is a unique element $u \in \Gamma_1$ at distance $d - 2$ from $u_1$ and $u_2$; namely, the element at distance $d - 3$ from their common neighbor in $\Gamma_{d-2}$. On the other hand, for $v \in \Gamma_d$, if $I_v$ is the set $\{v' \in \Gamma_1 : d(v, v') = d - 1\}$, then by our hypotheses, $I_v$ is a pair of representatives for the two components of $\Gamma_1$. And if $v \in H_{12}$ or $H_{21}$, one of these representatives will be $u$. Let $B$ be the component of $\Gamma_1$ not containing $u$. Then the distance from $u_1$ or $u_2$ to a vertex of $B$ is $d$. It follows that all vertices of $B$ will occur as the second vertex of $I_v$ for some $v_1 \in H_{12}$ and for some $v_2 \in H_{21}$. Therefore, we may choose pairs $(v_1, v_2)$ and $(v_1', v_1')$ with $v_1, v_1' \in H_{12}, v_2, v_2' \in H_{21}$, and $I_{v_1} = I_{v_2}$, while $I_{v_1'} \neq I_{v_2'}$. But as $(*, u_1, u_2, v_1, v_2)$ and $(*, u_1, u_2, v_1', v_2')$ are isometric, this contradicts homogeneity.

At this point the proof of Theorem 15 is complete. By the Lachlan-Woodrow classification (or by [Gar76]), in the finite primitive case $\Gamma_1$ is either an independent
set, a clique, a 5-cycle, or the line graph of \( K_{3,3} \), while in the imprimitive case we have \( G = K_m[I_n] \) or \( m \cdot I_n \), with \( 2 \leq m, n \leq \infty \). These cases have all been treated above.

6.3. The structure of \( \Gamma_1 \) in the generic case. We may continue on to say more about the possibilities for \( \Gamma_1 \) when \( \Gamma_1 \) is infinite and primitive. We have the difficult case in which \( \Gamma_1 \) is an infinite independent set, and the trivial case in which \( \Gamma_1 \) is a clique, and \( \Gamma \) is complete. Under the Lachlan/Woodrow classification, there are only three other cases to be considered:

1. \( \Gamma_1 \) is generic containing no independent set of size \( n \), with \( 2 \leq n < \infty \);
2. \( \Gamma_1 \) is generic containing no clique of size \( n \), with \( 2 \leq n < \infty \);
3. \( \Gamma_1 \) is generic (Rado’s graph).

The last two of the three possibilities occur naturally in metrically homogeneous graphs of all diameters, and should not be considered as exceptional. However the first of these possibilities does not occur.

Lemma 6.14. Let \( \Gamma \) be a metrically homogeneous graph of diameter at least 3 with \( \Gamma_1 \) infinite and primitive. Then \( \Gamma_1 \) contains an infinite independent set.

Proof. Suppose the contrary. Evidently \( \Gamma_1 \) contains an independent pair. By the Lachlan/Woodrow classification, if \( n \) is minimal such that \( \Gamma_1 \) contains no independent set of order \( n \), then for any finite graph \( G \) which contains no independent set of order \( n \), \( G \) occurs as an induced subgraph of \( \Gamma_1 \).

We consider a certain amalgamation diagram involving subspaces of \( \Gamma \). Let \( A \) be the metric space with three points \( a, b, x \) constituting a geodesic, with \( d(a, b) = 2 \), \( d(b, x) = 1 \), \( d(a, x) = 3 \). Let \( B \) be the metric space on the points \( a, b \) and a further set \( Y \) of order \( n - 1 \) with the metric given by

\[
\begin{align*}
&d(a, y) = 1, y \in Y \\
&d(y, y') = 2, y, y' \in Y \text{ distinct}
\end{align*}
\]

As the diameter of \( \Gamma \) is at least 3, the geodesic \( A \) certainly occurs as a subspace of \( \Gamma \). On the other hand, the metric space \( B \) embeds into \( \Gamma_1 \), and hence into \( \Gamma \). Therefore there is some amalgam \( G = A \cup B \) embedding into \( \Gamma \) as well.

Now for \( y \in Y \), the structure of \( (a, x, y) \) forces \( d(x, y) \geq 2 \). On the other hand, the element \( b \) forces \( d(x, y) \leq 2 \). Thus in \( G \), the set \( Y \cup \{x\} \) is an independent set of order \( n \). Furthermore this set is contained in \( \Gamma_1(b) \), so we arrive at a contradiction. □

We may sum up this section by repeating the statement of Theorem 9.

Theorem. Let \( \Gamma \) be a connected metrically homogeneous graph. Then one of the following occurs.

1. \( \Gamma \) has diameter at most 2, and is homogeneous as a graph (cf. §3);
2. \( \Gamma \) has degree 2, and is an \( n \)-cycle for some \( n \);
3. \( \Gamma_1 \) is finite or imprimitive, and \( \Gamma \) is one of the following.
   (a) Antipodal of diameter 3, and the result of doubling \( C_5 \), \( E(K_{3,3}) \), or a finite independent set;
   (b) A tree-like graph \( T_{r,s} \) as described by Macpherson.
4. \( \Gamma \) is of generic type, meaning that \( \Gamma_1 \) is one of the following three types:
   (a) An infinite independent set; and furthermore, for \( u \in \Gamma_2 \), the set of neighbors of \( u \) in \( \Gamma_1 \) is infinite;
(b) Generic omitting a complete graph $K_n$, for some $n \geq 3$ finite;
(c) Homogeneous universal (Rado’s graph).

7. Exceptional Bipartite Metrically Homogeneous Graphs

7.1. The Bipartite Case with $B\Gamma$ exceptional. If $\Gamma$ is a connected bipartite metrically homogeneous graph, then we consider the graph $B\Gamma$ induced on each half of the bipartition by the edge relation “$d(x, y) = 2$”, as described in §4. In addition to the exceptional graphs considered in the previous section, we wish to consider those for which $B\Gamma$ is itself exceptional in the sense of the previous section. The result in that case will be as follows.

**Theorem** (10). Let $\Gamma$ be a connected, bipartite, and metrically homogeneous graph, of diameter at least 3, and degree at least 3. Then one of the following occurs, writing $B\Gamma_1$ for $(B\Gamma)_1$.

1. $\Gamma$ is a tree;
2. $\Gamma$ has diameter 3, and $\Gamma$ is either the complement of a perfect matching, or a generic bipartite graph;
3. $\Gamma$ has diameter 4, $B\Gamma \cong K_\infty[I_2]$, and $\Gamma$ is a double cover of a generic bipartite graph, described in Lemma 7.2.
4. $\Gamma$ has diameter 4, and $B\Gamma_1$ is generic omitting an independent set of order $n + 1$, for some $n \geq 2$. With $n$ fixed, $\Gamma$ is determined up to isomorphism.
5. $\Gamma$ has diameter 5 and is antipodal; $B\Gamma_1$ is generic omitting an independent set of order 3. $\Gamma$ is determined up to isomorphism.
6. $B\Gamma_1$ is a homogeneous universal graph (Rado’s graph).

Each of these possibilities occurs.

If $\Gamma_1$ is finite then this was already covered in §6. So we may suppose $\Gamma_1 \cong I_\infty$. This means that $B\Gamma$ contains an infinite clique. In particular, $B\Gamma_1$ contains an infinite clique. Furthermore, it follows from the connectedness and homogeneity of $\Gamma$ that $B\Gamma$ is connected.

By the Lachlan/Woodrow classification this already reduces the possibilities to the following.

1. $B\Gamma_1$ is imprimitive of the form $m \cdot K_\infty$ or $K_\infty[I_m]$, with $1 \leq m \leq \infty$;
2. $B\Gamma_1$ is generic omitting $I_n$, for some finite $n \geq 2$;
3. $B\Gamma_1$ is generic.

We consider the various possibilities for $B\Gamma_1$, beginning with an infinite clique.

**Lemma 7.1.** Let $\Gamma$ be a connected, bipartite, and metrically homogeneous graph, of diameter $\delta \geq 3$. Suppose that $B\Gamma_1$ is an infinite clique. Then $\delta = 3$ and $\Gamma$ is either the complement of a perfect matching, or a generic bipartite graph (the Fraïssé limit of the class of finite bipartite graphs, in the category of graphs with a given bipartition).

**Proof.** As $B\Gamma_1$ is complete, $B\Gamma$ is also complete. Hence $\delta \leq 3$, and so $\delta = 3$.

Let $\Gamma'$ be the graph obtained from $\Gamma$ by switching edges and non-edges between the two halves of $\Gamma$. If $\Gamma'$ is disconnected, then by homogeneity of $\Gamma$, the relation “$d(x, y) = 3$” defines a bijection between the halves, and $\Gamma$ is the complement of a perfect matching. So suppose that $\Gamma'$ is connected. Then the graph metric on $\Gamma'$ results from the metric on $\Gamma$ by interchanging distance 1 and 3.
Any isometry $A \cong B$ in $\Gamma'$ corresponds to an isometry in $\Gamma$ and is therefore induced by an automorphism of $\Gamma$. These automorphisms preserve the partition of $\Gamma$, possibly switching the two sides. So they act as automorphisms of $\Gamma'$ as well. It follows that $\Gamma'$ is a metrically homogeneous graph.

If $\Gamma$ or $\Gamma'$ has bounded degree, we contradict Theorem 15. Thus if $A, B$ are the two halves of the partition of $\Gamma$, for $u \in A$ the set $B_u$ of neighbors of $u$ in $B$ is infinite, with infinite complement. In particular for any two disjoint finite subsets $B_1, B_2$ of $B$, there is $u \in A$ such that $u$ is adjacent to all vertices of $B_1$ and no vertices of $B_2$. This is the extension property which characterizes the Fräïssé limit of the class of finite bipartite graphs, so $\Gamma$ is generic bipartite.

Now suppose $\Gamma_1$ is finite and imprimitive.

By Theorem 9, and bearing in mind that $\Gamma_1$ is connected and contains $K_\infty$, we have the following alternatives.

1. $\Gamma_1$ is a graph of the form $K_\infty[I_m]$ with $2 \leq m \leq \infty$.
2. $\Gamma_1$ is a tree-like graph $T_{r,\infty}$ with $2 \leq r \leq \infty$.

We will consider each of these cases separately.

**Lemma 7.2.** Let $\Gamma$ be a connected, bipartite, and metrically homogeneous graph. Suppose that $\Gamma_1 \cong K_\infty[I_m]$ with $2 \leq m \leq \infty$. Then $m = 2$, and $\Gamma$ is obtained from a generic bipartite graph $\Gamma$ with partition $(A, B)$ as follows. Replace each vertex $v$ of $\Gamma$ by a pair $v_1, v_2$; and for each pair $u \in A, v \in B$, introduce a perfect matching by the rule

$$
\begin{cases}
   u_1 \leftrightarrow v_1, u_2 \leftrightarrow v_2 & \text{if } (u, v) \text{ is an edge} \\
   u_1 \leftrightarrow v_2, u_2 \leftrightarrow v_1 & \text{otherwise}
\end{cases}
$$

**Proof.** Let $A, B$ be the two halves of $\Gamma$, each isomorphic to $K_\infty[I_m]$ for some $m \leq \infty$, with induced metric the double of the graph metric.

We will refer to a maximal independent set in $A$ or $B$ as a component. In all cases $\Gamma$ will be the graph one the components induced by the edge relation of $\Gamma$, that is: there is an edge between two components if and only if there is at least one edge between them in $\Gamma$.

For $u \in A, v \in B$, the neighbors of $v$ in $A$ are all at distance 2 from one another, so at most one of them lies in the component of $u$ in $A$. Thus there is $v'$ adjacent to $v$ with $d(u, v') = 2$, and the diameter is 4.

Take $u \in A, v \in B$ adjacent, and let $A_u, B_v$ be their respective components. Then both distance 1 and distance 3 occur between pairs in $A_u$ and $B_v$. It follows by homogeneity that the same applies to any pair of components from $A$ and $B$, and furthermore, that whenever $u \in A, v \in B$ lie at distance 3, there is some $v'$ adjacent to $u$, lying in the component $B_v$ of $v$. In other words, between any two components of $A, B$, the graph $\Gamma$ induces a perfect matching.

In particular if $A_1, A_2$ are two components of $A$, and $v \in B$, then there is a $v$-definable bijection of $A_1$ with $A_2$, the composition of the two perfect matchings with $B_v$. Then considering the metric structure of $A_1 \cup A_2 \cup \{v\}$, it follows that $m = 2$.

Now we need to see that $\Gamma$ is universal homogeneous for this category, in other words that it satisfies the extension property: for any finite subgraph $G_0$ of $\Gamma$, and any extension of $G_0$ by an additional vertex $v$ to a graph $G$ satisfying the same conditions, the extension is realized by a vertex of $\Gamma$. 

\[\square\]
Let \( X = G \cap A, Y = G \cap B \). We may suppose both are nonempty, and that \( v \) is on the \( B \) side in the sense that its distances from vertices of \( X \) are odd. There is a canonical extension of the pair \((G_0, G)\) to a pair \( G'_0, G' \) for which \( X' = G'_0 \cap A \) is a union of components of \( A \), and \( G'_0 \) embeds in \( \Gamma \), so we may suppose \( X \) is itself a union of components of \( A \).

If there is a vertex \( y \in Y \) for which we require \( d(v, y) = 4 \), then there is no such vertex in \( G_0 \), and the structure of \( G \) is entirely determined by this condition, and is realized in \( \Gamma \).

So we may suppose that \( d(v, y) = 2 \) for all \( y \in Y \). Now the neighbors of \( v \) in \( X \) form a set of representatives \( X_0 \) for the components in \( X \), and it suffices to see that there are infinitely many vertices \( v \in \Gamma \) adjacent to the vertices of \( X_0 \).

By homogeneity, this follows from the fact that any \( v \) in \( \Gamma \) has infinitely many neighbors in \( A \).

\[ \square \]

**Lemma 7.3.** Let \( \Gamma \) be a connected, bipartite, and metrically homogeneous graph. Suppose that \( B \Gamma \cong T_{r, \infty} \) with \( 2 \leq r \leq \infty \). Then \( \Gamma \) is an infinitely branching tree.

**Proof.** Let \( A, B \) be the two halves of \( \Gamma \), and identify \( A \) with \( B \Gamma \). In particular, for \( u \in B \), the neighbors of \( u \) in \( A \) form a clique in the sense of \( B \Gamma \).

Suppose that vertices \( u, u' \) in \( B \) are adjacent to two points \( v_1, v_2 \) of \( A \). These points are contained in a unique clique of \( A \), so all the neighbors of \( u, u' \) lie in a common clique. Therefore this gives us an equivalence relation on \( B \), with \( u, u' \) equivalent just in case they have two common neighbors in \( A \). But the graph structure on \( B \) is also that of \( B \Gamma \), which is primitive, so this relation is trivial, and distinct vertices of \( B \) correspond to distinct cliques in \( A \). In particular, \( r = \infty \).

At the same time, every edge in \( B \) lies in the clique associated with some vertex in \( B \), so the neighbors of the vertices of \( B \) are exactly the maximal cliques of \( A \). Evidently the edge relation in \( B \) corresponds to intersection of cliques in \( A \). Thus \( B \) is identified with the “dual” of \( A \) with vertices corresponding to maximal cliques, and maximal cliques corresponding to vertices. At this point the structure of \( \Gamma \) has been recovered uniquely from the structure of \( T_{\infty, \infty} \), and must therefore be the infinitely branching tree. \( \square \)

What are the remaining possibilities for \( B \Gamma_1 \)? We still have the following possibilities, bearing in mind that \( B \Gamma \) contains an infinite complete subgraph.

1. \( B \Gamma \) is an infinite primitive homogeneous graph, containing no infinite independent set.
2. \( B \Gamma_1 \) is a generic graph (Rado’s graph).

We separate these two cases, as the diameter of \( \Gamma \) is at most 5 in the first case, and there are no direct analogs having larger diameter. We will narrow down this first case further. It will be convenient to have a notation for this which can be applied directly in \( \Gamma \), so let \( I_m^d \) represent a metric space on \( m \) points with constant distances \( d \), and call this a \( d \)-independent set; we are interested in the case \( d = 4 \), corresponding to \( d' = 2 \) in \( B \Gamma \).

**7.2. Bipartite graphs with \( B \Gamma \) an infinite primitive, omitting \( I_\infty \).** In this case, the diameter of \( \Gamma \) is 4 or 5. We take up the case of diameter 4 first.

**Lemma 7.4.** For each \( n \geq 2 \), there is a unique metrically homogeneous bipartite graph \( \Gamma \) of diameter 4 such that \( B \Gamma \) is the generic homogeneous graph omitting \( I_{n+1}^{(4)} \) (corresponding to \( I_{n+1}^{(4)} \) in \( \Gamma \)).
Proof. We have to deal with both existence and uniqueness. For existence, we use the Fraïssé theory. Let $\mathcal{A}$ be the class of finite metric spaces of diameter at most 4, omitting $I_{n+1}^{(4)}$, with $n \geq 2$, and containing no triangles of odd perimeter. We must show that this class is an amalgamation class, which reduces to the case of two-vertex amalgamations; that is, we consider structures $A_0, A_1, A_2$ in $\mathcal{A}$ with $A_i = A_0 \cup \{a_i\}$ for $i = 1, 2$, and we consider how to amalgamate these to get a graph on $A_1 \cup A_2$ extending $A_0$ and lying in $\mathcal{A}$. Once we have an amalgamation class, there is an associated homogeneous metric space, and since geodesics of length 4 are found in $\mathcal{A}$, the metric coincides with the graph metric for the edge relation $d(x, y) = 1$.

So consider a two-vertex amalgamation problem $(A_0, A_1, A_2)$; we need only determine $d(a_1, a_2)$ so as to have a metric compatible with our conditions. We may suppose $A_0 \neq \emptyset$ and therefore the parity of the distance is the same as that of $d(a, a_1) + d(a, a_2)$ for any $a \in A_0$. We will take $d(a_1, a_2)$ to be 2 if that parity is even, and 3 otherwise. It remains only to verify the triangle inequality for triangles $(a, a_1, a_2)$. One may inspect all the cases, but given that the distance in question is 2 or 3, this really comes down to the fact that forbidden “triangles” with distances $(1, 4, 2)$ or $(1, 1, 3)$ are ruled out by the parity constraint.

This establishes the existence of metrically homogeneous graphs of this type. We turn to uniqueness. Let $\Delta, \Delta'$ be the two halves of $\Gamma$, each isomorphic to $2B\Gamma$ (that is, isomorphic to $B\Gamma$ after rescaling the metric). We know the finite subspaces of $2B\Gamma$. Our claim is that for any finite $A, B \subseteq 2B\Gamma$ and any metric on the disjoint union $A \cup B$ in which all cross-distances between $A$ and $B$ are equal to either 1 or 3, there is an embedding of $A$ into $\Delta$ and $B$ into $\Delta'$ such that the metric of $\Gamma$ induces the specified metric on $A \cup B$. We will prove this by induction on the order of $B$.

Let $|B| = k$; we may suppose $k > 0$. We now proceed by induction on the number of pairs $u_1, u_2$ in $A$ with

$$d(u_1, u_2) = 4$$

Suppose there is such a pair, and fix one such $(u_1, u_2)$. Take $v \in B$. We may suppose $d(v, u_1) = 3$. We adjoin a vertex $a$ to $A$ as follows:

$$d(a, u_1) = 4, \quad d(a, u') = 2 \text{ for } u \in A, u' \neq u_1$$
$$d(a, v) = 1, \quad d(a, v') = 3 \text{ for } v \in A, v' \neq v$$

Now the configuration $(A \cup \{a\}, B \setminus \{v\})$ embeds into $\Gamma$ by induction on $|B|$, and the configuration $(A \setminus \{u_1\} \cup \{a\}, B)$ embeds into $\Gamma$ by induction on the number of pairs in $A$ at distance 4. So there is an amalgam $(A \cup \{a\}, B)$ of these two configurations embedding into $\Gamma$, and the metric on this amalgam agrees with the given metric on $(A, B)$ except possibly at the pair $(u_1, v)$. But we have $d(u_1, a) = 4, d(v, a) = 1$, so $d(u_1, v) \geq 3$, $d(u_1, v)$ is odd, and $d(u_1, v) \leq 4$. Thus $d(u_1, v) = 3$ also in the amalgam.

There remains the case in which there is no such pair, that is $A$ is an independent set in $\Gamma$ of the form $I_n^{(2)}$. In this case, let us first extend $(A, B)$ to a finite configuration $(A, B_1)$ with the following properties.

(1) Some vertex $b \in B_1$ is adjacent to all vertices of $A$.
(2) No two vertices of $A$ have the same neighbors in $B_1$. 

Now consider the configurations \((\{a\}, B_1)\) for \(a \in A\). If they all embed into \(\Gamma\), then some amalgam does as well, and this amalgam must be isomorphic to \((A, B_1)\) since the vertices of \(A\) must remain distinct and the metric is then determined. So it suffices to check that these configurations \((\{a\}, B_1)\) embed into \(\Gamma\).

However by symmetry we may equally well take \((B_1, \{a\})\), and hence conclude by induction unless \(k = 1\). So suppose \(|B| = 1\); by our first reduction, we may also suppose that \(A \cong I_n^{(2)}\) for some \(m\).

Take a basepoint \(*\) in \(\Gamma\) and a vertex \(u\) in \(\Gamma_1\). It suffices to show that the set \(I_u\) of neighbors of \(u\) in \(\Gamma_1\) is an infinite and co-finite subset of \(\Gamma_1\). By Theorem 9, the set \(I_u\) is infinite. Furthermore there are by assumption vertices \(u_1, u_2\) in \(\Gamma\) with \(d(u_1, u_2) = 4\), and we may suppose that the basepoint \(*\) lies at distance 2 from both. Then \(u_1, u_2 \in \Gamma_2\) and \(I_{u_1}, I_{u_2}\) are disjoint. Thus they are co-finite. This completes the uniqueness proof. \(\square\)

Now we turn to diameter 5. The claim in this case is as follows.

**Proposition 7.5.** Let \(\Gamma\) be a metrically homogeneous bipartite graph of diameter 5, and suppose that \(B\Gamma_1\) omits \(I_5^{(2)}\). Then \(\Gamma\) is antipodal, and as it is not a cycle, it is therefore the second graph described in Corollary 5.14.

**Lemma 7.6.** Let \(\Gamma\) be a metrically homogeneous bipartite graph of diameter 5, omitting \(I_3^{(4)}\). Then \(\Gamma\) is antipodal.

**Proof.** Suppose \(|\Gamma_5| \geq 2\). We will show first that there is a triple \((a, b, c)\) in \(\Gamma\) with \(d(a, b) = 4\), \(d(a, c) = 5\), and \(d(b, c) > 1\).

Take a pair \(u_1, u_2 \in \Gamma_5\). Then \(d(u_1, u_2) = 2\) or 4, and if it is 4 then our triple \((a, b, c)\) can be \((u_1, u_2, *)\) with \(*\) the chosen basepoint. If \(d(u_1, u_2) = 2\) then extend \(u_1, u_2\) to a geodesic \((u_1, u_2, u_3)\) with \(d(u_2, u_3) = 1\), \(d(u_1, u_3) = 3\). As \(d(u_2, u_3) = 1\) we find \(u_3 \in \Gamma_4\) and therefore the triple \((*, u_3, u_1)\) will do.

Now fix a triple \((a, b, c)\) with \(d(a, b) = 4\), \(d(a, c) = 5\), and \(d(b, c) = 3\) or 5. Take a triple \((b, c, d)\) with \(d(c, d) = 1\) and \(d(b, d) = 4\); this will be a geodesic of length 4 or 5, and therefore exists in \(\Gamma\) by homogeneity.

Now \(d(a, b) = d(b, d) = 4\), and consideration of the path \((a, c, d)\) shows that \(d(a, d) \geq 4\), and \(d(a, b)\) is even, so \(d(a, d) = 4\) as well, and we have \(I_3^{(4)}\) in \(\Gamma\), a contradiction. \(\square\)

It remains to show that in an infinite metrically homogeneous graph \(\Gamma\) of diameter 5 for which \(B\Gamma_1\) contains an independent set of order 3, \(B\Gamma_1\) contains arbitrarily large independent sets. We will subdivide this case according to the structure of \(\Gamma_5\). Since in this case \(\Gamma\) is not antipodal, it follows that \(\Gamma_5\) is infinite.

**Lemma 7.7.** Suppose that \(\Gamma\) is bipartite of diameter 5, and \(\Gamma_5 = I_3^{(4)}\). Then \(B\Gamma\) is the universal homogeneous graph (Rado’s graph).

**Proof.** Our claim is that \(I_n^{(4)}\) embeds into \(\Gamma\) for all \(n\). We proceed by induction. As \(\Gamma\) is not antipodal, we have the result for \(n = 3\).

Suppose \(I_n^{(4)}\) embeds into \(\Gamma\), with \(n \geq 3\). Let \(I \cong I_{n-1}^{(4)}\) be a metric subspace of \(\Gamma\). We aim to embed subspaces \(A = I \cup \{a\}\) and \(B = I \cup \{b, u\}\) into \(\Gamma\), with \(I \cup \{a\} \cong I \cup \{b\} \cong I_n^{(4)}\), and with \(u\) chosen so that

\[
\begin{align*}
    d(u, a) &= 1 & d(u, b) &= 5 \\
    d(u, x) &= 3 & (x \in I)
\end{align*}
\]
Supposing we have this, considering \((a, u, b)\) we see that \(d(a, b) \geq 4\) and hence \(I \cup \{a, b\} \cong I_{n+1}^{(4)}\).

We treat the second factor \(I \cup \{b, u\}\) first. Consider the metric space \(I \cup \{b, b'\}\) in which \(b'\) lies at distance 2 from each point of \(I \cup \{b\}\). The corresponding configuration in \(B I\) is a point \(b'\) adjacent to an independent set of order \(n\), and this we have in \(B I\). Thus the space \(I \cup \{b, b'\}\) embeds into \(I\).

By hypothesis, there is also a triple \((u, b, b')\) with \(b, b' \in \Gamma_5(u)\). Amalgamate \(I \cup \{b, b'\}\) with \((u, b, b')\) over \(b, b'\). For \(x \in I\), considering \((u, b', x)\), we see that \(d(u, x) \geq 3\), and that \(d(u, x)\) is odd. Our hypothesis on \(\Gamma_5\) implies that \(d(u, x) \neq 5\), so \(d(u, x) = 3\) for all \(x \in I\). Thus \(I \cup \{b, u\}\) is as desired.

The construction of the factor \(I \cup \{a, u\}\) is more elaborate. Consider the metric spaces \(A = I \cup \{a\} \cup J\) and \(B = J \cup \{a, u\}\), where \(J \cong I_{n}^{(2)}\), and \(J\) may be labeled as \(\{v^* : v \in I\}\) in such a way that

\[
\begin{align*}
    d(a, v^*) &= d(v, v^*) = 4 \quad (v \in I) \\
    d(v, w^*) &= 2 \quad v, w \in I \text{ distinct} \\
    d(u, v) &= 5 \quad (v \in J)
\end{align*}
\]

Supposing that \(A\) and \(B\) embed into \(I\), take their amalgam over \(J \cup \{a\}\). Then for \(v \in I\) the triple \((u, a, v)\) shows that \(d(u, v) \geq 3\), and the distance is odd, while the triple \((u, v', v)\) and the hypothesis on \(\Gamma_5\) shows that this distance is not 5. Thus the space \(I \cup \{a, u\}\) will have the desired metric. It remains to construct \(A\) and \(B\).

Consider \(B = J \cup \{a, u\}\). The graph \(I\) contains an edge \((a, u)\) as well as a copy of \(J \cup \{u\}\), the latter by the hypothesis on \(\Gamma_5\). Furthermore, in any amalgam of \((a, u)\) with \(J \cup \{u\}\), the only possible value for the distance \(d(a, v)\), for \(v \in J\), is 4. So this disposes of \(B\).

Now consider \(A = I \cup J \cup \{a\}\), in which all distances are even. So we need to look for the rescaled graph \((1/2)A\) in \(B I\). It suffices to check that the maximal independent sets of vertices in \((1/2)A\) have order at most \(n\). This is the case for \(I \cup \{a\}\), and any independent set meeting \(J\) would have order at most 3. Since \(n \geq 3\), we are done. \(\Box\)

**Lemma 7.8.** Suppose that \(\Gamma\) is bipartite, metrically homogeneous, not antipodal, and of diameter 5. Then \(\Gamma_5\) contains a subspace of the form \(I_{\infty}^{(2)}\); in other words, \((1/2)\Gamma_5\) contains an infinite clique.

**Proof.** Supposing the contrary, for each \(u \in \Gamma_4\), the set \(I_u\) of neighbors of \(u\) in \(\Gamma_5\) is finite and nonempty, of fixed order \(k\). Since any subset of \(\Gamma_5\) isomorphic to \(I_{n+1}^{(2)}\) would have a common neighbor \(u \in \Gamma_4\), it follows that the \(I_u\) represent maximal cliques of \((1/2)\Gamma_5\).

As \(B I\) is either generic omitting \(I_{n+1}\) for some \(n \geq 3\), or universal homogeneous, it follows that \(\Gamma_4\) is primitive and contains both edges and nonedges. Now the map \(u \rightarrow I_u\) induces an equivalence relation on \(\Gamma_4\) which can only be equality, that is the map is a bijection. Since \(\Gamma_4\) contains both edges and non-edges, it follows that \(\Gamma_5\) is primitive. As each vertex \(v \in \Gamma_5\) has infinitely many neighbors in \(\Gamma_4\), we have \(|I_u| > 1\) for \(u \in \Gamma_4\). On the other hand if \(|I_u| \geq 3\) then for \(u, u' \in \Gamma_4\) we have the possibilities \(|I_u \cap I_{u'}| = 0, 1, 2\) while there are only two distances occurring in \(\Gamma_4\). So \(|I_u| = 2\). That is, \((1/2)\Gamma_5\) is generic triangle-free, and the vertices of \(\Gamma_4\) correspond to edges of \((1/2)\Gamma_5\). It follows that vertices of \(\Gamma_4\) lie at distance two iff the corresponding edges meet, that is \((1/2)\Gamma_4\) is the line graph of \((1/2)\Gamma_5\). But
there are pairs of vertices in the latter graph at distance greater than 2, so \((1/2)\Gamma_4\) is not homogeneous, and we have a contradiction.

Lemma 7.9. Suppose that \(\Gamma\) is bipartite, metrically homogeneous, and of diameter \(5\), and \(\Gamma_5\) contains a pair of vertices at distance \(4\). Then the relation 

\[
\text{“}d(x, y) = 0 \text{ or } 4\text{”}
\]

is not an equivalence relation on \(\Gamma_5\).

Proof. Supposing the contrary, we have

\[
\Gamma_5 \cong I_\infty^{(2)}[I_k^{(4)}]
\]

for some \(k\) with \(2 \leq k \leq \infty\).

Suppose first \(k \geq 3\). Fix two equivalence classes \(C, C'\) in \(\Gamma_5\), and choose a triple \(u_1, u_2, u_3\) in \(C\) and a vertex \(u_1'\) in \(C'\). Choose \(v \in \Gamma\) with \(d(v, u_1) = d(v, u_1') = 1\), and let \(d_i = d(v, u_i)\) for \(i = 2, 3\). We may then choose \(u_2', u_3'\) in \(C'\) so that \(d(v, u_2') = d(v, u_3') = d_i\) for \(i = 2, 3\).

Now the permutation of the \(u_i, u_i'\) which switches \(u_i'\) and \(u_i'\) and fixes the other elements is an isometry, so there is an element \(v'\) with \(d(v', u) = d(v, u)\) for \(u = u_1, u_2, u_3, u_3'\), but with \(d(v', u_2') = d_2, d(v', u_3') = 1\).

As \(u_1\) is adjacent to \(v, v'\) we have \(d(v, v') = 2\). Now \(u_3, v, v'\) is isometric with \(u_3', v, v'\), and the equivalence class of \(u_3\) contains a common neighbor of \(v, v'\); therefore the equivalence class of \(u_3'\) contains a common neighbor of \(v, v'\). But \(v\) can have at most one neighbor in an equivalence class, so this contradicts the choice of \(v'\).

So we are left with the case \(k = 2\):

\[
\Gamma_5 \cong I_\infty^{(2)}[I_2^{(4)}]
\]

In this case we will consider a specific amalgamation.

Let \(\gamma = (u, v, w)\) be a geodesic with

\[
d(u, v) = 1; d(v, w) = 4; d(u, w) = 5
\]

Let \(A = \gamma \cup \{a\}, B = \gamma \cup \{b\}\), with the metrics given by

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If \(A, B\) embed into \(\Gamma\), then their relation to \(v\) prevents them from being identified in the amalgam. However \(a, b, u \in \Gamma_5(w)\) and \(d(a, u) = d(b, u) = 4\). So \(d(a, b) = 4\) by our assumption, and this contradicts \(k = 2\).

Lemma 7.10. Suppose that \(\Gamma\) is bipartite, metrically homogeneous, and of diameter \(5\). Then \(\Gamma_i\) is connected with respect to the edge relation given by \(d(x, y) = 2\), for \(1 \leq i \leq 5\).

Proof. This is true for \(\Gamma_1\) automatically. It is true for \(i = 2\) or \(4\) in view of the structure of \(B\Gamma\). It remains to prove it for \(i = 3\) or \(5\).

If \(\Gamma_i\) is disconnected with respect to this relation, then for \(u \in \Gamma_{i-1}\), the set \(I_u\) of neighbors of \(u\) in \(\Gamma_i\) is contained in one of the equivalence classes of \(\Gamma_i\), and there is more than one such class. Thus we have a function from \(\Gamma_{i-1}\) to the quotient of \(\Gamma_i\). As \(i - 1\) is even, in view of the structure of \(B\Gamma\) we know \(\Gamma_{i-1}\) is primitive, so as \(\Gamma_i\) contains more than one equivalence class, this function is \(1 - 1\). Then the
sets $I_u$ for $u \in \Gamma_{i-1}$ must be exactly the equivalence classes of $\Gamma_i$, and $\Gamma_{i-1}$ is in bijection with the quotient. In particular, only one distance occurs in $\Gamma_{i-1}$. But in view of the structure of $B\Gamma_i$, this is not the case. \hfill \Box

**Corollary 7.11.** Suppose that $\Gamma$ is bipartite, metrically homogeneous, and of diameter 5, and $\Gamma_5$ contains a pair of vertices at distance 4. Then $\Gamma_5$ is primitive, infinite, and contains a copy of $I_{5}^{(2)}$. 

**Lemma 7.12.** Suppose that $\Gamma$ is bipartite, metrically homogeneous, and of diameter 5, that $\Gamma_5$ contains a pair of vertices at distance 4, and that $B\Gamma_i$ is generic omitting $I_{n+1}$. Then $I_{n-1}$ embeds into $\Gamma_5$.

**Proof.** Let $k$ be maximal so that $I_k^{(4)}$ embeds into $\Gamma_5$, and suppose $k \leq n - 2$.

Let $I \cong I_n^{(4)}$, and suppose $a, b, u$ are additional vertices with $I \cup \{a\} \cong I \cup \{b\} \cong I_n^{(4)}$, and with $d(u, a) = 1$, $d(u, b) = 5$. If $I \cup \{a, u\}$ and $I \cup \{b, u\}$ embed into $\Gamma$, then so does an amalgam $I \cup \{a, b, u\}$, and the auxiliary vertex $u$ forces $d(a, b) = 4$, and $I \cup \{a, b\} \cong I_n^{(4)}$, a contradiction. So it suffices to embed $I \cup \{a, u\}$ and $I \cup \{b, u\}$ into $\Gamma$.

**Construction of $I \cup \{a, u\}$.**

Introduce a metric space $J = \bigcup_{v \in I} J_v$ with $J_v \cong I_k^{(4)}$ and with $d(x, y) = 2$ for $x \in J_v$, $y \in J \setminus J_v$. Extend to a metric on $I \cup J$ by taking

\[
d(v, x) = \begin{cases} 4 & \text{if } x \in J_v \\ 2 & \text{if } x \in J \setminus J_v \end{cases}
\]

for $v \in I$.

Give $J \cup \{a, u\}$ the metric with $d(a, x) = 4$, $d(u, x) = 5$ for $x \in J$. We claim that $I \cup J \cup \{a\}$ and $I \cup \{a, u\}$ embed into $\Gamma$. Now $B\Gamma$ is generic omitting $I_{n+1}$, and $(1/2)G_5$ is generic omitting $I_{k+1}$. Since the space $I \cup J \cup \{a\}$ does not contain $I_n^{(4)}$, and all its distances are even, it embeds into $\Gamma$. Since $J$ does not contain $I_n^{(4)}$, it embeds into $\Gamma_5$, so $J \cup \{u\}$ embeds into $\Gamma$. In any amalgam of $J \cup \{u\}$ with $\{a, u\}$ we have $d(a, x) = 4$ for $x \in J$, so $J \cup \{a, u\}$ embeds into $\Gamma$ as well.

Thus an amalgam of $I \cup J \cup \{a\}$ and $J \cup \{a, u\}$ embeds into $\Gamma$. For $v \in I$, consideration of $(u, a, v)$ shows that $d(u, v)$ is 3 or 5, and consideration of $J_v \cup \{u, v\}$ shows that $d(u, v)$ is not 5. Thus we have $d(u, v) = 3$ for all $v \in I$ in our amalgam, and thus $I \cup \{a, u\}$ embeds isometrically into $\Gamma$.

**Construction of $I \cup \{b, u\}$.**

Let $J' = \bigcup_{v \in I} J'_v$ with $J'_v \cong I_k^{(4)}$. Put a metric on $I \cup J' \cup \{b, u\}$ by taking $d(u, x) = 5$, $d(b, x) = 4$ for $x \in J'$, while for $v \in I$ we take $d(v, x) = 4$ for $x \in J'_v$, and $d(v, x) = 2$ for $x \in J \setminus J'_v$.

Introduce an auxiliary vertex $b'$ with $d(b', u) = 5$, $d(b', x) = 2$ for $x \in I \cup J' \cup \{b\}$.

We claim that $I \cup J' \cup \{b, b'\}$ and $J' \cup \{a, u, b, b'\}$ embed isometrically in $\Gamma$. For $I \cup J' \cup \{b, b'\}$ we use the structure of $B\Gamma$, together with the condition $k + 1 < n$, and for $J' \cup \{a, u, b, b'\}$ we use the structure of $I_n^{(4)}$ to check that $J \cup \{b, b'\}$ embeds into $\Gamma_5$.

Therefore some amalgam $I \cup J' \cup \{a, b, b'\}$ embeds into $\Gamma$. Let $v \in I$. In the amalgam, the auxiliary vertex $b'$ ensures that $d(u, v)$ is 3 or 5. Consideration of $J'_v \cup \{u, b, v\}$ shows that $d(u, v)$ is not 5. Thus $d(u, v) = 3$ for $v \in I$, and $I \cup \{b, u\}$ embeds isometrically in $\Gamma$. \hfill \Box
We will need some additional amalgamation arguments to complete our analysis, beginning with the following preparatory lemma.

**Lemma 7.13.** Let \( \Gamma \) be bipartite of diameter 5, and not antipodal. Suppose \( B\Gamma \) is generic omitting \( I_{n+1}^{(4)} \), with \( n \geq 3 \), and \( \Gamma_5 \) contains a pair of vertices at distance 4. Then the following hold.

1. \( I_n^{(4)} \) embeds in \( \Gamma_3 \);
2. \( \Gamma_3 \) is primitive.

**Proof.**

1. \( I_n^{(4)} \) embeds in \( \Gamma_3 \):

   We show inductively that \( I_m^{(4)} \) embeds into \( \Gamma_3 \) for \( m \leq n \).

   Let \( I \cong I_{m-1}^{(4)} \). Form extensions \( I \cup \{ u \} \) and \( I \cup \{ v \} \) with \( d(u, x) = 5 \), \( d(v, x) = 3 \) for \( x \in I \). Then \( I \cup \{ u \} \) embeds into \( \Gamma \) since \( m - 1 \leq n - 1 \), while \( I \cup \{ v \} \) embeds into \( \Gamma \) by induction on \( m \). So some amalgam \( I \cup \{ u, v \} \) embeds into \( \Gamma \) with \( d(u, v) \) either 2 or 4.

   Consider a geodesic \( \{ u, v, w \} \) with \( d(u, w) = 1 \), \( d(v, w) = 3 \), and \( d(u, v) \) as specified. There is an amalgam \( I \cup \{ u, v, w \} \) of \( I \cup \{ u, v \} \) with \( \{ u, v, w \} \) over \( u, v, w \), and consideration of \( \{ w, u, x \} \) for \( x \in I \) show that \( I \cup \{ w \} \cong I_{m}^{(4)} \). As \( I \cup \{ w \} \subseteq \Gamma_3 \), the induction is complete.

2. \( \Gamma_3 \) is primitive:

   By Lemma 7.10 we have \((1/2)\Gamma_3 \) connected. Suppose now that \( \Gamma_3 \) is disconnected with respect to the edge relation “\( d(x, y) = 4 \)”.

   Fix two connected components \( C, C' \) with respect to this relation. By (1) these have order \( n \), and by assumption \( n \geq 3 \). Fix \( u \in C \) and \( u_1', u_2' \in C' \), and \( v_1 \in \Gamma_2 \) adjacent to \( u, u_1' \). With \( * \) the chosen basepoint for \( \Gamma \), consider the isometry of \( C \cup C' \cup \{ * \} \) which interchanges \( u_1' \) and \( u_2' \) and fixes the remaining vertices. Then this extends to an isometry \( C \cup C' \cup \{ *, v \} \cong C \cup C' \cup \{ *, v' \} \) for some vertex \( v' \).

   Take \( u_3' \in C' \), distinct from \( u_1, u_2 \). Then the map \((*, u_3', v, v') \mapsto (*, u_3', u_3, v, v') \) is an isometry and therefore extends to \( \Gamma \); its extension interchanges \( C \) and \( C' \) and fixes \( v, v' \). However \( d(v, x') = d(v', x) \) for \( x \in C \), so the same applies to \( C' \). But \( d(v, u'_1) = 1 \), \( d(v', u'_2) = 1 \), and \( d(u'_1, u'_2) = 4 \), so this is impossible. \( \square \)

Now we can assemble these ingredients.

**Lemma 7.14.** If \( \Gamma \) is bipartite of diameter 5 and not antipodal, then \( B\Gamma \) is the universal homogeneous graph (Rado’s graph).

**Proof.** The alternative is that \( B\Gamma \) is generic omitting \( I_{n+1}^{(4)} \) for some \( n \geq 3 \). By Lemma 7.7, we may suppose that \( \Gamma_5 \) contains a pair of vertices at distance 4, and hence \( I_{n-1}^{(4)} \) embeds in \( \Gamma_5 \) by Lemma 7.12.

To get a contradiction, we will aim at an amalgamation of the following form.

Let \( I \cong I_{n-1}^{(4)} \), let \( I \cup \{ a \} \cong I \cup \{ b \} \cong I_n^{(4)} \), and adjoin a vertex \( u \) such that

\[
d(u, a) = 1; \quad d(u, b) = 5; \quad d(u, x) = 3 \quad \text{for} \quad x \in I
\]

We will embed \( I \cup \{ a, u \} \) and \( I \cup \{ b, u \} \) in \( \Gamma \), and then in their amalgam we will have \( I \cup \{ a, b \} \cong I_n^{(4)} \), a contradiction. Each of the factors \( I \cup \{ a, u \} \) and \( I \cup \{ b, u \} \) will require its own construction.

The first factor, \( I \cup \{ a, u \} \). Let \( I = I_0 \cup \{ c \} \) and introduce a vertex \( v \) with

\[
\begin{align*}
d(v, a) &= 1, & d(v, c) &= 5, & d(v, u) &= 2, \\
d(v, x) &= 4 & (x \in I_0 \cup \{ a \})
\end{align*}
\]
If $I_0 \cup \{a, u, v\}$ and $I_0 \cup \{c, u, v\}$ embed into $\Gamma$, then in their amalgam $I_0 \cup \{a, c, u, v\}$ we have $d(a, c) = 4$ and thus the desired metric space $I \cup \{a, u\}$ is embedded into $\Gamma$.

Construction of $I_0 \cup \{a, u, v\}$.

We first embed $I_0 \cup \{a, u\}$ into $\Gamma$. Introduce a vertex $a'$ with

$$d(a', u) = 1;\ d(a', a) = 2;\ d(a', x) = 2$$

for $x \in I_0$

On the one hand, the geodesic $(a, u, a')$ embeds into $\Gamma$; on the other hand, the metric space $I_0 \cup \{a, a'\}$ embeds into $BT$. So some amalgam $I_0 \cup \{a, a', u\}$ embeds into $\Gamma$, and for $x \in I_0$, consideration of the paths $(u, a, x)$ and $(a, a', x)$ shows that $d(u, x) = 3$, as required. Thus $I_0 \cup \{a, u\}$ embeds into $\Gamma$.

Take $a$ as basepoint. Then $u \in \Gamma_1$, $I_0 \subseteq \Gamma_4$, and $d(u, x) = 3$ for $x \in I_0$. Let $I_1 \subseteq I_0$ be obtained by removing one vertex, so $I_1 \cong I_n^{(4)}$. Consider the sets

$$A = \{u \in \Gamma_1 : d(u, x) = 3\ \text{for}\ x \in I_1\}$$

$$B = \{u \in \Gamma_4 : d(u, x) = 4\ \text{for}\ x \in I_1\}$$

The partitioned metric space $(A, B)$ is homogeneous with respect to the metric plus the partition. We consider the structure of $(A, B)$.

We show first that $A$ is infinite. Assuming the contrary, consider a configuration $I_1 \cup I_2$ in $I_2 \cong I_n^{(2)}$, and $I_1 \cup \{x\} \cong I_n^{(4)}$ for each $x \in I_2$. This configuration embeds into $\Gamma_4$. There is a pair $x, y \in I_2$ such that $I_1 \cup \{x\}$ and $I_1 \cup \{y\}$ have the same vertices at distance 3 in $\Gamma_1$. Now $\Gamma_4$ is generic omitting $I_n^{(4)}$. By homogeneity it follows easily that any two subsets of $\Gamma_4$ isomorphic to $I_n^{(4)}$ have the same vertices at distance 3 in $\Gamma_1$. This yields a nonempty subset of $\Gamma_1$ definable without parameters, and a contradiction. So $A$ is infinite.

Now $B$ is generic omitting $I_3^{(4)}$. In particular $B$ is primitive. Furthermore, each vertex of $B$ lies at distance 3 from some vertex of $A$. By primitivity, this vertex cannot be unique. Take $c \in B$ and $u, v \in A$ so that $d(c, u) = d(c, v) = 3$. Then $I_0 \cup \{c, a, u, v\}$ has the desired structure.

Construction of $I_0 \cup \{c, u, v\}$.

We introduce a vertex $d$ with

$$d(d, u) = d(d, v) = 1;\ d(d, c) = 4$$

The relation of $d$ to $I_0$ will be determined below.

In any amalgam of $I_0 \cup \{c, d, u\}$ with $I_0 \cup \{c, d, v\}$ over $I_0 \cup \{c, d\}$ we have $d(u, v) = 2$. It remains to construct $I_0 \cup \{c, d, u\}$ and $I_0 \cup \{c, d, v\}$.

We claim first that $I_0 \cup \{c, u\}$ embeds into $\Gamma$, in other words that $I_0 \cup \{c\}$ embeds into $\Gamma_3(u)$. This holds by Lemma 7.13. Now we may form $I_0 \cup \{c, d, u\}$ by amalgamating $I_0 \cup \{c, u\}$ with $\{c, d, u\}$ (a geodesic) to determine the metric on $I_0 \cup \{d\}$; all distances $d(d, x)$ will be even for $x \in I_0$. This amalgamation determines the structure of $I_0 \cup \{d\}$ and thereby completes the determination of the second factor $I \cup \{c, d, v\}$ as well.

We claim that $I \cup \{c, d, v\}$ embeds into $\Gamma$. Since the distance $d(c, d) = 4$ is forced in any amalgam of $I_0 \cup \{v, c\}$ with $I_0 \cup \{v, d\}$, we consider these two metric spaces separately.
Now \( I_0 \cup \{v, d\} \cong I_0 \cup \{u, d\} \), so this is not at issue, and we are left only with \( I_0 \cup \{c, v\} \). This last embeds into the second factor \( I \cup \{b, u\} \), so we may turn finally to a consideration of this second factor.

The second factor, \( I \cup \{b, u\} \).

We introduce another vertex \( v \) satisfying

\[
d(v, b) = 1; \quad d(v, u) = 4; \quad d(v, x) = 5 \quad \text{for} \quad x \in I
\]

This will force \( d(b, x) = 4 \) for \( x \in I \). So it will suffice to embed \( I \cup \{u, v\} \) and \( \{b, u, v\} \) separately into \( \Gamma \). Since \( \{b, u, v\} \) is a geodesic, we are concerned with \( I \cup \{u, v\} \).

Introduce a vertex \( d \) with

\[
d(d, u) = 1; \quad d(d, v) = 5; \quad d(d, x) = 2 \quad \text{for} \quad x \in I
\]

Then amalgamation of \( I \cup \{d, u\} \) with \( I \cup \{d, v\} \) forces \( d(u, v) = 4 \). It remains to embed \( I \cup \{d, u\} \) and \( I \cup \{d, v\} \) into \( \Gamma \).

The second of these, \( I \cup \{d, v\} \), has a simple structure with \( I \cup \{d\} \subseteq \Gamma_5(v) \), and since \( I \cup \{d\} \) has order \( n \), with all distances even, it embeds into \( \Gamma_5 \) by Lemma 7.12. So we need only construct \( I \cup \{d, u\} \).

Taking \( d \) as base point, and \( I \) contained in \( \Gamma_2 \), we are looking for a vertex \( u \in \Gamma_1 \) at distance 3 from all elements of \( I \). For \( v \in I \), let \( I_v \) be the set of neighbors of \( v \) in \( \Gamma_1 \). Any vertex \( u \in \Gamma_1 \) which is not in \( \bigcup_{v \in I} I_v \) will do. So it remains to be checked that \( \bigcup_{v \in I} I_v \neq \Gamma_1 \).

The sets \( I_v \) for \( v \in I \) are pairwise disjoint. Suppose they partition \( \Gamma_1 \). We may take a second set \( J \cong I_n^{(4)} \) in \( \Gamma_2 \) overlapping with \( I \) so that \( |I \cap J| = n - 2 \), and then the \( I_v \) for \( v \in J \) will also partition \( \Gamma_1 \); so the vertices \( v_1 \in I \setminus J \) and \( v_2 \in J \setminus I \) have the same neighbors in \( \Gamma_1 \). As \( \Gamma_2 \) is primitive, it follows that all vertices of \( \Gamma_2 \) have the same neighbors in \( \Gamma_1 \), a contradiction. \( \square \)

At this point, the proof of Proposition 7.5, and also Theorem 10, is complete. We review the analysis.

Proof. \( B \Gamma_1 \) falls under the Lachlan/Woodrow classification.

If \( \Gamma_1 \) is finite, then Theorem 15 applies, and as we assume diameter at least 3 and valence at least 3, we arrive at either the complement of a perfect matching or a tree in this case.

With \( \Gamma_1 \) infinite, \( B \Gamma \) contains an infinite clique, and hence so does \( B \Gamma_1 \). As noted at the outset, \( B \Gamma \) is connected. So by the Lachlan/Woodrow classification, \( B \Gamma \) is either imprimitive of the form \( K_\infty[I_m] \) or \( m \cdot K_\infty, 2 \leq n \leq \infty \), or generic omitting \( I_n \) for some finite \( n \geq 2 \), or universal homogeneous (Rado’s graph).

When \( B \Gamma_1 \) is imprimitive, the classification in Theorem 9 applies to \( B \Gamma \), and as \( B \Gamma \) contains an infinite clique, the result is that \( B \Gamma \) is either an imprimitive homogeneous graph of the form \( K_\infty[I_m] \) \( (2 \leq m \leq \infty) \), or one of the tree-like graphs of Macpherson, \( T_{r, \infty} \) with \( 2 \leq r \leq \infty \).

When \( B \Gamma \) is of the form \( K_\infty[I_m] \) with \( m \geq 2 \), Lemma 7.2 applies. When \( B \Gamma \) is tree-like, Lemma 7.3 applies, and \( \Gamma \) is a tree.

Thus we may suppose that \( B \Gamma_1 \) is primitive. We have set aside the case in which \( B \Gamma_1 \) is universal homogeneous as a distinct (and typical) case. So we are left with the possibility that \( B \Gamma_1 \) is generic omitting \( I_n \) with \( 2 \leq n < \infty \). In view of Theorem 9, \( B \Gamma \) must have diameter at most 2, and be homogeneous as a graph. Then our hypothesis on \( B \Gamma_1 \) implies that \( B \Gamma \) is also generic omitting \( I_n \).
In case $n = 2$, Lemma 7.1 applies. If $n > 2$ then the diameter of $\Gamma$ is 4 or 5. If the diameter is 4, then Lemma 7.4 applies, and leads to case 4 of the theorem.

This leaves us with the case taken up in Proposition 7.5: $\Gamma$ has diameter 5, and $B\Gamma$ is generic omitting $I_n$. As the diameter is 5, we have $n \geq 3$. If $\Gamma$ is antipodal, then Corollary 5.14 applies. If $\Gamma$ is not antipodal, then Lemma 7.14 applies. □

8. Graphs of small diameter

In the Appendix to [Che98], we gave an exhaustive list of certain amalgamation classes for highly restricted languages. The languages considered were given by a certain number of irreflexive binary relations, symmetric or asymmetric, and the structures considered were those in which every pair of elements is related by one and only one of the specified relations. The cases of interest here are the languages with either 3 or 4 symmetric irreflexive binary relations. The amalgamation classes $\mathcal{A}$ under consideration were those satisfying the following three conditions:

1. The class $\mathcal{A}$ is determined by a set of forbidden triangles.
2. The Fraïssé limit of the class is primitive.
3. The class in question is not a free amalgamation class.

This last point means that there is no single relation $R(x, y)$ such that every amalgamation problem $A_0 \subseteq A_1, A_2$ can be completed by taking $R$ to hold between $A_1 \setminus A_0$ and $A_2 \setminus A_0$. This excludes some readily identified metrically homogeneous graphs of diameter 3, but none of greater diameter.

For example, in the case of three symmetric relations $A(x, y), B(x, y), C(x, y)$, the only such class (up to a permutation of the language) is the one given by the following constraints:

$$(AAB), (ACC), (AAA)$$

In this notation, $AAB$ represents a triple $x, y, z$ with $A(x, y), A(x, z), B(y, z)$.

Now for the Fraïssé limit of this class to correspond to a metric space of diameter 3, one of the forbidden configurations must correspond to the triangle (113). Since the configuration corresponding to (113) may be either $(AAB)$ or $(ACC)$, there are two distinct homogeneous metric spaces of this type, with the following constraints:

$$(113), (122), (111) \text{ or } (233), (113), (333)$$

The first of these has no triangle of odd perimeter 5 or less, that is $K_1 = K_2 = 3, C_0 = 10, C_1 = 11$. The second has no triangle of perimeter 8 or more, that is $K_1 = 1, K_2 = 3, C_0 = 8, C_1 = 9$. We recognize these falling within our previous classification with $C > 2\delta + K_1$ and $C' = C + 1$.

Of course, any of our examples of type $\mathcal{A}^{3}_{K_1, K_2, C_0, C_1}$ will be a free amalgamation class unless one of the forbidden triangles involves the distance 2, which means:

$$K_1 = 3 \text{ or } C \leq 8$$

And as our list was confined to the primitive case it omits bipartite and antipodal examples.

We view the classification of the amalgamation classes determined by triangles as a natural ingredient of a full catalog of “known” types, and a natural point of departure for an attempt at a full classification. In [AMP10] (a working draft) the problem is taken from the other end, in the case of diameter 3, and it is shown that in the triangle-free case (i.e., $K_3$-free), the classification does indeed reduce
to identifying the combinations of triangle constraints and (1, 2)-space constraints which (jointly) define amalgamation classes.

8.1. Diameter 4. The explicit classification of the metrically transitive graphs of diameter 4 whose minimal forbidden configurations are triangles is more complex. Once the diameter $\delta$ exceeds 3, the possibility of “free amalgamation” falls by the wayside, as one can use an amalgamation to force any particular distance strictly between 1 and $\delta$, using the triangle inequality. So in diameter 4, the table in [Che98] covers all imprimitive homogeneous structures with 3 symmetric 2-types which can be given in terms of forbidden triangles.

There are 27 such (up to a permutation of the language) of which 17 correspond to homogeneous metric spaces, some in more than one way (permuting the distances matters to us, if not to the theory). We will exhibit those classes in a number of formats. Table 4, at the end of the paper, gives all 27 classes in the order they were originally given, using the symbols $A, B, C, D$ for the binary relations involved. The numbering of cases used in the next two tables conforms to the numbering given in that table.

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Table 1. 22 metric spaces, with duplication

In Table 1 we have converted $A, B, C, D$ to distances, usually in the order $1, 2, 3, 4$, or $4, 2, 3, 1$, and in one case in the order $2, 4, 3, 1$, for those cases in which the result is a metric space. In checking the possibilities, begin by identifying the forbidden triangle $(1, 2, 4)$, involving three distinct distances; in all 27 cases this can only be the triple $(ABD)$, by inspection. Thus $C$ corresponds to distance 3.
After that, look for the forbidden triangle (113): by inspection, this is either $CDD$ or $AAC$ in each case. Thus $A$ or $D$ corresponds to distance 1, after which there is at most one assignment of distances that produces the constraint (114).

All primitive metrically transitive graphs of diameter 4 whose constraints are all of order 3 are listed in the resulting table. We omit the columns corresponding to the non-geodesic triangles of types $(1, 2, 4), (1, 1, 3), (1, 1, 4)$, which are of course present as constraints in all cases.

In Table 2 we list these metric spaces together with their defining parameters $K_1, K_2, C, C'$.

<table>
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<tr>
<th>#</th>
<th>$K_1$</th>
<th>$K_2$</th>
<th>$C$</th>
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Table 2. 20 metric spaces, sorted

Let us compare the outcome to the statement of Theorem 12.

The table contains no examples with $C \leq 2\delta + K_1$. The case $C' > C + 1$ and $3K_2 = 2\delta - 1$ is impossible with $\delta = 4$, while the case $K_1 = \infty$ is imprimitive and omitted. On the other hand, when $C' = C + 1$, if $C = 2\delta + 1$ then again the graph is imprimitive, while if $C \geq 2\delta + 2$ then $K_1 \geq 2$, and the condition $K_1 + 2K_2 \leq 2\delta - 1 = 7$ gives $K_2 = K_1 = 2$, and hence $C = 2(K_1 + K_2) + 1 = 2\delta + 1$ after all.

So what we see here is the range of possibilities illustrating the third case under Theorem 12: $K_1 < \infty$, $C > 8 + K_1$, $K_1 + 2K_2 \geq 7$, $3K_2 \geq 8$, and if $K_1 + 2K_2 = 7$ then $C \geq 10 + K_1$, while if $C' > C + 1$ then $C \geq 8 + K_2$.

If $C' = C + 1$ these constraints amount to $1 \leq K_1 \leq K_2$, $K_2 = 3$ or 4, $C \geq 9 + K_1$, and if $K_1 = 1$ and $K_2 = 3$ then $C \geq 11$. This corresponds to the first two sections of the table, arranged according to increasing $K_2$. 
If \( C' > C + 1 \) then there is the added constraint \( C \geq 8 + K_2 \) and this implies \( C = 11, C' = 14, K_2 = 3 \). Since \( C > 2\delta + K_1 \) we find \( K_1 \leq 2 \). This corresponds to the last two lines of the table.

### Part 3. Finite Approximations to the Generic Triangle Free Graph

#### 9. The Finite Model Problem

A structure has the finite model property if every first order sentence true in the structure is true in some finite structure. A slightly stronger property is the finite submodel property, where the finite approximation should be taken to lie within the original structure. There is little difference between the notions in the cases of most immediate concern here. We focus mainly on the finite model property for the generic triangle free graph, and as “triangle free” is part of the first order theory of this structure, the finite model property and the finite submodel property are equivalent here. We do not necessarily expect the finite submodel property to be true. There is little evidence either way. But however that may be, it is a concrete problem of a familiar type in graph theory, and we will take some pains to put it in an explicit form.

This is an attractive problem which has attracted the attention of a number of combinatorialists and probabilists, but there is not much to show for it and, in particular, not much in the literature. So my main task here is to assemble and to document such information as has come my way about the problem (and to respond more fully to some occasional queries that have come my way as well).

There is some literature on lower bounds for finite approximations to Fraïssé limits, in cases where probabilistic considerations guarantee their existence, e.g. [Sz2, Bon09]. We would like to encourage such exploration in cases where the existence of any such approximation remains in question, and so we will include some relatively anecdotal information to give a clearer picture of where those explorations might usefully begin.

#### 9.1. Overview

In general, the theory of a homogeneous locally finite structure is axiomatized by two kinds of axioms: negative axioms defining the “forbidden substructures,” e.g. triangles in the case that most concerns us, and extension properties stating that for each \( k \), any \( k \)-generated subset of \( k \) elements may be extended to a \((k + 1)\)-generated subset in any way not explicitly ruled out by the negative constraints. We will drop all further mention of \( k \)-generated structures here, and suppose that the language is purely relational, so that the issue is one of extending \( k \) given elements by one more element.

In the context of triangle free graphs, the natural extension properties are the following three.

- \( E_n \): for any set \( A \) of at most \( k \) vertices, and any subset \( B \) of \( A \) consisting of independent vertices, there is a vertex \( v \) adjacent to all vertices of \( B \), and to none of \( A \setminus B \).
- \( E'_n \): any maximal independent set contains at least \( k \) vertices; for any set \( A \) of \( k \) independent vertices, and any subset \( B \) of \( A \), there is a vertex \( v \) adjacent to all vertices of \( B \), and to none of \( A \setminus B \).
- \( Adj_n \): Any set of at most \( n \) independent vertices has a common neighbor.
Here the properties $E_n$, together with the axiom stating that triangles do not occur, gives the full axiomatization of the generic triangle free graph. Therefore the problem of the finite model property for the generic triangle free graph is simply the question, whether for each $n$ some finite triangle free graph has the extension property $E_n$.

The equivalence of $E_n$ and $E'_n$ in triangle free graphs is straightforward, and with the former more easily applied, and the latter more easily checked. The mutual adjacency condition $\text{Adj}_n$ is manifestly weaker, but only because it allows some relatively degenerate examples: the complete bipartite graph, and some less obvious ones—these can all be satisfactorily classified. As a result, we can replace the property $E'_n$ by a mild strengthening of $\text{Adj}_n$, giving us the simplest version to check.

Our first order of business will be to sort out the force of these extension properties. Along the way we will find it useful to classify explicitly all the triangle free graphs which satisfy the condition $\text{Adj}_n$ and $E_2$ but not $E_n$. For this, a description of triangle free graphs satisfying $\text{Adj}_3$ in terms of combinatorial geometries is useful. The geometries in question are formed simply by fixing a vertex $v$ in the graph, taking its neighbors as points, and the non-neighbors as blocks, with the edge relation in the graph providing the incidence relation in the geometry. This slight shift in point of view will simplify both the analysis and the construction of such graphs. This corresponds to the usual construction of the Higman-Sims graph, and while none of our geometries will have a geometry with the elegance of that graph’s, that point of view will still be useful.

We refer to triangle free graphs satisfying condition $E_n$ as $n$-e.c., which stands for “$n$-existentially complete” (for the category of triangle free graphs). This terminology comes from model theory.

We will see that there are a number of infinite families of geometries all corresponding to 3-e.c. graphs, but that the families known to date are neither varied nor robust. In the $M_{22}$ geometry which is associated with the Higman-Sims graph, every block has 6 points; but in the infinite families of 3-e.c. graphs known to us, there is always at least one block of order 2 (and possibly just one). At the other extreme, in a geometry associated with a 4-e.c. graph, the minimal block size is at least 19. Barring some breakthrough taking us to finite 4-e.c. triangle free graphs and beyond, it would be interesting to make the acquaintance of more robust 3-e.c. graphs. We will give explicit descriptions of some infinite families, in the hopes that this may stimulate someone to find better constructions.

Of the five known non-trivial strongly regular graphs, two provide interesting examples of 3-e.c. graphs. The question as to whether there are more such to be found, and possibly an infinite family, seems to be tied up with fundamental problems in that subject. However following a suggestion of Peter Cameron, we can eliminate the possibility that a strongly regular triangle free graph could be 4-e.c.

The following general principle is immediate.

**Remark 9.1.** If $M$ is a structure with the finite model property, and $M'$ is a structure which can be interpreted in $M$, then $M'$ inherits the finite model property.

In particular if the generic $K_n$-free graph has the finite model property for one value of $n$, then so does the generic $K_m$-free for $m \leq n$, interpreting the latter as the graph induced on the set of vertices adjacent to $(n - m)$ vertices of the former.
Thus the finite model property for the generic triangle free graph is the weakest instance of the problem still open in the case of homogeneous graphs.

One would of course like to have general methods for settling the finite model property in homogeneous structures, but since the most concrete individual cases of the problem remain challenging, we will have to leave that aside.

9.2. Probabilistic methods. One knows the finite model property for the Rado graph and similarly unconstrained homogeneous structures by a simple probabilistic argument. Just as a random (countable) graph will be isomorphic to the Rado graph with probability one, a large finite random graph will have the and one of the appropriate extension properties with asymptotic probability 1 [Fag76]. But probabilistic constructions behave poorly in the presence of constraints. In particular, with a direct use of counting measure on the set of triangle free graphs of a given size, a random one will be bipartite with high probability [KPR87], so that the theory of the random finite triangle free graph does not approximate the theory of the generic triangle free graph at all well.

Some time ago Vershik raised the question of a Borel measure invariant under the full infinite symmetric group and concentrating on the generic triangle free graph, a question answered positively in [PV08] (with a classification of the measures in question). But this does not seem to help with the finite model property.

9.3. The Extension properties $E'_2$, $E_n$, and $E'_n$. We begin by making the relationships between the natural extension properties explicit.

**Lemma 9.2.** For triangle free graphs $G$ the following properties are jointly equivalent to $E'_2$:

1. $G$ is maximal triangle free;
2. $G$ is indecomposable;
3. $G$ contains an independent set of order 3.

Maximality means that the adjunction of any additional edge would create a triangle, which is the same as the mutual adjacency condition: any two independent vertices have a common neighbor.

A graph is decomposable if it carries a nontrivial congruence, that is, an equivalence relation such that for any two classes $C_1, C_2$ either all pairs in $C_1 \times C_2$ are edges, or none are.

*Proof.* Assume (i – iii). The first point follows from (i, ii). For the second, use (i – ii) to show that a counterexample must be a 5-cycle, and invoke (iii). □

Next we check that the properties $E'_n$ and $E_n$ are equivalent.

**Lemma 9.3.** For $G$ triangle free, and $n$ arbitrary, properties $E_n$ and $E'_n$ are equivalent.

*Proof.* Let $A$ be a vertex set of order at most $n$, and $B$ an independent subset of $A$. We must show that there is a vertex adjacent to all vertices of $B$ and no vertices of $A \setminus B$. We proceed by induction on $n$, and on $|A \setminus B|$.

If there is an edge $(a, b)$ with $a \in A \setminus B$ and $b \in B$, then let $A_0 = A \setminus \{a\}$, and apply induction to $n$ to get $v$ adjacent to all vertices of $B$, and no vertices of $A_0 \setminus B$, and hence to no vertex of $A \setminus B$. 
So we may suppose there is no edge connecting $A \setminus B$ and $B$, but that there is an edge in $A$. In particular $|A \setminus B| \geq 2$. We claim then: \textit{There is a vertex $u \notin A$ which is adjacent to some vertex $a \in A$ and to no vertex of $B$.}

Let $A_0 \subseteq A$ be a maximal independent subset of $A$ containing $B$, and $u$ a vertex not adjacent to any vertex of $A_0$. Then $u$ is not in $A$, and if $u$ is adjacent to some vertex of $A$ we have our claim. So suppose $u$ is adjacent to no vertex of $A$. Take $a \in A \setminus B$, and a vertex $v$ adjacent to $u$ and $a$, and to no vertex in $B$. Then $v \notin A$, and $v$ meets the conditions of our claim.

Now applying the claim with $u$ adjacent to $a \in A$ and to no vertex of $B$, let $B_1 = B \cup \{u\}$. Let $A_1 = A \setminus \{a\} \cup \{u\}$. By induction on $
 |A \setminus B|$ we find $v$ adjacent to all vertices of $B_1$ and no vertex of $A_1 \setminus B_1$. Then the set of neighbors of $v$ in $A$ is $B$, as required.

\section{The strength of $\text{Adj}_3$.}

Our main interest will be in showing that with few exceptions graphs having the properties $E_2$ and $\text{Adj}_n$ satisfy the full $n$-e.c. property $E_n$. The delicate case arises when $n = 2$, and we first dispose of the others.

\begin{lemma}
If $G$ is a 3-e.c. triangle free graph with the mutual adjacency property $\text{Adj}_3$, then $G$ is $n$-e.c.
\end{lemma}

\begin{proof}
It suffices to verify the condition $E'_n$. We proceed by induction on $n$.

Fix $A$ an independent set of order at most $n$. We may assume $|A| = n$ or conclude by induction. Our objective is to show that every subset of $A$ occurs as the set of neighbors in $A$ of some vertex of $G$.

Let us write $A_v$ for the set of neighbors in $A$ of the vertex $v$. Then for $v'$ chosen adjacent to $v$ and to all vertices of $A \setminus A_v$, we find $A_{v'} = A \setminus A_v$. So this collection of sets is closed under complementation in $A$.

Fix $X \subseteq A$. We will show that $X$ is $A_v$ for some vertex $v$. Taking complements if necessary, suppose $|X| \leq n/2$. Take $a \in A \setminus X$ and $u$ a vertex whose neighbors in $A \setminus \{a\}$ are the vertices in $X$. We may suppose that $A_u = X \cup \{a\}$.

Let $Y$ be the complement $A \setminus (X \cup \{a\})$ and let $v$ be a vertex with $A_v = Y$. If $|Y| \geq 1$ then taking $u'$ (inductively) whose set of neighbors in $(A \setminus Y) \cup \{v\}$ is $X \cup \{v\}$, we finish.

We conclude that $|X| = n - 2$, so $n = 4$ and $|X| = 2$. In particular any singleton occurs as $A_v$ for some $v$.

Write $A = \{a_1, a_2, b_1, b_2\}$ and let $X = \{a_1, a_2\}$. Let $b'_1$ be a vertex with $A_{b'_1} = \{b_1\}$. Let $b_2$ be a vertex whose unique neighbor in $\{a_1, a_2, b'_1, b_2\}$ is $b_2$. Let $u$ be a vertex adjacent to $a_1, a_2, b'_1, b_2$. Then $A_u = X$. \qed

Now we take up the graphs satisfying $E_2$ and $\text{Adj}_3$, but not $E_3$, and we begin with a construction.

\section{The Linear Order Geometries.}

Given any triangle free graph $G$ with the mutual adjacency property $\text{Adj}_3$ we associate a combinatorial geometry to each vertex $v$ of $G$ by taking as the set of points $P$ the neighbors of $v$, and as the blocks of the geometry its non-neighbors. A point $p$ lies on a block $b$ if the pair $(p, b)$ is an edge. The geometry obtained may depend on the vertex chosen, but given one such geometry, the graph $G$ may be entirely reconstructed as follows. As the vertex set for $G$ we take $P \cup B \cup \{v_0\}$ where $v_0$ is an additional vertex. We take as edges all pairs $(v_0, p)$ with $p$ in $P$, all pairs $(p, b)$ with $p$ on $b$, and all pairs $(b_1, b_2)$ with $b_1, b_2$ disjoint when viewed as subsets of $P$; and we symmetrize. This agrees
with the original graph $G$: in particular, if $b_1, b_2$ are not adjacent in $G$, then by the property $\text{Adj}_3$ there is a point $p$ lying on both, and hence they are not adjacent in the reconstructed version of $G$.

The following uses extremely weak assumptions, but then we intend to apply it to an extremely weak geometry.

**Lemma 9.5.** Let $(P, B)$ be a combinatorial geometry on at least 3 points satisfying the following conditions.

1. No three blocks are pairwise disjoint.
2. No pair of distinct blocks correspond to the same subset of $P$.
3. For every block $b$ and every point $p$ not in $b$, there is a block containing $p$ and disjoint from $b$.
4. For every pair of points there is a block containing one but not the other.
5. No block is incident with every point of $P$.

Then the associated graph $G$ is triangle free and 2-e.c.

**Proof.** One checks that $G$ is maximal triangle free, indecomposable, with an independent set of size at least three.

One point that requires checking is that no block is empty; this is part of the verification that $G$ is maximal triangle free. For this, use the assumptions to get two nonempty blocks which are disjoint, and observe that the empty block would extend this to a pairwise disjoint triple.  

Let $L$ be a linear order. Let $B$ be a set of proper initial segments of $L$ and proper terminal segments of $L$ satisfying the conditions:

1. For all $a < b$ in $L$, there is an initial segment in $B$ containing $a$ and not $b$, and a terminal segment in $B$ containing $b$ and not $a$.
2. If $I$ is an initial segment and $a \in B \setminus I$ is a lub for $I$, then the terminal segment $[a, \infty)$ is in $B$; and dually.

One way to meet these conditions is to let $B$ consist of the proper segments of the form $(-\infty, a]$ and $[a, \infty)$, and this is the only way to achieve even the first of them if $L$ is finite. Let us call such a geometry a *linear geometry*.

**Lemma 9.6.** Let $(P, B)$ be a linear geometry on at least 3 points. Then the associated graph satisfies the conditions $E_2$ and $\text{Adj}_n$ for all $n$ (if $P$ is finite, this is vacuous for $n$ larger than $|P|$).

**Proof.** The conditions on the geometry have been written to ensure that our criterion for the property $E_2$ applies.

Now suppose $A$ is any independent subset of the associated graph. If $A$ contains no points, then $A$ consists of some blocks, and possibly the base point, and as the blocks all meet pairwise, it suffices to take a point common to the minimal initial segment in $A$, and the minimal terminal segment in $A$. That point is then a vertex adjacent to all vertices of $A$.

If $A$ contains a point $p$, then it cannot contain both initial segments and terminal segments, as they would be separated by $p$. So suppose for example $A$ contains initial segments, and let $I$ be the greatest among them. Let $a \in A$ be the least point. Then $a \notin I$. One of our two conditions on $B$ then applies to give a terminal segment disjoint from $I$, and containing $a$.

Of course if $A$ consists exclusively of points, the base point will suffice as a common neighbor.  

If \( L \) is finite of size \( n \), then the resulting graph \( G \) has order \( 3n - 1 \) and can be construed as follows. The elements of \( G \) are the integers \( 0, 1, \ldots, 3n - 1 \); the edge relation is defined by \( |i - j| \equiv 1 \mod 3 \). The value \( n = 2 \), which is not permitted as we require 3 points, corresponds to the pentagon, and the first legitimate example is a graph of order 8. The maximal independent sets have size \( n \) and are the points of the geometry, with respect to a basepoint which is their common neighbor. The geometry obtained is independent of the base point.

Our next point is that the converse holds: a graph with property \( E_2 \) and property \( \text{Adj}_n \) which does not have property \( E_n \) must be obtained in this way from a linear geometry, and in particular if it is finite then it isomorphic to the graph just described explicitly. It will suffice to treat the case \( n = 3 \), since once \( E_3 \) is satisfied, \( \text{Adj}_n \) implies \( E_n \).

9.6. Graphs with \( E_2 \), \( \text{Adj}_3 \), and not \( E_3 \).

**Definition 9.7.** An independent set of vertices \( I \) in a graph \( G \) will be said to be shattered if every subset of \( I \) occurs as \( \{a \in I : a, v \text{ are adjacent}\} \) for some vertex \( v \).

We will be concerned for the present with graphs which are 2-e.c. but not 3-e.c., and therefore contain independent triples which are not shattered. We want to show that within the independent set of all neighbors of any fixed vertex of \( G \), if one triple is shattered, than all are. We arrive at this gradually by considering various special cases.

**Lemma 9.8.** Let \( G \) be a triangle free graph with properties \( E_2 \) and \( \text{Adj}_3 \), and suppose that \( a, b, c \) is a shattered independent triple. If \( a', b, c \) is another independent triple with \( a, a' \) adjacent, then \( a', b, c \) is also shattered.

**Proof.** Since we have property \( \text{Adj}_3 \), and the collection of sets that can be realized as sets of neighbors in \( a', b, c \) of vertices of \( G \) is closed under complementation, it suffices to consider a single vertex \( u \in \{a', b, c\} \) and to show that \( u \) occurs as the unique neighbor of some vertex among \( a', b/c \).

For \( u = a' \) we take \( a \) as the witnessing vertex. For \( u = b \) we take \( u' \) adjacent to \( a, b \) and not \( c \). For \( u = c \) proceed similarly.

**Lemma 9.9.** Let \( G \) be a triangle free graph with properties \( E_2 \) and \( \text{Adj}_3 \), and suppose that the triple \( (a, b, c) \) is independent and not shattered. Then there is a unique vertex \( u \) in \( \{a, b, c\} \) which does not occur as the unique neighbor among \( a, b, c \) of a vertex in \( G \).

**Proof.** There must be at least one such vertex, say \( b \). Taking \( u \) adjacent to \( b \) and not to \( a \), the neighbors of \( u \) among \( a, b, c \) will be \( b, c \); and then a common neighbor of \( u \) and \( a \) will have \( a \) as its unique neighbor among \( a, b, c \); similarly \( c \) will occur as the unique neighbor of some vertex among \( a, b, c \).

**Lemma 9.10.** Let \( G \) be a triangle free graph with properties \( E_2 \) and \( \text{Adj}_3 \), and suppose that \( I = \{a, b, c, d\} \) is an independent quadruple with \( a, b, c \) shattered, while some vertex \( u \) has \( a \) and \( d \) as its only neighbors in \( I \). Then \( (b, c, d) \) is shattered.

**Proof.** By the previous lemma, the triple \( u, b, c \) is shattered, and by another application of the lemma, \( b, c, d \) is shattered.
Lemma 9.11. Let $G$ be a triangle free graph with properties $E_2$ and $\text{Adj}_3$, and suppose that $I = \{a, b, c, d\}$ is an independent quadruple with $a, b, c$ shattered, while no vertex $u$ has precisely two neighbors in $I$. Then the triple $b, c, d$ is shattered.

Proof. Fix vertices $a', b', c'$ having as their unique neighbors in $a, b, c$ the vertices $a, b, c$ respectively. By our hypothesis, none of the vertices $a', b', c'$ is adjacent to $d$.

Now at least two of the vertices $a', b', c'$ are nonadjacent; let $u_1, u_2$ be two such. Take $v$ adjacent to $u_1, u_2, a, b, c, d$. Then $v$ is adjacent to at most two vertices of $a, b, c, d$ and hence, by our hypothesis, to at most one; that is, $v$ is adjacent to $d$ and not to $a, b, c$.

Suppose now that $u_1 = a'$. Then $I' = \{a', b, c, d\}$ is an independent quadruple with $a', b, c$ shattered, and $v$ has only $a'$ and $d$ as its neighbors in $I'$. By the previous lemma, $b, c, d$ is shattered.

So we may suppose that $u_1 = b'$ and $u_2 = c'$. Then $b', c'$, and $v$ has as their unique neighbors in $b, c, d$ the vertices $b, c, d$ respectively, and so $b, c, d$ is shattered.

\[ \square \]

Lemma 9.12. Let $G$ be a triangle free graph with properties $E_2$ and $\text{Adj}_3$, and suppose that $I = \{a, b, c, d\}$ is an independent quadruple with $a, b, c$ shattered. Suppose there are vertices $b', c'$ in $G$ having as their neighbors among $a, b, c, d$ the pairs $a, b$ and $a, c$ respectively. Then $b, c, d$ is shattered.

Proof. The triple $b', c', d$ is independent. Let $d'$ be adjacent to $b', c', d$. Then the vertices $b', c', d'$ show that the triple $b, c, d$ is shattered.

\[ \square \]

Lemma 9.13. Let $G$ be a triangle free graph with properties $E_2$ and $\text{Adj}_3$, and suppose that $I = \{a, b, c, d\}$ is an independent quadruple with $a, b, c$ shattered. Suppose there are vertices $u, u'$ in $G$ having as their neighbors the pairs $a, b$ and $c, d$, and no other pairs from $a, b, c, d$ occur in this fashion. Then $b, c, d$ is shattered.

Proof. We show first

\[(5)\]

$a, b, d$ is shattered

This follows by applying Lemma 9.10 to the triples $a, b, c, a, b, u'$, and $a, b, d$.

Now if $a, c, d$ is shattered, we argue similarly that $b, c, d$ is shattered by looking at the sequence of triples $a, c, d, u, c, d, b, c, d$.

Take a vertex $a'$ adjacent to $a$ and not adjacent to $b, c$. By our hypothesis, $a'$ is not adjacent to $d$ either. Now take a vertex $v$ adjacent to $a', b, d$. Then $v$ is not adjacent to $a$, so by our hypothesis $v$ is adjacent to $c$.

We now consider two cases. First, if there is a vertex $d'$ adjacent to $d$ but not to $a, b, c$, then we apply Lemma 9.10 to the series of independent triples $a, b, d, a, b, d', a, v, d', a, c, d', a, c, d$, $a, c, d$ to conclude.

Now suppose that there is no such vertex $d'$. Then there is no vertex whose unique neighbor in $b, c, d$ is $d$. By Lemma 9.9, there is a vertex $c'$ whose unique neighbor among $b, c, d$ is $c$. Hence, by our hypothesis, $c$ is the only neighbor of $c'$ among $a, b, c, d$. If $(a, c')$ is an edge, we conclude by applying Lemma 9.10 to the sequence of independent triples

\[(a, b, d); (a', b, d); (c', b, d); (c, b, d)\]
Lemma 9.14. Let $G$ be a triangle free graph with properties $E_2$ and $\text{Adj}_3$, and let $I$ be an independent set containing some shattered triple. Then all triples of vertices from $I$ are shattered.

Proof. By the foregoing lemmas, if $(a, b, c, d)$ is any independent quadruple containing a shattered triple, then all of its triples are shattered. The general case follows. □

Proposition 9.15. Let $G$ be a triangle free graph with properties $E_2$ and $\text{Adj}_3$, but not $E_3$. Let $v$ be a vertex of $G$. Then the geometry $(P, B)$ associated to the vertex $v$ in $G$ is a linear geometry, and $G$ is the associated graph.

Proof. $G$ is certainly the associated graph, so everything comes down to recognizing the geometry on $(P, B)$, with $P$ the set of neighbors of $v$ and $B$ the set of non-neighbors.

We first choose $v$ to be a vertex of $G$ having a triple of neighbors which is not shattered, and let $(L, B)$ be the associated geometry. Once we verify that this is a linear geometry, the structure of $G$ is determined, and it follows that the same applies to the geometry at any vertex of $G$.

We next look for a linear betweenness relation on $L$. This is a ternary relation $\beta(x, y, z)$, irreflexive in the sense that it requires $x, y, z$ to be distinct, which picks out for each triple $x, y, z$ a unique element which is between the other two, i.e., $\beta(x, y, z)$ implies $\beta(z, y, x)$ and not $\beta(y, z, x)$ or $\beta(z, x, y)$. In addition to these basic properties we have the axiom:

For $x, y, z, t$ distinct, $\beta(x, y, z)$ implies $\beta(x, y, t)$ or $\beta(t, y, z)$

Any linear order gives rise to a linear betweenness relation, and the reverse order gives the same betweenness relation; conversely, a betweenness relation determines a unique pair of linear orders which give rise to it (assuming there are at least two points).

We define $\beta(x, y, z)$ on $L$ as follows: $\beta(x, y, z)$ holds if any vertex adjacent to $y$ is adjacent to $x$ or $z$; equivalently (taking complements) any vertex adjacent to $x$ and $z$ is adjacent to $y$. This is symmetric in $x$ and $z$, and by Lemma 9.9, the relation picks out each independent triple $(x, y, z)$ which is not shattered, a unique $y$ satisfying $\beta(x, y, z)$ and $\beta(z, y, x)$. Furthermore by our choice of $v$ and Lemma 9.14, none the triples in $L$ are shattered. So we only have to check the critical axiom: assuming $\beta(x, y, z)$, with $t$ a fourth vertex in $L$, we claim that $\beta(x, y, t)$ or $\beta(t, y, z)$ holds.

Suppose $\beta(x, y, z)$ holds and $\beta(x, y, t)$ fails. We will show that $\beta(t, y, z)$ holds. As $\beta(x, y, t)$ fails, we have $\beta(x, t, y)$ or $\beta(t, x, y)$.

Suppose $\beta(x, t, y)$ holds. Take a vertex $u$ adjacent to $y$ and not $t$. Then by $\beta(x, t, y)$, $u$ is not adjacent to $x$. By $\beta(x, y, z)$, $u$ is adjacent to $z$. This proves $\beta(t, y, z)$, as claimed.

Now suppose $\beta(t, x, y)$ holds, and take a vertex $u$ adjacent to $y$, but not adjacent to $z$. Then by $\beta(x, y, z)$ we have $u$ adjacent to $x$, and by $\beta(x, t, y)$ we have $u$ adjacent to $t$. Thus $\beta(t, y, z)$ holds.
Accordingly, \( \beta \) is a linear betweenness relation on \( L \) and we may fix a linear ordering giving rise to this relation. By the definition of \( \beta \), the blocks of \( B \) are convex, and are not bounded both above and below. Therefore they are initial and terminal segments of \( L \) (reversing the order will of course interchange these two notions). It remains to check that the blocks are proper, pairwise distinct, and sufficiently dense in \( L \) to satisfy our axioms for a linear geometry. This all follows from the assumption that \( G \) satisfies \( E_2 \), now that the general shape of the geometry has been established.

With this result in hand we can give a reasonably efficient axiomatization of the geometries associated with 3-e.c. graphs.

9.7. Geometries associated with 3-e.c. graphs.

**Definition 9.16.** An \( E_3 \)-geometry is a combinatorial geometry \((P, B)\) satisfying the following axioms.

1. There are no three disjoint blocks.
2. No block is contained in any other.
3. There are at least two points. For any two distinct points, there is a block containing exactly one of them, and a block containing neither.
4. If \( b \) is a block and \( p, q \) are points not in \( b \), then there is a block disjoint from \( b \) containing \( p \) and \( q \).
5. If \( b, b' \) are blocks which intersect, and \( p \) is a point outside their union, then some block containing \( p \) is disjoint from \( b \) and \( b' \).
6. If three blocks intersect pairwise, then they either have a point in common, or some block is disjoint from their union.

The \( E_3 \)-geometries are just the geometries associated with 3-e.c. triangle free graphs, as we shall show. We avoid the seemingly natural term “3-e.c. geometry”, as the natural interpretation for that term would be a considerably stronger set of conditions in which, notably, in axiom 6 we would require both a point in common and a disjoint block, that is, we would apply the 3-e.c. condition directly to the geometry, specifying the type of the element of \( P \cup B \) realizing the giving condition. The thrust of this is considerably more like 4-e.c. in the corresponding graph; in fact, it is 4-e.c. restricted to quadruples including the base point. No doubt this is an interesting class of geometries in its own right, and more tractable than those associated with 4-e.c. graphs, but still a good deal beyond anything we can construct, or analyze, at present.

**Lemma 9.17.** The geometry associated to any vertex of a 3-e.c. triangle free graph is an \( E_3 \)-geometry, and conversely the graph constructed in the usual way from an \( E_3 \)-geometry is a 3-e.c. triangle free graph.

**Proof.** One can read off all these axioms directly from the 3-e.c. property (with the triangle free condition accounting for the first of them). The point is to check that conditions \((I -- V)\) are strong enough. For that, we use the analysis of the previous subsection. By Axiom II, we exclude the linear geometries of the previous section, and therefore it suffices to check that the associated graph is triangle free, 2-e.c., and satisfies the adjacency condition \( \text{Adj}_3 \), which is more or less what the axioms assert. \( \square \)
There is still a little redundancy in the set of axioms. It is not actually necessary to assume explicitly that any two points belong to a block, as long as we assume there are at least three points, since other axioms contain very similar (and in some ways stronger) conditions. Let us use the term “weak $E_3$-geometry” momentarily for a geometry satisfying the modified axiom system where we drop the requirement on the existence of blocks containing two specified points, but we add the explicit requirement that there be at least three points.

**Lemma 9.18.** Let $(P, B)$ be a weak $E_3$-geometry. Then the union of two blocks is never $P$.

*Proof.* Suppose first that $b_1, b_2$ are two blocks which meet, and let $p \in b_2 \setminus b_1$. There is a block $b'$ containing $p$ and disjoint from $p_1$, and as $b'$ cannot be contained in $b$, it follows that $b_1 \cup b_2 \neq P$.

Now suppose that $b_1, b_2$ are disjoint and their union is $P$. We may suppose that $|b_1| \geq 2$. Take two points of $b_1$ and a block $b$ containing just one of them. As $b$ is not contained in $b_1$ and $b_1 \cup b_2$ is $P$, $b$ meets $b_2$. By construction $b \cup b_2$ is not $P$, so there is a block $b''$ disjoint from $b \cup b_2$. Then $b''$ is a proper subset of $b_1$ and we have a contradiction. □

**Lemma 9.19.** A weak $E_3$-geometry is an $E_3$-geometry.

*Proof.* Call two points of $P$ collinear if they lie in a common block. We claim this is an equivalence relation on $P$.

Suppose that $p, q \in b_1$, and $q, r \in b_2$. Take a point $a \notin b_1 \cup b_2$, and a block $b$ disjoint from $b_1 \cup b_2$ containing $a$. As $p, r$ lie outside $b$, Axiom IVa applies, and $p, r$ are collinear.

By Axiom I, there are at most two equivalence classes for the collinearity relation, and we claim there is only one.

Suppose there are two collinearity classes $P_1, P_2$. We may suppose $|P_1| \geq 2$. Take a block $b$ meeting (and hence contained in) $P_1$. As there are no inclusions between blocks, it follows from Axiom III that $b$ is a proper subset of $P_1$. Therefore by Axiom IVa we have a block meeting $P_1$ and $P_2$, a contradiction. □

At this point we are through sorting through the basic axioms and we can begin to look more closely at examples of 3-e.c. triangle free graphs and their associated geometries. We begin with a quick look at the Higman-Sims graph and strongly regular triangle free graphs in general.

**9.8. The Higman-Sims Graph.** This is constructed from the $M_{22}$ geometry, defined as follows. Let $P_0$ be the projective plane over the field of order 4, with 21 points. Adjoin an additional point $\infty$ to get $P = P \cup \{\infty\}$. The associated blocks will be of two kinds. The first kind are obtained by extending an arbitrary line $\ell$ of $P_0$ by the point $\infty$: $\ell^* = \ell \cup \{\infty\}$. The second kind are called hyperovals. A hyperoval is a set of 6 points in $P_0$ which meets any line of $P_0$ in an even number of points. There are 168 hyperovals, and on this set the relation “$|O_1 \cap O_2|$ is even” is an equivalence relation, with three classes of 56 hyperovals each, permuted among themselves by the automorphism group of the base field. Any one class of 56 hyperovals may be taken, together with the extended lines, as the set of blocks $B$ for the $M_{22}$ geometry, on 22 points. Thus there are 77 blocks, each with 6 points, and 100 vertices in the associated graph, the Higman-Sims graph. Its automorphism
group is vertex transitive and edge transitive, so we get the same geometry from any base point, and any adjacent point can play the role of the “new” point ∞.

We check that this graph is a 3-e.c. triangle free graph. Given three independent vertices, one may be taken to be the base point \( v \), and the other two will then represent two intersecting blocks of the geometry. We may suppose that they both contain the point ∞ in the associated geometry, and therefore they represent two extended lines, whose intersection has order 2. So we have the condition Adj_3 with multiplicity two, that is there are two points meeting the adjacency conditions in every case. Since the graph visibly satisfies the \( E_2 \) condition, and there are no containments between blocks, this completes the verification that it is 3-e.c. On the other hand, it is not 4-e.c. As we know, this comes down to the 4-adjacency property \( \text{Adj}_4 \). Taking a triple of points lying on a projective line \( \ell \) in \( P_0 \), and another line meeting \( \ell \) in a different point, a common neighbor of the four vertices involved would be a block containing the given three points and disjoint from the given line; but there is only one block containing three given points, so this is impossible.

The Higman-Sims graph is an example of a strongly regular triangle free graph. In general, a graph on \( n \) vertices is strongly regular with parameters \((n, k, \lambda, \mu)\) if it is regular of degree \( k \), and any pair of vertices \( v, v' \) has \( \lambda \) common neighbors if \( v, v' \) are adjacent, and \( \mu \) common vertices otherwise. In the case of triangle free graphs (\( \lambda = 0 \)), leaving aside the complete bipartite graphs and the pentagon, there are five known examples, which go by the names of the Petersen, Clebsch, Hoffman-Singleton, Gewirtz, \( M_{22} \), and Higman-Sims graphs. Two of these graphs are 3-e.c., the Clebsch graph and the Higman-Sims graph.

While there are no other known strongly regular triangle free graphs, there are many “feasible” sets of parameters, that is combinations of parameters which are compatible with all known constraints on such graphs. Following a suggestion of Peter Cameron, we will use that theory to show that there are no 4-e.c. strongly regular triangle free graphs, leaving entirely open the problem whether there are any more, or infinitely many more, 3-e.c. strongly regular triangle free graphs.

We will also take a closer look at the known strongly regular triangle free graphs, notably the Clebsch graph, which serves as the basis for the simplest construction of an infinite family of 3-e.c. graphs, first proposed by Michael Albert.

9.9. **Strongly regular graphs and properties** \( E_2, E_3 \). Leaving aside the complete bipartite graphs and the pentagon, the known strongly regular triangle free graphs have the following parameters.

1. Petersen: \((10, 3, 0, 1)\);
2. Clebsch: \((16, 5, 0, 2)\);
3. Hoffman-Singleton: \((50, 7, 0, 1)\);
4. Gewirtz: \((56, 10, 0, 2)\);
5. \( M_{22} \): \((77, 16, 0, 4)\);
6. Higman-Sims: \((100, 22, 0, 6)\)

The “0” here simply says that the graph is triangle free. The Gewirtz graphs and the \( M_{22} \) graph can be seen naturally inside the Higman-Sims graph (the Hoffman-Singleton graph, less naturally): the Gewirtz graph is the graph on the hyperovals of the \( M_{22} \) geometry, which could be viewed as the set of vertices in Higman-Sims nonadjacent to two vertices lying on an edge. The \( M_{22} \) graph is the graph on the
blocks of the $M_{22}$ geometry, and appears as the constituent of Higman-Sims on the non-neighbors of a fixed vertex.

In the Higman-Sims graph, any independent triple of vertices has exactly two common neighbors, as noted previously. In particular if the vertices represent hyperovals with two common points, then their common neighbors are both represented by points, and hence do not lie in the $M_{22}$ graph. Thus neither the Gewirtz graph nor the $M_{22}$ graph can satisfy the condition $\text{Adj}_3$.

The Hoffman-Singleton graph does not fit into this framework at all: there is no useful geometry induced on the set of neighbors of a fixed vertex. The same applies to the Petersen graph, but that graph can be viewed as the set of blocks in the geometry associated to the Clebsch graph, and the latter is indeed a 3-e.c. graph. In the case of the Clebsch graph, the geometry is extremely degenerate: it consists of all pairs from a set of order 5. Nonetheless this geometry is an $E_3$-geometry.

Thus the Clebsch graph and Higman-Sims graph both are 3-e.c., and the Petersen, Gewirtz, and $M_{22}$ graphs are naturally represented as descriptions of part or all of the associated geometries.

Any strongly regular graph other than the complete bipartite graphs and the pentagon graph will satisfy the condition $E_2$, and have no proper inclusion between blocks, so in all other cases the condition $E_n$ will be equivalent to the adjacency condition $\text{Adj}_n$.

However the condition $\text{Adj}_4$ is already incompatible with strong regularity for triangle free graphs, as we now show, following the notation of [Big09], which relies on the eigenvalue theory for strongly regular graphs, expressing everything in terms of the minimal eigenvalue for the adjacency matrix of the graph. So we begin by reviewing that material.

9.10. **Eigenvalues and $E_4$ in the strongly regular case.** Let $G$ be strongly regular with parameters $(n, k, \lambda, \mu)$. Let $A$ be the $n \times n$ adjacency matrix for $G$, with 0 entries for non-adjacent pairs of vertices, and 1 for adjacent pairs. With $J$ the $n \times n$ matrix consisting entirely of 1’s, the condition of strong regularity, with the specified parameters, translates into the matrix condition

$$A^2 + (\mu - \lambda)A - (k - \mu)I = \mu J$$

and as $J$ has the eigenvalues $n$ with multiplicity 1 and 0 with multiplicity $n - 1$, $A$ has three eigenvalues: $k$ with multiplicity 1, and two eigenvalues $\alpha, \beta$ which are roots of the quadratic equation

$$x^2 + (\mu - \lambda)x - (k - \mu) = 0$$

with multiplicities $m_\alpha, m_\beta$ satisfying

$$m_\alpha + m_\beta = n - 1; m_\alpha \cdot \alpha + m_\beta \cdot \beta = 0$$

since the trace of $A$ is zero. Specializing to the case $\lambda = 0$ this gives the following formulas in terms of the parameter $s = \sqrt{\Delta}$, $\Delta$ being the discriminant $\mu^2 + 4(k - \mu)$, an integer in the nontrivial cases (leaving aside the pentagon and complete bipartite graph):

$$\alpha = \frac{s - \mu}{2}, \beta = \frac{-s - \mu}{2}$$
and following [Big09] we write $q$ for the eigenvalue of minimal absolute value ($\alpha$, above) and express everything in terms of $q$ and $\mu$ as follows.

\[
\begin{align*}
k &= (q + 1)\mu + q^2 \\
n &= (q^2 + 3q + 2)\mu + (2q^3 + 3q^2 - q) + (q^4 - q^2)\mu \\
&\quad - (q + 2)k + (q^3 + q^2 - q) + (q^4 - q^2)/\mu \\
\mu &\leq q(q + 1)
\end{align*}
\]

The inequality on $\mu$ is not obvious but is derived rapidly from elementary considerations of linear algebra in [Big09].

We note that the extremal values $\mu = q(q + 1)$, $k = q^3 + 3q^2 + q$, $n = q^2(q + 3)^2$ satisfy all known feasibility constraints and give $k \approx (q + 1)^3$, $n \approx (q + 1\frac{1}{2})^4$, so that $n^4$ and $k^3$ are fairly close, and as we will see in a moment this makes the refutation of condition $E_4$ a little delicate. Namely, given $E_4$, or what amounts to the same thing, $\text{Adj}_4$, our condition is that the collection of independent 4-tuples of vertices $(u_1, u_2, u_3, u_4)$ should be covered by the independent 4-tuples lying in neighborhoods of the vertices, with the former being slightly less than $n^4$, the latter approximately $n \cdot k^4$, leading to an estimate of roughly the form $k^4 > n^3$. At least for large values of $n$ it is clear that this will not be satisfied at the extreme values and is less likely to hold lower down. But we will work through this more precisely to get the following.

**Proposition 9.20.** There is no strongly regular triangle free graph with the property $E_4$.

**Proof.** Begin with the estimates

\[
\begin{align*}
n &= (q + 2)k + (q^3 + q^2 - q) + (q^4 - q^2)/\mu \\
&\geq (q + 2)k + (q^3 + q^2 - q) + (q^2 - q) \\
&\quad - (q + 2)k + (q^3 + 2q^2 - 2q) \\
&\geq (q + 3)k - (\mu + 2q) \\
&\geq (q + 3)k - (q^2 + 3q)
\end{align*}
\]

Now the number of independent quadruples of vertices in our graph $G$ is at least $n[n - (k + 1)][n - 2(k + 1) + \mu][n - 3(k + 1) + \mu]$ and assuming $E_4$, they all occur in neighborhoods of individual vertices, so the number is at most $nk(k-1)(k-2)(k-3)$. So we have

\[
[n - (k + 1)][n - 2(k + 1) + \mu][n - 3(k + 1) + \mu] \leq k(k - 1)(k - 2)(k - 3)
\]

and we show this is impossible.

We have

\[
[n - (k + 1)] \geq (q + 2)k - (q^2 + 3q + 1) \\
\geq (q + 2)(k - (q + 1))
\]

and

\[
(n - 2(k + 1) + \mu)(n - 3(k + 1) + \mu) \geq [(q + 1)k - (2q + 2)][qk - (2q + 3)] \\
\geq q(q + 1)(k - 2)(k - (2 + 3/q))
\]

Furthermore $q(q + 1)(q + 2) \geq k + q$, so

\[
[n - (k + 1)][n - 2(k + 1) + \mu][n - 3(k + 1) + \mu] \geq (k + q)(k - (q + 1))(k - 2)[k - (2 + 3/q)]
\]
Suppose $\mu \geq q + 1$. Then $k \geq (q + 1)^2 + q^2 \geq 2q(q + 1)$, so
\[(k + q)(k - (q + 1)) = k^2 - k - q(q + 1) \geq k(k - \frac{1}{2})\]
If we take $q \geq 6$ as well then we have the estimate
\[(n - 2(k + 1) + \mu)(n - 3(k + 1) + \mu) \geq k(k - \frac{1}{2})(k - 2\frac{1}{2}) > k(k - 1)(k - 2)(k - 3)\]
ruled out this case.

For $q < 6$ we may consult the tables in [Big09]. The two cases in which the necessary inequality holds are one for $q = 2$, namely the Higman-Sims graph, already ruled out, and one for $q = 3$, with parameters $n = 324$, $k = 57$, $\mu = 12$, where oddly enough the two sides are exactly equal. One way to eliminate this is to show that the block size is too small: later we will give a lower bound of 19 for the minimal block size in a geometry associated with a 4-e.c. graph.

There remains the marginal case $\mu \leq q$. In this case as $q \leq 2q^2 + q$ we have $n \geq (q + 3)k + q^3 - 3q \geq (q + 3)k + 3$ for $q \geq 3$ and thus we can use the crude estimate
\[n - (k + 1)(n - 2(k + 1)) = n - 3(k + 1) \geq \frac{(q + 2)k}{k^2} \geq k^4\]
to reach a contradiction. \hfill \Box

Since we have quoted a lower bound for the block sizes in geometries associated with 4-e.c. triangle free graphs, we will give that next.

9.11. $E_4$-geometries: Block size. We will refer to a geometry associated with a 4-e.c. triangle free graph as an $E_4$-geometry. We will not try to write out the axioms explicitly. These would consist of conditions encoding the 2-e.c. property as was done in the case of $E_3$-geometries, the condition that no block is contained in another, to eliminate the degenerate case of a linear geometry, and finally the main axioms which correspond to the adjacency condition $\text{Adj}_4$ which takes on various forms in the geometrical context depending on how the various vertices are interpreted in the geometry. We may use any instance of the 4-e.c. condition, and are not confined to the special cases corresponding directly to our reduced set of axioms.

We omit the elementary proofs of the next few lemmas.

**Lemma 9.21.** Let $(P, B)$ be an $E_4$-geometry, let $b, b_1$ be intersecting blocks, and let $b_2$ be any other block. Then 
\[|(b \cap b_1) \setminus b_2| \geq 2\]

**Lemma 9.22.** Let $(P, B)$ be an $E_4$-geometry, let $b, b_1$ be intersecting blocks, Then 
\[|b \cap b_1| \geq 5\]

**Lemma 9.23.** Let $(P, B)$ be an $E_4$-geometry, and let $b, b_1, b_2$ be blocks with a point in common, and $b_3$ a block meeting $b$ but disjoint from $b_2, b_3$. Then 
\[|b \setminus (b_1 \cup b_2 \cup b_3)| \geq 5\]

**Proposition 9.24.** Let $(P, B)$ be an $E_4$-geometry. Then any block contains at least 19 points.
Proof. Let \( b \) be a block. We claim first that there are \( b_1, b_2 \) with \( b \cap b_1 \cap b_2 \neq \emptyset \), and with \(|(b \cap b_2) \setminus b_1| \geq 4\).

Begin with \( b \cap b_1 \cap b_2 \) nonempty and with the three blocks distinct, and suppose \(|(b \cap b_2) \setminus b_1| \leq 3\). Take a block \( b_3 \) so that:

\[
|b_3 \cap [(b \cap b_2) \setminus b_1]| = 1; b_3 \cap b_1 = \emptyset
\]

Then \(|b \cap b_2 \cap b_3| = 1\) and \((b \cap b_3) \setminus b_2| \geq 4\).

So fix blocks \( b_1, b_2 \) with \( b \cap b_1 \cap b_2 \neq \emptyset \), and with \(|(b \cap b_2) \setminus b_1| \geq 4\). Then \(|b \cap (b_1 \cup b_2)| \geq 9\). Now take \( b_3 \) disjoint from \( b_1, b_2 \) and meeting \( b \). Then \(|b \cap (b_1 \cup b_2 \cup b_3)| \geq 14\). And then by the previous lemma \(|b| \geq 19\). \(\square\)

It would be good to have a more sophisticated lower bound here. We can convert this bound into a crude but decent lower bound for the number of points in such a geometry.

Lemma 9.25. Let \((P, B)\) be an E\(_4\)-geometry. Then there are intersecting blocks \( b_1, b_2 \) with \(|b_1 \cup b_2| \geq 33\).

Proof. Let \( m = \min(|b_1 \cap b_2|: b_1 \cap b_2 \neq \emptyset)\). If \( m = 5 \) we are done.

Suppose \( m \geq 6\). Take any two intersecting blocks \( b_1, b_2 \). Take a block \( b \) meeting \( b_1 \) and disjoint from \( b_2 \). Take a point \( p \) in \( b_1 \setminus (b_2 \cup b) \), and a block \( b' \) disjoint from \( b_2 \) containing \( p \).

Then \( b_1, b, b' \) meet pairwise and hence by the 4-e.c. condition have a point in common. Furthermore \( b_1 \) meets \( b_2 \) and \( b, b' \) are disjoint from \( b_2 \). So by Lemma 9.23, \(|b_1 \setminus (b \cup b' \cup b_2)| \geq 5\). Furthermore \(|(b_1 \cap b') \setminus b| \geq 2\). So \(|b_1 \setminus (b \cup b_2)| \geq 7\) and \(|b_1 \setminus b_2| \geq m + 7 \geq 13\).

Also, in the proof of the previous lemma, the lower bound obtained on the block size is actually \( m + 4 + m + 5 \) which with \( m \geq 6\) would give a block size of at least 21 and a lower bound for \(|b_1 \cup b_2|\) of at least 34 in this case. \(\square\)

Lemma 9.26. Let \((P, B)\) be an E\(_4\)-geometry, \( n = |P| \). Then \( n \geq 66\).

Proof. Take intersecting blocks \( b_1, b_2 \) with \(|b_1 \cup b_2| \geq 33\). Let \( m = \min(|b_3 \cap b_4|: b_3 \cap b_4 \neq \emptyset, (b_3 \cup b_4) \cap (b_1 \cup b_2) = \emptyset)\). If \( m = 5 \) the result is immediate, so take \( m = 6 \) and argue as in the previous lemma. \(\square\)

Another way of stating all of this is as follows.

Corollary 9.27. Let \( G \) be a 4-e.c. graph. Then every vertex has degree at least 66, every pair of independent vertices has at least 19 common neighbors, and every triple of independent vertices has at least 5 common neighbors.

In the Higman-Sims graph the corresponding numbers are 22, 6, and 2.

It might be of interest to explore the known feasible parameter sets for strongly regular triangle free graphs to see which seem compatible with the \( E_3 \)-condition. At the extreme value \( \mu = q(q + 1) \), the ratio of \( k(k - 1)(k - 2) \) to \(|n - (k + 1)][n - 2(k + 1) + \mu]\) is \( q \), which in the two known cases of the Clebsch and Higman-Sims graphs actually corresponds to the condition \( \text{Adj}_3 \) with multiplicity \( q \). There are many other cases where the necessary inequality is satisfied with smaller values of \( \mu \). In fact, the majority of the cases listed in the appendix to [Big09] meet this condition. The only case consistent with this inequality in which the \( \text{Adj}_3 \) condition is known to fail is the case of the \( M_{22} \) graph.
In a similar vein, dropping the $E_4$ condition, we add some comments on the relationship between the multiplicity with which $\text{Adj}_3$ is satisfied, and the multiplicity with which $\text{Adj}_2$ is satisfied.

Let $G$ be a 3-e.c. triangle free graph. Define $\mu_n(G)$ as the minimum over all independent sets $I \subseteq G$ of order $n$ of the cardinality of the set of common neighbors of $I$. Thus for example in the Higman-Sims graph, $\mu_2(G) = 6$ and $\mu_3(G) = 2$.

**Lemma 9.28.** Let $G$ be a 3-e.c. triangle free graph. If $\mu_3(G) \geq 2$ then $\mu_2(G) \geq 5$.

**Proof.** Fix two vertices $u_1, u_2 \in G$, and $v$ adjacent to both. With $v$ as base point work in the associated geometry. We look for 4 blocks containing the points $u_1, u_2$.

Let $b_1$ be a block containing $u_1, u_2, u_3$ ($\mu_3(G) = 2$), and $u_4 \in P \setminus b_1 \cup b_2$. Let $b_3$ be a block containing $u_1, u_2, u_4$. It suffices to show that $b_1 \cup b_2 \cup b_3 \neq P$.

Let $b$ be a block containing $u_4$ and disjoint from $b_1, b_2$. Then $b \setminus b_3 \subseteq P \setminus (b_1 \cup b_2 \cup b_3)$. \square

So we have what appears to be a sharply descending series of successive weakenings of the $E_4$ condition, with no known examples of even the weakest condition other than subgraphs of the Higman-Sims graph.

1. $G$ is 4-e.c.
2. $\mu_3(G) \geq 5$
3. $\mu_3(G) \geq 2$
4. $\mu_2(G) \geq 5$
5. $\mu_2(G) \geq 3$
6. In the geometry associated to some base point, every block contains at least 3 points.

What we can do, as mentioned, is produce an infinite family of $E_3$-geometries with a unique block of order 2, but even in this case we get no bound on the number of blocks of order 2 for other geometries associated with the same graph, at different basepoints.

We return now to the case of $E_3$-geometries. The first examples of an infinite family of 3-e.c. triangle free graphs was given by Michael Albert. An examination of the corresponding geometries leads naturally to the consideration of a more general class of geometries with rather special properties. From one point of view these examples are degenerate: on the other hand, examples can be constructed naturally from projective geometries.

### 10. Some $E_3$-Geometries

#### 10.1. Albert geometries.

We first present Michael Albert’s original construction. Observe that the Clebsch graph can be represented as a collection of 4 copies of a 4-cycle, related systematically to one another: a vertex in one copy will be connected only to one vertex in any other copy, namely the vertex corresponding to the diagonally opposed to it. Evidently the same recipe can be extended to any number of copies of a 4-cycle, and any triple of vertices in one of these extended graphs embeds into a copy of the Clebsch graph; so the “stretched” Clebsch graph inherits the 3-e.c. property from the Clebsch graph.

In terms of the associated combinatorial geometry, the Clebsch graph corresponds to the geometry on 5 points in which every pair is a block. The stretched
Clebsch graphs correspond to a geometry on \( n \) points, \( n \geq 5 \), in which the blocks are of two sorts: (i) all the pairs containing either of two fixed points; (ii) all the sets of points of order \( n - 3 \) not containing either of those two points.

**Definition 10.1.**

1. A point \( p \) in a combinatorial geometry \((P, B)\) will be said to be isolated if every pair of points containing \( p \) is a block.

2. An \( E_3 \)-geometry will be called an Albert geometry if it has at least one isolated point.

**Lemma 10.2.** For \( n \geq 5 \), there is a unique Albert geometry on \( n \) points having two isolated points.

**Proof.** If \( p, q \) are two isolated points, and \( b \) is a block not containing \( p \) or \( q \), then \( |b| \geq n - 3 \) as otherwise there will be three pairwise disjoint blocks.

Let \( a \in P \), \( a \neq p, q \). There is a block \( b \) disjoint from the blocks \{\( a, p \)\} and \{\( a, q \)\}, and as \( |b| \geq n - 3 \), \( b = P \setminus \{a, p, q\} \). So the identification of the geometry is complete. \( \square \)

We will look at some examples of Albert geometries with a unique isolated point. We do not expect that one can classify these without some further restrictions. In general, the geometry in the associated graph will depend on the base point selected, so it is noteworthy that if one of these geometries is an Albert geometry, then they all are. We note that even the number of points in the geometry may depend on the base point, in other words the corresponding graphs are not regular in general. For the specific case of the geometry with two isolated points just described, the corresponding graph is vertex transitive (as is clear from Albert’s original description of it), so the same geometry is obtained from any base point.

If we remove an isolated point from an Albert geometry and look at the geometry introduced on the remaining points, we get a reasonable class of geometries. This point of view is useful for the construction of examples.

**Definition 10.3.** Let \((P, B)\) be an Albert geometry, \( a \) an isolated point. The derived geometry \((P_0, B_0)\) with respect to \( p \) has point set \( P_0 = P \setminus \{a\} \), and blocks \( B_0 = \{b \in B : a \notin b\} \).

The reconstruction of \((P, B)\) from \((P_0, B_0)\) is immediate. We can phrase the \( E_3 \)-conditions directly in terms of the derived geometry as follows.

I-D There are at least two points.

II-D For any three points \( p, q, r \) there is a block containing \( p, q \), and not \( r \).

III-D There is no inclusion between distinct blocks.

IV-D For any block, its complement is a block.

V-D If \( b_1, b_2, b_3 \) are blocks intersecting pairwise, but with no common point, then \( b_1 \cup b_2 \cup b_3 \neq P_0 \).

**Lemma 10.4.** If \((P, B)\) is an Albert geometry with \( a \) an isolated point then the derived geometry with respect to \( a \) satisfies Axioms I-D through V-D. Conversely, if \((P_0, B_0)\) is a derived geometry, then the combinatorial geometry on \( P = P_0 \cup \{a\} \) whose blocks are the blocks of \( B_0 \) together with the pairs containing \( a \) is an Albert geometry.

We omit the verification. Using this result, we can give examples in a very convenient form.
Examples 1.

1. Let \((P_0, H)\) be a projective geometry with \(H\) the set of hyperplanes. Let \((P_0, B_0)\) have as its blocks the blocks of \(H\) and their complements. This is a derived geometry.

2. Let \((P_0, L)\) be a projective plane and let \(L'\) be a set of lines satisfying one of the following conditions:
   (a) Every point lies on at least 3 lines of \(L'\);
   (b) \(L'\) contains all the line lines of \(L\) not passing through some fixed point \(p\), and two of the lines passing through \(p\).

   Let \(L^*\) consist of the lines in \(L'\) and their complements. Then \((P_0, L^*)\) is a derived geometry.

3. Let \((P_0, L)\) be a projective plane and \(\ell \in L\) a fixed line. Let \(L_\ell\) consist of the line \(\ell\) together with the sets \(\ell \cup \ell_1 \setminus \ell \cap \ell_1\) as \(\ell_1\) varies over the remaining lines. Taking these sets and their complements as blocks, we get an Albert geometry, to which we return below.

4. Let \((P_0, H)\) be as in (1) and let \(\infty\) be an additional point. Extend \((P_0, H)\) to \((P_1, H_1)\) with \(P_1 = P_0 \cup \{\infty\}\), \(H_1 = \{h \cup \{\infty\} : h \in H\}\), and let \(H_1^*\) consist of the blocks in \(H_1\) and their complements. \((P_1, H_1^*)\) Then \((P_1, H_1^*)\) is a derived geometry.

5. Let \((P_0, B_0)\) be a derived geometry, and \(a \in P_0\). Let \(P_a = \{b \in B_0 : a \not\in b\} \cup \{\infty\}\) with \(\infty\) a new point. Let \(B_a = P_0 \setminus \{a\}\). For \(p \in B_a\) let \(b_p = \{b \in B_0 : a \not\in b\} \cup \{\infty\}\). Let \(B_a^*\) consist of \(B_a\) and its complements in \(P_a\). Then \((P_a, B_a^*)\) is a derived geometry, as we shall see below. If \(|P_0| = n\) and \(|B_0| = 2n'\), we will say that the derived geometry has parameters \((n, n')\). Then \((P_a, B_a)\) has parameters \((n' + 1, n - 1)\). If we repeat this construction to form the geometry \(P_{\infty, B_{\infty}}\), we recover \((P_0, B_0)\).

   For example: the derived geometry associated with a projective geometry has type \((n, n)\) and the new geometry \((P_a, B_a)\) thus has parameters \((n + 1, n - 1)\).

As we are ultimately interested in graphs, we need to consider the effect of a change of basepoint.

Proposition 10.5. Let \((P, B)\) be an Albert geometry, \(G\) the associated graph, and \(u \in G\) any vertex. Then the geometry \((P_a, B_a)\) associated to the graph with respect to the base point \(u\) is again an Albert geometry.

Proof. Since \(G\) is connected, it suffices to prove the claim when \(u\) is a point of the original Albert geometry, in other words \(u\) is adjacent to the base point \(v\) of \(G\).

   The case in which \(u\) is an isolated point of \((P, B)\) must be handled separately. In this case, there is an involution \(i \in \text{Aut}(G)\) defined by
   \[
   v \leftrightarrow u; a \leftrightarrow \{a, u\}; b \leftrightarrow P \setminus b \text{ on blocks}
   \]

   Thus in this case the geometry associated with \(u\) is isomorphic to the original geometry \((P, B)\).

   Suppose now that \(p_0 \in P\) is an isolated point, and \(u \neq p_0\). Let \(P_1 = \{p_0, u\} \subseteq P_a\). We claim that \(p_1\) is an isolated point of \(P_a\).

   The vertex \(p_0 \in B_a\) is incident with \(\{v, p_1\}\) in \(P_a\). For any other \(b \in P_a \setminus \{v, p_1\}\), we have \(b \in B_0\), \(u \not\in b\), and \(b' = P_0 \setminus b\) is in \(B_a\), with \(b'\) incident with \(p_1\) and \(b\).

   Thus \(p_1\) is an isolated point. \(\square\)
Tracing through the construction of the associated geometry at the level of the derived geometries, we arrive at our fourth example above. Hence that construction takes derived geometries to derived geometries.

**Lemma 10.6.** Let \((P, B)\) be an Albert geometry, \(p_0 \in P\) an isolated point, and \((P_0, B_0)\) the corresponding derived geometry with parameters \((n, n')\) where \(n = |P_0|\) and \(n' = |B_0|/2\). Let \(G\) be the graph associated with \((P, B)\), with base point \(v\). Then for any vertex \(u \in B_0 \cup \{p_0, v\}\), the associated geometry with base point \(u\) has the same parameters \((n, n')\), while for \(u \in P_0 \cup (B \setminus B_0)\), the associated geometry has parameters \((n' + 1, n - 1)\). Thus at most two vertex degrees in the associated graph.

**Proof.** Let \(g\) be the order of the graph \(G\). We have \(g = 1 + (n + 1) + (2n' + n) = 2n + 2n' + 1\). For \(p \in P_0\), the geometry \((P, B_p)\) has parameters \((n_p, n'_p)\) where \(n_p = |P_p| - 1\) with \(P_p\) the set of neighbors of \(p\) in \(G\), namely \(v\), \(\{p, p_0\}\), and the blocks in \(B_0\) which contain \(p\): so \(n_p = n' + 1\), and therefore \(n'_p = n - 1\).

Making use of the involution \(v \leftrightarrow p_0\) which interchanges \(B \setminus B_0\) and \(P_0\), the same parameters are associated with \(u \in B \setminus B_0\).

On the other hand, for a block \(b \in B_0\), the set \(P_b\) of neighbors of \(b\) consists of the points \(p\) belonging to \(b\), the pairs \(\{p_0, p\}\) with \(p \in P_0\) not belonging to \(b\), and the complement \(b'\) of \(b\) in \(P_0\), leading to \(n_b = |P_b| - 1 = |B_0| = n\) and thus \(n'_b = n'\).

This raises the question as to what sorts of geometries are associated on the one hand with regular graphs, and on the other hand with graphs having just two vertex degrees. These conditions do not seem to be very restrictive, and it may be of interest to impose similar conditions generalizing strong regularity, perhaps allowing some further use of algebraic methods. As an example, if we begin with the Albert geometry based on a projective geometry with a single hyperplane removed, we get a regular graph.

**Example 1.** Let \(G\) be the graph associated to the Albert geometry whose derived geometry comes from the projective plane. Let \(v\) be the base point, \(p_0 \in P\) the isolated point, and \(\ell \in B\) a line. The associated geometry has points \(P_\ell\) consisting of:

\[
\begin{cases}
\text{points } p \in P_0 \text{ lying on } \ell \\
\text{co-points } \{p, p_0\} \text{ with } p \notin \ell
\end{cases}
\]

The for \(\ell_1 \neq \ell\) a line, the points of \(P_\ell\) incident with \(\ell\) as a block in \(B_\ell\) are

\[
\{p \in \ell \setminus \ell_1\} \cup \{\hat{p} : p \in \ell_1 \setminus \ell\}
\]

In other words, this corresponds to the union of the fixed line \(\ell\) with \(\ell_1\), with their common point removed. The block associated with the base point \(v\) is \(\{p : p \in \ell\}\).

In the specific case of the projective plane of order 2, the resulting geometry again comes from the projective plane of order 2. Otherwise, the geometry is a different one, and the automorphism group of the graph leaves the pair \(\{v, p_0\}\) invariant, and is \(\text{Aut}(P_0, B_0) \times \mathbb{Z}_2\). But for \(q = 2\) the group is transitive on \(B_0 \cup \{p_0, v\}\).

10.2. **Another series of \(E_7\)-geometries.** Moving away from Albert geometries, what we would like to see next is an infinite family of 3-e.c. graphs \(G\) with \(\mu_2(G) \to \infty\), ideally even \(\mu_3(G) \to \infty\). But we are far from this. Leaving aside the \(M_{22}\) geometry with its remarkably good properties \((\mu_2(G) = 6, \mu_3(G) = 2)\), in **infinite families** the best we have done to date is to reduce the number of blocks of order
2 to a single one, while all other blocks can be made arbitrarily large. But in this construction we deal with single geometry, rather than the set of geometries associated with a given graph. So there is much to be improved on even at this weak level.

**Example 2.** With $m_1, m_2, m_3 \geq 2$, let the geometry $A(m_1, m_2, m_3)$ be defined as follows. Our point set consists of three disjoint sets $P_1, P_2, P_3$ with $|P_i| = m_i$, together with two distinguished points $p', p^-$, and the following blocks.

1. $b_0 = \{p, p^-\}$ (size 2);
2. For $a \in P_i$: $a' = P_i \setminus \{a\} \cup \{p'\}$ (size $m_i$);
3. For $a \in P_i$: $a^+ = \{a\} \cup P_{i+1}$ (addition modulo 3) (size $m_{i+1} + 1$);
4. For $a \in P_i$: $a^- = \{a\} \cup P_{i-1} \cup \{p^-\}$ (size $m_{i-1} + 2$).

If $m_1, m_2, m_3 \geq 2$ then $A(m_1, m_2, m_3)$ is an $E_3$-geometry, and if $m_1, m_2, m_3 \geq 3$ then it has a unique block of order 2, with the other blocks of order at least $\min(m_1, m_2, m_3)$. Let $G(m_1, m_2, m_3)$ be the associated graph.

**Lemma 10.7.** The dihedral group of order 8 acts on $G = G(m_1, m_2, m_3)$ as a group of automorphisms, extending the natural action on the 4-cycle $\langle v, p', b_0, p^- \rangle$, with $v$ the base point, $b_0 = \{p', p^-\}$. In this action the classes $P_0 = P_1 \cup P_2 \cup P_3$, $B' = \{a' : a \in P_0\}$, $B^+ = \{a^+ : a \in P_0\}$, and $B^- = \{a^- : a \in P_0\}$ are permuted.

**Proof.** We define two involutions in $\text{Aut}(G)$ by:

$$
\iota' : \quad v \leftrightarrow p' \quad b_0 \leftrightarrow p^- \quad a \leftrightarrow a' \quad a^+ \leftrightarrow a^-
$$

$$
\iota^- : \quad v \leftrightarrow p^- \quad b_0 \leftrightarrow p' \quad a \leftrightarrow a^- \quad a^+ \leftrightarrow a'^-
$$

□

**Lemma 10.8.** The graph $G(m_1, m_2, m_3)$ is regular.

**Proof.** It suffices to check the degree of a vertex $a \in P_i$. The neighbors of $a$ are $v, a^+, a^-$ and $a'_e$ for $a_1 \in A_1 \setminus \{a\}$, $b^-$ for $a \in A_{i-1}$, $c^+$ for $c \in A_{i-1}$, for a total of $3 + (m_i - 1) + m_{i-1} + m_{i+1}$ which is the degree of the base point. □

The geometries associated with $G(m_1, m_2, m_3)$ (i.e., giving an isomorphic graph) are not well behaved.

**Lemma 10.9.** Let $G = G(m_1, m_2, m_3)$ corresponding to $A(m_1, m_2, m_3)$ with point set $P_1 \cup P_2 \cup P_3 \cup \{p', p^-\}$, and take $a \in P_i$. Then the geometry $(P_a, B_a)$ associated with the base point $a$ has at least $m_i + 1$ blocks of order 2, and exactly $2(m_i - 1)$ blocks of order $m_1 + m_2 + m_3 - 1 = n - 2$ with $n = |P|$ the degree of $G$. On the set $P^*_i = \{a'_1 : a_1 \neq a, a_1 \in P_i\} \cup \{a^+, a^-\}$ of order $m_i + 1$, the induced geometry is the Albert geometry with two isolated points $a^+, a^-$, in the sense that all pairs $\{a^+, a'_1\}$ and $\{a^-, a'_1\}$ occur as blocks of the associated geometry $(P_a, B_a)$, while all subsets of order $n - 3$ contained in $P^*_i$ which are disjoint from $\{a^+, a^-\}$ and which contain $P^*_i \setminus P^*_i$ also occur as blocks.

Note that in the “restricted” geometry on $P^*_i$ we are taking as blocks, those which lie within $P^*_i$, and those which contain its complement.

**Proof.**

$$
P_a = \{v, a^+, a^-, b'(b \in P_i, b \neq a), c^-(c \in P_{i+1}), d^+(d \in P_{i-1})\}
$$
The block associated with \( a' \) is \( \{a^+, a^-\} \). The block associated with \( b^+ \) for \( b \in A_i \), \( b \neq a \) is \( \{a^-, b'\} \). The block associated with \( b^- \) for \( b \in P_i \) is \( \{a^+, b'\} \).

Finally, the block associated with \( b \in P_i \) (\( b \neq a \)) is

\[
\{v\} \cup P_{i+1}^+ \cup A_{i-1}^- \cup \{b'_1 : b_1 \in P_i, b_1 \neq a, b\}
\]

\[\square\]

We observe that the smallest geometry for which we are able to get a single block of order 2 has order 11, namely \( A(3,3,3) \), and this is sharp. We will give some additional information concerning small geometries.

10.3. **E\(_3\)**-Geometries of order at most 7.

**Lemma 10.10.** Let \((P, B)\) be an \( E_3 \)-geometry, and \( n = |P| \).

1. The maximal block size is at most \( n - 3 \).
2. If there is a block of order \( n - 3 \) then all pairs lying in its complement are blocks.
3. \( n \geq 5 \).
4. If \( p \) is a point which is contained in at least \( n - 3 \) blocks of order 2 then \( p \) is isolated.
5. If \( n = 5 \) or 6 then \((P, B)\) is the Albert geometry with two isolated points.

**Proof.** The first three points are immediate. For the fourth, let \( q, q' \) be the two points not known to occur together with \( p \) as a block of order 2. Take a block containing \( p, q \) and not \( q' \); it must be \( \{p, q\} \). Similarly \( \{p, q'\} \) is block.

For the last point, the case \( n = 5 \) is immediate, so take \( n = 6 \). Then there is a pair of points \( p, q \) which do not constitute a block, and therefore they are contained in two blocks of order 3, say \( \{p, q, r\} \) and \( \{p, q, s\} \). Let \( t \notin \{p, q, r, s\} \). Then the pairs containing \( t \) and neither of \( p, q \) are blocks, by our second point. So \( t \) is an isolated point. So the geometry has two isolated points, and we are done. \[\square\]

We will also carry through the analysis for the case \( n = 7 \), finding in this case that the geometry is necessarily an Albert geometry, and that there are only two possibilities: the Albert geometry with two isolated points, and one other.

**Example 3.** We construct a derived geometry \( B'(n_1, n_2) \) as follows. Let \( A_1, A_2 \) be sets of orders \( n_1, n_2 \) respectively, and \( c \) an additional point. Take as blocks:

\[
a_1A_2(a_1 \in A_1); a_2A_1(a_2 \in A_2); cA'_i(A'_i \subseteq A_i, |A_i \setminus A'_i| = 1)
\]

Let \( B(n_1, n_2) \) be the corresponding Albert geometry, with \( n_1 + n_2 + 2 \) points.

If \( n_1 = n_2 = 2 \) then this is the Albert geometry with two isolated points on 6 vertices. Otherwise, it is an Albert geometry with one isolated vertex. In particular we have the case \( n_1 = 2, n_2 = 3 \) of order 7.

** Lemma 10.11.** Let \((P_0, B_0)\) be the derived geometry associated with an Albert geometry, with \( n_0 = |P_0| > 4 \). Call the blocks of order 2 in \( B_0 \) edges, and view \( P_0 \) as a graph with respect to these edges. Then all edges in \( P_0 \) have a common vertex.

**Proof.** As \( n_0 > 4 \) there can be no disjoint edges. So we need only eliminate the possibility that there is a triangle.

Suppose \( p, q, r \) form a triangle: any pair is a block. Take further points \( s, t \), and a block \( b \) containing \( r, s \) but not \( t \). Then \( b \) is disjoint from \( p, q \) and hence the complement of \( b \) is \( p, q \). But then \( t \) is in \( b \), a contradiction. \[\square\]
Lemma 10.12. An Albert geometry on 7 points with one isolated point must be isomorphic with $B(2, 3)$.

Proof. We work in the derived geometry $(P_0, B_0)$ on 6 points, and consider the graph on $P_0$ whose edges are the pairs occurring as blocks in $B_0$. If there are three or more edges then their common vertex is a second isolated point, a contradiction.

If there are exactly two edges we identify the geometry $B(2, 3)$ as follows. Let the vertex $v$ common to the edges be called $c$, and let the other vertices on the edges be $A = \{a_1, a_2\}$, while the remaining vertices are $B = \{b_1, b_2, b_3\}$. As there are just the two edges, the other blocks containing $c$ are triples of the form $\{c, b, b'\}$ with $b, b' \in B$, and as we may exclude any element of $B$, all such triples occur. Thus we know all the blocks containing $c$ and taking complements, we have all the blocks.

Suppose there is at most one edge, and take four points $A = \{a_1, a_2, a_3, a_4\}$ containing no edge, and let $c_1, c_2$ be the other points. Then every block containing $c_1$ and meeting $A$ is a triple, thus the blocks containing $c_1$ meet $A$ in a certain set of pairs $E_1$, and for any two points $a, a' \in A$ there is a pair in $E_1$ containing $a$ and not $a'$. If $E_1$ contains no disjoint pairs, it follows that the edge set $E_1$ forms a triangle in $A$; and if we define $E_2$ similarly, and $E_2$ contains no disjoint pairs, then $E_2$ forms a triangle in $A$. However $E_1 \cup E_2$ covers all pairs in $A$. So we may suppose that $E_1$ contains two disjoint pairs, say $\{a_1, a_2\}$ and $\{a_3, a_4\}$. We may suppose then that $E_2$ contains the pair $\{a_1, a_3\}$, and hence $B_0$ contains the blocks $\{c_1, a_1, a_2\}, \{c_1, a_3, a_4\}, \{c_2, a_1, a_3\}$ which meet pairwise but have no point in common. But as the union of these blocks is $P_0$, we contradict our axioms. □

And lastly we claim that up to this point no non-Albert geometry occurs.

Lemma 10.13. Let $(P, B)$ be an $E_3$-geometry on $n \leq 7$ points. Then $(P, B)$ is an Albert geometry.

Proof. We have dealt with the cases $n < 7$ and we suppose $n = 7$.

We call a block of order 2 an edge in $P$.

Suppose first that there is some block $A$ of order 4. We claim that any $a \in A$ lies on an edge.

Let $A = \{a, a_1, a_2, a_3\}$ and take blocks $b_1, b_2$ with $a_i, a_3 \in b_i, a \notin b_i$ for $i = 1, 2$. Take $b$ disjoint from $b_1 \cup b_2$ with $a \in b$. Since any pair disjoint from $A$ is an edge, $b$ must also be an edge.

Now as $|A| = 4$ we can find two points $a_1, a_2$ in $A$ for which there is are edges $\{p, a_1\}, \{p, a_2\}$ with a common neighbor $p \in P \setminus A$. Then $p$ lies on 4 edges and is therefore an isolated point.

From now on suppose that there is no block of order 4, and we will arrive at a contradiction.

We show first that every point lies on an edge. Suppose the point $p$ lies on no edge, so that every block containing $p$ has order 3. Take two blocks $b_1, b_2$ whose intersection is $\{p\}$. Then the complement of $b_1 \cup b_2$ is an edge $e$.

Take $q \in b_1, r \in b_2$ with $q, r \neq p$, and with $q, r$ not an edge, using the fact that there are no three disjoint edges. Take a block $b_3$ containing $q, r$, and not $p$. Then $b_1, b_2, b_3$ meet pairwise but have no common point, so they are disjoint from a block, which must be $e$. Thus $b_3 \subseteq b_1 \cup b_2$. There is a point $s \notin b_3 \cup \{p\} \cup e$. Form a block $b$ containing $q, s$ and not $p$, and take a block $b'$ disjoint from $b \cup b_3$ and containing $p$. Then $p \in b' \subseteq \{p\} \cup e$ and thus $b'$ is an edge containing $p$. 


Now consider the graph on $P$ formed by the edges. Every vertex lies on an edge, there are no three disjoint edges, and furthermore no vertex has degree greater than 3, as it would then be an isolated point of the geometry.

By the first two conditions, some vertex $p$ must have degree at least 3, and hence exactly 3. Then it follows by inspection that there is some vertex $q$ not adjacent to $p$ such that every point of $P$ other than $p, q$ is adjacent to one of the two points $p, q$. Consider a block $b$ containing $p, q$. Since $b$ cannot contain any edge at $p$ or $q$, $b$ is $\{p, q\}$, so these points are adjacent and we have a contradiction. □

10.4. **A small non-Albert geometry.** We record some further information about small $E_3$-geometries. The smallest non-Albert geometry lives on a set with 8 points, and is unique up to isomorphism. This geometry has 7 blocks of order 2.

The smallest geometry in which one has a unique block of order 2 is the geometry $A(3,3,3)$ on 11 points. If one takes the geometry of lines and hyperovals in the projective plane over a field of order 4, one gets an example of order 21. On the other hand, we have checked that such a geometry must have 13 points.

By brute force search, all of the $E_3$-geometries of order 8 may be identified. There are such 10 geometries, corresponding to 7 graphs, four of which correspond to two geometries of order 8, two correspond to one geometry of order 8, and the last corresponds to one geometry of order 8 and one of order 9. All of these geometries are Albert geometries except for one pair of geometries corresponding to a single graph. We list them as follows, including the block counts (the number of blocks of each size, from the minimum size 2 up to the maximum size). We will refer to one geometry as a “variant” of another if it defines an isomorphic graph.

$E_3$-geometries of order 8:

1. The Albert geometry with two isolated points. Block count $(13, 0, 0, 6)$.
2. Albert geometries with unique isolated points:
   a. The geometry whose derived geometry comes from a projective plane minus a line, with a vertex transitive automorphism group; Block count $(7, 6, 6)$.
   b. The geometry whose derived geometry comes from a projective plane. In the associated graph, there is also a geometry on 9 points. Block count $(7, 7, 7)$.
   c. The geometry $B(2, 4)$ with block count $(9, 4, 4, 2)$ and a variant with block count $(7, 6, 6)$.
   d. The geometry $B(3, 3)$ with block count $(7, 6, 6)$ and a variant with block count $(9, 4, 4, 2)$.
   e. An Albert geometry with block count $(8, 5, 5, 1)$ and a variant with block count $(7, 6, 6)$.
3. A pair of non-Albert geometries with block counts $(6, 10, 1, 2)$ and $(7, 6, 6)$.

11. **Appendix: Amalgamation Classes (Data from [Che98])**

The following table shows the full list of 27 amalgamation classes determined by constraints on triangles, not allowing free amalgamation, and with the associated Fraissé limit primitive. The structures are assumed to have three nontrivial 2-types, all of them symmetric (e.g., one may think of these structures as complete graphs, with 3 colors of edges). The data are taken from [Che98].
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Table 3. 27 amalgamation classes [Che98]. See §8

References


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