METRICALLY HOMOGENEOUS GRAPHS

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Abstract. We give a catalog of the known metrically homogeneous graphs, and with proofs of existence, mainly via Fraïssé theory. We also give some classification results.

Introduction

A connected graph may be considered as a metric space with the graph metric, and a disconnected graph may be taken similarly as an extended metric space with some distances infinite. A graph is metrically homogeneous if the associated extended metric space is homogeneous. In other words, the connected components are homogeneous as metric spaces, and are isomorphic to one another.

We give a catalog of the known metrically homogeneous graphs, with proofs of existence, and classifications of some of the exceptional types.

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1. THE KNOWN METRICALLY HOMOGENEOUS GRAPHS: A CATALOG

We present a catalog of the known metrically homogeneous graphs, and some classification theorems. We confine ourselves largely to the connected case. The general metrically homogeneous graph is a disjoint union of isomorphic connected metrically homogeneous graphs. If the diameter of the connected components is $\delta$, it may be useful to regard these as graphs of diameter $\delta + 1$, particularly when $\delta = 1$, in which case the ordinary homogeneous graphs $m \cdot K_n$ and $I_\infty$ arise.

1.1. The Catalog. The main class of graphs will be those we denote by $\Gamma^\delta_{K_1,K_2;C,C';S}$ (or similar, abbreviated forms). These are defined as the Fraissé limits of amalgamation classes $A^\delta_{K_1,K_2;C,C';S}$ defined below. These amalgamation classes are determined by triangle constraints and forbidden $(1, \delta)$-subspaces.

**Notation.** If $\Gamma$ is metrically homogeneous then for $v \in \Gamma$ we denote by $\Gamma_i(v)$ the set of vertices at distance $i$ from $v$, with the induced metric; this is a homogeneous metric but does not necessarily come from a graph metric, and in fact the distance 1 may not even be represented in $\Gamma_i(v)$. Since the isomorphism type of $\Gamma_i(v)$ is independent of the choice of $v$, we often write $\Gamma_i$ rather than $\Gamma_i(v)$.

If the distance 1 is represented in $\Gamma_i$ and $\Gamma_i$ is connected, then the metric on $\Gamma_i$ is the graph metric (see [Cam98]).

A metrically homogeneous graph $\Gamma$ is called *imprimitive* if it has a non-trivial $\text{Aut}(\Gamma)$-invariant equivalence relation, and is called *primitive* otherwise.

The first three groups in our catalog will be the metrically homogeneous graphs of diameter at most 2, those with $\Gamma_1$ finite or imprimitive, and those which are themselves imprimitive. The “typical” graphs from our point of view will be the graphs $\Gamma^\delta_{K_1,K_2;C,C';S}$ alluded to above, which provide all known examples falling outside the three exceptional groups, as well as most examples of the third kind.

The complete and edgeless graphs will be called *defective*, since from a model theoretic point of view they might as well be considered as structures in the empty language.

I. $\delta \leq 2$.
(a) Finite primitive nondefective: $C_5$, $L[K_3,3]$.
(b) Defective or imprimitive: $m \cdot K_n$, $K_m[I_m]$.
(c) Infinite primitive, not defective: the Henson graphs $G_n$ and their complements $G^c_n$, and the Rado graph $G_\infty$; (these are also of the form $\Gamma^2_{1,2,3,4;S}$ for suitable $S$).

II. $\delta \geq 3$, $\Gamma_1$ finite or imprimitive.
(a) An $n$-gon with $n \geq 6$.
(b) Antipodal double of one of the graphs $C_5$, $L[K_3,3]$, or a finite independent set.
(c) A tree-like graph $T_{r,s}$ as described by Macpherson in [Mph82], where $2 \leq r, s \leq \infty$, and if $s = \infty$ then $r \geq 3$.
These will be described in detail subsequently.

III. $\delta \geq 3$, $\Gamma_1$ infinite and primitive, and $\Gamma$ imprimitive.
(a) $\Gamma$ bipartite
   (i) An infinitely branching regular tree;
   (ii) $\Gamma = \Gamma_{\infty;0;2\delta+1,C_0;S}$ with $C_0$ even, $2\delta + 2 \leq C_0 \leq 3\delta + 2$, and:
   
   If $\delta$ is odd or $C_0 \leq 3\delta$ then $S$ is empty.

(b) $\Gamma$ antipodal, not bipartite
   $\Gamma_{a,n}^\delta$: $\delta \geq 4$, or $\delta = 3$, $n = \infty$. (a variation on the graph $\Gamma_{1,\delta;2\delta+1}$, with more delicate constraints).

IV. $\delta \geq 3$, $\Gamma$ and $\Gamma_1$ both primitive and infinite.
$\Gamma_{K_2,K_\delta;C,C';S}$ with an admissible choice of parameters, $K_1 < \infty$ and $C' > 2\delta + 2$.

Notes on the bipartite cases:
(1) When $C_0 = 2\delta + 1$ we have the antipodal bipartite case. In this case $B\Gamma \cong \Gamma_{1,\delta';2\delta'+\epsilon}^\delta$ with $\delta' = [\delta/2]$ and $\epsilon = 1$ if $\delta$ is even and 2 if $\delta$ is odd.

   Example: $\delta = 4$, $B\Gamma \cong K_\infty[I_2]$.

(2) When $\delta$ is even, $C_0 = 3\delta + 2$, and $S = \{I_n^\delta\}$ then $B\Gamma \cong \Gamma_{1,\delta';3\delta'+1;\{I_n^\delta\}}$ is generic omitting $I_n^\delta$.

   E.g. $\delta' = 2$, $B\Gamma \cong G_3^\infty$.

(3) When $\delta$ is odd, $B\Gamma \cong \Gamma_{1,\delta';C_0/2}$. But for $C_0 = 3\delta - 1$ or $3\delta + 1$, since $C_0/2 > 3\delta'$ this imposes no constraint.

   For example, the graphs $\Gamma_{\infty;0;C_0}^5$ with $C_0 = 14, 16$ both have $B\Gamma \cong G_\infty$; they differ in the structure of $\Gamma_5$.

(4) In case $\delta = 3$ this gives $\Gamma_{3;0;7}^\infty$ which is the complement of a perfect matching (antipodal) and $\Gamma_{3;0;7;10}^3$ which is generic bipartite.

$G_n$ is the generic graph omitting $K_n$ (Henson’s graph) and $G_n^c$ is its complement. We write $I_n^1$ for a set of $n$ points at distance $i$; for $i = 1$ this is a clique $K_n$, and for $i > 1$ it is a particular type of independent set of vertices. We also write $I_n$ for $I_n^2$.

We still have to explain the following notions:
(1) The graphs $T_{r,s}$ of [Mph82].
(2) The antipodal double of an ordinary graph.
(3) The notation $\Gamma_{K_1,K_2;C,C';S}^\delta$

for graphs constructed via the Fraïssé theory from a suitable amalgamation class.
1.2. The Tree-like Graphs $T_{r,s}$.

**Definition 1.1.** For $2 \leq r, s \leq \infty$, we may construct an $r$-tree of $s$-cliques $T_{r,s}$ as follows. Take a tree $T(r,s)$ partitioned into two sets of vertices $A, B$, so that each vertex of $A$ has $r$ neighbors, all in $B$, and each vertex of $B$ has $s$ neighbors, all in $A$. Consider the graph induced on $A$ with edge relation given by “$d(u,v) = 2$”. This is $T_{r,s}$ (and the corresponding graph on $B$ is $T_{s,r}$).

**Lemma 1.2.** For any $r, s$ the tree $T(r,s)$ is homogeneous as a metric space with a fixed partition into two sets, and the graph $T_{r,s}$ is metrically homogeneous.

**Proof.** For any finite subset $A$ of a tree $T$, one can see that the metric structure on $A$ induced by $T$ determines the structure of the convex closure of $A$, the smallest subtree of $T$ containing $A$. Given that, a map between two finite subsets of $T(r,s)$ that respects the partition will extend first to the convex closures and then to the whole of $T(r,s)$.

This applies in particular to the two halves of $T(r,s)$. □

With $r, s < \infty$ these graphs are locally finite (that is, the vertex degrees are finite). Conversely:

**Fact 1.3** (Macpherson, [Mph82]). Let $G$ be an infinite locally finite distance transitive graph. Then $G$ is $T_{r,s}$ for some finite $r, s \geq 2$.

The proof uses a result of Dunwoody on graphs with nontrivial cuts given in [Dun82].

Eventually we will prove the following.

We will prove the following.

**Theorem 1** (Theorem 7). Let $\Gamma$ be a connected metrically homogeneous graph of diameter $\delta$, and suppose $\Gamma_1$ is finite or imprimitive. Then one of the following occurs.

1. $\delta \leq 2$, $\Gamma$ is found under the Lachlan/Woodrow classification.
2. $\Gamma$ has degree 2, a cycle.
3. $\delta = 3$, $\Gamma$ is obtained by doubling $C_5$, $E(K_3,3)$, or an independent set.
4. $\delta = \infty$, $\Gamma = T_{r,s}$ for some $r, s \geq 2$.

1.3. The amalgamation classes $A^{\delta}_{K_1,K_2,C,C',S}$.

**Notation.** Let $\delta > 0$ be fixed

1. $\mathcal{M}^{\delta}$ is the collection of all finite integral metric spaces of diameter at most $\delta$.
2. A triangle is a metric space with three points. The type of a triangle is the triple of distances between its points, taken in any order. The perimeter of a triangle is the sum of these distances.
3. A $(1,\delta)$-space is a metric space in which all distances are equal to 1 or to $\delta$. 

(4) For $1 \leq K_1 \leq K_2 \leq \delta$ or $K_1 = \infty$, $K_2 = 0$, let $A_{K_1,K_2}^\delta$ be the class of $X \in \mathcal{M}^\delta$ such that for any triangle of type $(i, j, k)$ embedding in $X$, if the perimeter $P = i + j + k$ is odd then it satisfies:

$$P \geq 2K_1 + 1 \text{ and } 2P \leq 2K_2 + \min(i, j, k)$$

(5) For any $C_0, C_1 \geq 2\delta + 1$ with $C_0$ odd and $C_1$ even, let $B_{C_0,C_1}^\delta$ be the class of $X \in \mathcal{M}^\delta$ such that for any triangle of type $(i, j, k)$ embedding in $X$, if the perimeter $P = i + j + k$ has parity $\epsilon$ (i.e. $P \equiv \epsilon \mod 2$ and $\epsilon = 0$ or 1) then

$$P < C_\epsilon$$

(6) For any set $S$ of $(1, \delta)$-spaces, let $C_S^\delta$ be the class of $X \in \mathcal{M}^\delta$ such that no space in $S$ embeds isometrically into $X$.

(7) With $\delta, K_1, K_2, C_0, C_1, S$ as above let $A_{K_1,K_2}^\delta$ be the intersection

$$A_{K_1,K_2}^\delta \cap B_{C_0,C_1}^\delta \cap C_S^\delta$$

(8) Given parameters $\delta, K_1, K_2, C_0, C_1, S$ we also set

$$C = \min(C_0, C_1) \text{ and } C' = \max(C_0, C_1)$$

and we also write

$$A_{K_1,K_2;C,C'}^\delta : S$$

for $A_{K_1,K_2;C_0,C_1;S}^\delta$ and since $C_0, C_1$ may be distinguished by their parity, there is no ambiguity in this notation.

(9) A choice of parameters $\delta, K_1, K_2, C, C', S$ is irredundant if for every proper $A \in S$ we have

$$A \in A_{K_1,K_2;C,C'}^\delta \setminus \{A\}$$

In other words, $S$ has no superfluous elements.

(10) When $C' = C + 1$ we omit it and write $A_{K_1,K_2;C;S}^\delta$; when $S = \emptyset$ we omit it.

We need to determine the parameters $\delta, K_1, K_2, C, C', S$ for which the associated class $A_{K_1,K_2;C,C';S}^\delta$ is an amalgamation class.

**Definition 1.4.** A choice of parameters $\delta, K_1, K_2, C, C', S$ is admissible if $\delta \geq 3$, either $1 \leq K_1 \leq K_2 \leq \delta$ or $K_1 = \infty$ and $K_2 = 0$, $2\delta + 1 \leq C < C' \leq 3\delta + 2$, with one of $C, C'$ even and the other odd, $S$ is a set of finite $(1, \delta)$-spaces of order at least 3, and one of the following holds.

1. $K_1 = \infty$:
   
   $$K_2 = 0, C_1 = 2\delta + 1, \text{ and }$$
   
   $$S \text{ is } \left\{ \begin{array}{ll}
   \text{empty} & \text{if } \delta \text{ odd, or } C_0 \leq 3\delta \\
   \text{a set of } \delta \text{-cliques} & \text{if } \delta \text{ even, } C_0 = 3\delta + 2
   \end{array} \right.$$
(2) $K_1 < \infty$ and $C \leq 2\delta + K_1$:

\[
C = 2K_1 + 2K_2 + 1, \quad K_1 + K_2 \geq \delta, \text{ and } K_1 + 2K_2 \leq 2\delta - 1
\]

If $C' > C + 1$ then $K_1 = K_2$ and $3K_2 = 2\delta - 1$.

If $K_1 = 1$ then $S$ is empty.

(3) $K_1 < \infty$, and $C > 2\delta + K_1$:

\[
K_1 + 2K_2 \geq 2\delta - 1 \text{ and } 3K_2 \geq 2\delta.
\]

If $K_1 + 2K_2 = 2\delta - 1$ then $C \geq 2\delta + K_1 + 2$.

If $C' > C + 1$ then $C \geq 2\delta + K_2$.

If $K_2 = \delta$ then $S$ cannot contain a triangle of type $(1, \delta, \delta)$.

If $K_1 = \delta$ then $S$ is empty.

If $C = 2\delta + 2$, then $S$ is empty.

Eventually we will prove the following.

**Theorem 2.** Let $\delta, K_1, K_2, C, C', S$ be an admissible choice of parameters. Then $A_{\delta K_1 K_2 C C', S}$ is an amalgamation class. Conversely, if $\delta, K_1, K_2, C, C', S$ is an irredundant choice of parameters and $A_{\delta K_1 K_2 C C', S}$ is an amalgamation class, then $\delta, K_1, K_2, C, C', S$ is an admissible choice of parameters.

### 1.4. Antipodal Variations

The graph $\Gamma$ of diameter $\delta$ is **antipodal** if the relation

$$d(x, y) = 0 \text{ or } \delta$$

is an equivalence relation. Eventually we will prove the following.

**Proposition 1.5.** Let $\Gamma$ be a connected metrically homogeneous and antipodal graph, of diameter $\delta \geq 3$. Then for each vertex $u \in \Gamma$, there is a unique vertex $u' \in \Gamma$ at distance $\delta$ from $u$, and we have the law

$$d(u, v) = \delta - d(u', v)$$

for $u, v \in \Gamma$. In particular, the map $u \mapsto u'$ is an automorphism of $\Gamma$.

Note however that in diameter 2, the complete multipartite graph $K_m[I_n]$ is antipodal for any $m, n \leq \infty$.

We consider modifications of our definitions in the previous section which allow us to include arbitrary $(1, \delta)$-constraints when the associated graph is antipodal and $K_1 = 1$. In this case the associated amalgamation class may have additional constraints which are neither triangles nor $(1, \delta)$-spaces.

**Definition 1.6.** Let $\delta \geq 4$ be finite and $2 \leq n \leq \infty$, or $\delta = 3$ and $n = \infty$. Then

\[
(1) \quad A_{1,\delta; 2\delta + 2, 2\delta + 1; 0} \text{ is the set of finite integral metric spaces in which no triangle has perimeter greater than } 2\delta.
\]
(2) $\mathcal{A}^\delta_{a,n}$ is the subset of $\mathcal{A}^\delta_a$ containing no subspace of the form $I_2^{\delta-1}[K_k,K_\ell]$ with $k + \ell = n + 1$; here $I_2^{\delta-1}$ denotes a pair of vertices at distance $\delta - 1$ and $I_2^{\delta-1}[K_k,K_\ell]$ stands for the corresponding composition, namely a graph of the form $K_k \cup K_\ell$ with $K_k,K_\ell$ cliques (at distance 1), and $d(x,y) = \delta - 1$ for $x \in K_k$, $y \in K_\ell$. In particular, with $k = n + 1$, $\ell = 0$, this means $K_{n+1}$ does not occur.

Eventually we will prove the following.

**Theorem 3.** If $\delta \geq 4$ is finite and $2 \leq n \leq \infty$, then $\mathcal{A}^\delta_{a,n}$ is an amalgamation classes. If $n \geq 3$ then the associated Fraissé limit is a connected antipodal metrically homogeneous graph which is said to be generic for the specified constraints.

Here the parameter $n$ stands in place of the set $S$; since there are no triangles of perimeter greater than $2\delta$, the only relevant $(1,\delta)$-spaces are 1-cliques.

1.5. **Bipartite Graphs.** If $\Gamma$ is a connected metrically homogeneous bipartite graph, we write $B\Gamma$ for the graph induced on either of the two equivalence classes for the congruence (2); these are isomorphic, with respect to the edge relation $R_2$: $d(x,y) = 2$. This is called a halved graph for $\Gamma$, and $\Gamma$ is a doubling of $B\Gamma$.

**Fact 1.7** (cf. [AH06, Theorem 2.3]). Let $\Gamma$ be a connected metrically homogeneous bipartite graph. Then $B\Gamma$ is metrically homogeneous.

**Proof.** Since $\text{Aut}(\Gamma)$ preserves the equivalence relation whose classes are the two halves of $\Gamma$, the homogeneity condition is inherited by each half. $\square$

Eventually we will prove the following, which refers to some specific constructions we will explain shortly.

**Theorem 4** (8). Let $\Gamma$ be a connected, bipartite, and metrically homogeneous graph, of diameter at least 3, and degree at least 3. Then one of the following occurs, writing $B\Gamma_1$ for $(B\Gamma)_1$.

1. $\Gamma$ is a tree;
2. $\Gamma$ has diameter 3, and $\Gamma$ is either the complement of a perfect matching, or a generic bipartite graph;
3. $\Gamma$ has diameter 4, $B\Gamma \cong K_\infty[I_2]$, and $\Gamma$ is a bipartite twisted double cover of a generic bipartite graph.
4. $\Gamma$ has diameter 4, and $B\Gamma_1$ is generic omitting an independent set of order $n + 1$, for some $n \geq 2$. With $n$ fixed, $\Gamma$ is determined up to isomorphism.
5. $\Gamma$ has diameter 5 and is antipodal; $B\Gamma_1$ is generic omitting an independent set of order 3. $\Gamma \cong (B\Gamma)^{ab}$.
6. $B\Gamma_1$ is a homogeneous universal graph.
Definition 1.8. Let $\Gamma$ be a metrically homogeneous graph of diameter $\delta$. The antipodal bipartite double cover $\Gamma^{ab}$ of $\Gamma$ is the graph with vertex set $V(\Gamma) \times \mathbb{Z}_2$ and edge relation $\sim$ given by $(u, i) \sim (v, j)$ iff:

$$d(u, v) = \delta \text{ and } i \neq j$$

We will prove the following.

Theorem (6). Let $\Gamma'$ be a metrically homogeneous graph of diameter $\delta'$, with the following properties.

1. $\Gamma'$ is primitive.
2. No triangle in $\Gamma'$ has perimeter greater than $2\delta' + 1$.

The antipodal bipartite double cover $\Gamma = \tilde{\Gamma}'$ is a metrically homogeneous bipartite antipodal graph of diameter $\delta = 2\delta_0 + 1$.

Conversely, if $\Gamma$ is a bipartite antipodal metrically homogeneous graph of odd diameter $\delta$, then $\Gamma$ is the double cover of $\Gamma' = B\Gamma$, which has diameter $\delta = (\delta - 1)/2$ and the two stated properties.

1.6. More Doubling. We define some additional doubling constructions.

Definition 1.9. Let $G$ be a graph.

1. The twisted double cover $\Gamma = 2 \ast G$ of $G$ is the graph on $V(G) \times \mathbb{Z}_2$ with edges given by $(u, i) \sim (v, j)$ iff

$$\begin{cases}
    u \sim v & \text{if } i = j \\
    u \neq v \text{ and } u \neq v & \text{if } i \neq j
\end{cases}$$

2. The antipodal double cover $\Gamma = \hat{G}$ of $G$ is the twisted double cover of the graph $G^*$ obtained from $G$ by adding one additional vertex $*$ adjacent to all vertices of $G$.

3. If $G$ is a bipartite graph with a specified bipartition $(A, B)$, the bipartite twisted double cover of $G$ with respect to this bipartition is obtained from the twisted double cover by taking only those edges connecting $A \times \mathbb{Z}_2$ with $B \times \mathbb{Z}_2$.

Eventually we will prove the following.

Theorem 5. Let $G$ be one of the following graphs: the pentagon (5-cycle), the line graph $L[K_{3,3}]$, an independent set $I_n$ ($n \leq \infty$), or the random graph $\Gamma_\infty$. Then the antipodal double cover $\Gamma = \hat{G}$ is a metrically homogeneous antipodal graph of diameter 3. Conversely, any connected metrically homogeneous antipodal graph of diameter 3 is of this form.

2. Smith's Theorem

Before turning to proofs of existence, it will be convenient to clarify the structure of imprimitive metrically homogeneous graphs in general. This is covered in the finite case by Smith's Theorem [AH06, BCN89, Smi71], which indeed applies in that context to distance transitive graphs and even
more generally. The main points for us, in our present context, will be the following.

1. The imprimitive metrically homogeneous graphs are bipartite or antipodal, or both.
2. The classification of bipartite metrically homogeneous graphs reduces largely to the classification of non-bipartite graphs by the operation of “halving.”
3. In the antipodal case, the relation \( d(x, y) = \delta \) defines a pairing \( v \leftrightarrow v' \) on the vertices of \( \Gamma \), which is an automorphism.

### 2.1. Imprimitive metrically transitive graphs.

If \( \Gamma \) is a distance transitive graph, then any binary relation \( R \) invariant under \( \text{Aut}(\Gamma) \) is a union of relations \( R_i \) defined by \( d(x, y) = i \), \( R = \bigcup_{i \in I} R_i \) for some set \( I \), with \( \delta \) the diameter (possibly infinite). We denote by \( \langle t \rangle \) the union \( \bigcup_{i \mid t} R_i \) taken over the multiples of \( t \) (including 0). The first point is the following.

#### Fact 2.1 (cf. [AH06, Theorem 2.2]).

Let \( \Gamma \) be a connected distance transitive graph of diameter \( \delta \), and let \( E \) be an \( \text{Aut}(G) \)-invariant equivalence relation on \( \Gamma \). Then the following hold.

1. \( E = \langle t \rangle \) for some \( t \).
2. If \( 2 < t < \delta \), then \( \Gamma \) has degree 2.
3. If \( t = 2 \) then either \( \Gamma \) is bipartite, or \( \Gamma \) is a complete regular multipartite graph, of diameter 2.

In particular, if the diameter of \( \Gamma \) is at least 3, then \( \Gamma \) is either bipartite or antipodal (and possibly both).

Of course, the exceptional case of diameter 2 has already been noticed within the Lachlan/Woodrow classification, where it occurs as the complement of \( m \cdot K_n \), with \( m, n \leq \infty \).

If \( \Gamma \) is a connected distance transitive bipartite graph, we write \( B\Gamma \) for the graph induced on either of the two equivalence classes for the congruence \( \langle 2 \rangle \); these are isomorphic, with respect to the edge relation \( R_2: d(x, y) = 2 \). This is called a halved graph for \( \Gamma \), and \( \Gamma \) is a doubling of \( B\Gamma \). In the finite case one also considers a folding operation on antipodal graphs, which we do not find useful in our context.

#### Fact 2.2 (cf. [AH06, Theorem 2.3]).

Let \( \Gamma \) be a connected metrically homogeneous bipartite graph. Then \( B\Gamma \) is metrically homogeneous.

**Proof.** Since \( \text{Aut}(\Gamma) \) preserves the equivalence relation whose classes are the two halves of \( \Gamma \), the homogeneity condition is inherited by each half. \( \square \)

### 2.2. The Antipodal Case.

All graphs considered under this heading are connected and of finite diameter.

**Proposition 2.3.** Let \( \Gamma \) be a connected metrically homogeneous and antipodal graph, of diameter \( \delta \geq 3 \). Then for each vertex \( u \in \Gamma \), there is a unique
vertex $u' \in \Gamma$ at distance $\delta$ from $u$, and we have the antipodal law
\[ d(u, v) = \delta - d(u', v) \]
for $u, v \in \Gamma$. In particular, the map $u \mapsto u'$ is an automorphism of $\Gamma$.

For $v \in \Gamma$, $\Gamma_i(v)$ denotes the graph induced on the vertices at distance $i$ from $v$, and since the isomorphism type is independent of $v$, this will sometimes be denoted simply by $\Gamma_i$, when the choice of $v$ is immaterial. In particular, $\Gamma_\delta \cong K_n^{(\delta)}$, the complete graph with edge relation given by $R_\delta$, for some $n$, with $1 \leq n \leq \infty$. Our main claim is that $n = 1$.

**Lemma 2.4.** Let $\Gamma$ be a connected distance transitive antipodal graph of diameter $\delta$, and let $C_1, C_2$ be two equivalence classes for the antipodality relation $R_\delta$. Then the set of distances $d(u, v)$ for $u \in C_1, v \in C_2$ is a pair of the form $\{i, \delta - i\}$ (which is actually a singleton if $i = \delta/2$ with $\delta$ even).

**Proof.** Since there are geodesics $(u, v, u')$ with $d(u, v) = i$, $d(v, u') = \delta - i$, and $d(u, u') = \delta$, whenever we have $d(u, v) = i$ we also have $d(u', v) = \delta - i$ for some $u'$ antipodal to $u$, by distance transitivity.

We claim that for $u \in C_1, v, w \in C_2$, with $d(u, v) = i$, $d(u, w) = j$ and $i, j \leq \delta/2$, we have $i = j$. If $i < j$, take $u' \in C_2$ with $d(u, u') = \delta - j$. Then $d(v, u') \leq i + (\delta - j) < \delta$, so $w = u'$ and $i = \delta - j$, in which case $i = j = \delta/2$.

For $u \in C_1$ the set of distance $d(u, v)$ with $v \in C_2$ has the form $\{i, \delta - i\}$, and for $v \in C_2$ the same applies with respect to $C_1$, with the same pair of values. This implies our claim. \qed

**Lemma 2.5.** Let $\Gamma$ be a connected distance transitive antipodal graph of diameter $\delta$, and $(abc)$ a triangle in $\Gamma$ (that is, a triple of vertices, with the induced metric). Then the perimeter
\[ d(a, b) + d(a, c) + d(b, c) \]
is at most $2\delta$.

**Proof.** Write $i = d(a, b), j = d(a, c), k = d(b, c)$. In the triangle $(a', b, c)$ we have the distances $\delta - i, \delta - j, k$ and by the triangle inequality we find
\[ k \leq (\delta - i) + (\delta - j) \]
and our claim follows. \qed

**Lemma 2.6.** Let $\Gamma$ be a metrically homogeneous and antipodal, of diameter $\delta$. Suppose $u, u' \in \Gamma$, and $d(u, u') = \delta$. Then for $i < \delta/2$, the relation $R_\delta$ defines a bijection between $\Gamma_i(u)$ and $\Gamma_i(u')$, while $\Gamma_{\delta/2}(u) = \Gamma_{\delta/2}(u')$.

**Proof.** Fix $i < \delta/2$, and $v \in \Gamma_i(u)$. We work with the equivalence classes $C_1, C_2$ of $u$ and $v$ respectively, with respect to the relation $R_\delta$. As $d(u, v) = i$, $d(u, u') = \delta$, $i \leq \delta/2$, and $d(v, u') \in \{i, \delta - i\}$, we have $d(v, u') = \delta - i$.

Now $(v, u')$ extends to a geodesic $(v, v', v'')$ with $d(v, v') = \delta$, $d(u', v') = i$, and we claim that $v'$ is unique. If $(v, u', v'')$ is a second such geodesic then we have $d(v, v') = d(v, v'') = \delta$ and $d(v', v'') \leq 2(\delta - i) < \delta$, so $v' = v''$. \qed
Lemma 2.7. Let $\Gamma$ be a metrically homogeneous and antipodal graph, of diameter $\delta \geq 3$, and let $\Gamma_\delta \cong K_n^{(\delta)}$ with $1 < n \leq \infty$. Then $\delta \geq 5$.

Proof. Take $a_1, a_2, a_3$ at mutual distance $\delta$, and take $u_1, v_1 \in \Gamma_1(a)$ with $d(u_1, v_1) = 2$.

Suppose the diameter is 3. Using the previous lemma, take vertices $u_2, v_2 \in \Gamma_1(a_2)$, $u_3, v_3 \in \Gamma_1(a_3)$, with $u_1, u_2, u_3$ and $v_1, v_2, v_3$ triples with pairwise distances all equal to $\delta$.

Then $\delta(u_1, v_3) = \delta - d(u_1, v_1) = \delta - 2$ and similarly $\delta(u_2, v_3) = \delta - 2$, so $\delta = d(u_1, u_2) \leq 2(\delta - 2)$, a contradiction.

So now suppose the diameter is 4. Consider $v_2 \in \Gamma_1(a_2)$ with $d(v_1, v_2) = \delta$. Observe that the triangles $(a_3, u_1, v_1)$ and $(a_3, u_2, v_2)$ are isometric. Now apply metric homogeneity to find an isometry carrying $(a_3, u_1, v_1, a_1)$ to $(a_3, u_1, v_2, b_1)$ for some $b_1$. Then $u_1, v_2 \in \Gamma_1(b_1)$ and $d(b_1, a_3) = 4$. But then $d(a_1, b_1), d(a_2, b_1) \leq 2$ while $a_1, a_2, a_3, b_1$ are all in the same antipodality class, forcing $a_1 = b_1 = a_2$, a contradiction.

Now we need to extend Lemma 2.4 to some cases involving distances which may be greater than $\delta/2$.

Lemma 2.8. Let $\Gamma$ be a metrically homogeneous and antipodal, of diameter $\delta \geq 3$, and let $\Gamma_\delta \cong K_n^{(\delta)}$ with $1 < n \leq \infty$. Suppose $d(a, a') = \delta$ and $i < \delta/2$. Suppose $u \in \Gamma_1(a)$, $u' \in \Gamma_1(a')$, with $d(u, u') = \delta$. If $v \in \Gamma_1(a)$ and $d(u, v) = 2i$, then $d(u', v) = \delta - 2i$.

Proof. We have $d(a, u') = \delta - i$. Take $v_0 \in \Gamma_1(a)$ so that $(a, v_0, u')$ is a geodesic, that is $d(v_0, u') = \delta - 2i$. As $u, v_0 \in \Gamma_1(a)$ we have $d(u, v_0) \leq 2i$. On the other hand $d(u, u') = \delta$ and $d(v_0, u') = \delta - 2i$, so $d(u, v_0) \geq 2i$. Thus $d(u, v_0) = 2i$.

So we have at least one triple $(a, u, v_0)$ with $v_0 \in \Gamma_1(a)$, $d(u, v_0) = 2i$, and with $d(v_0, u') = \delta - 2i$. Let $(a, u, v)$ be any triple isometric to $(a, u, v_0)$. Then the quadruples $(a, u, v, a')$ and $(a, u, v_0, a')$ are also isometric since $u, v, v_0 \in \Gamma_1(a)$ with $i < \delta/2$. But as $a, u$ together determine $u'$, we then have $(a, u, v, a', u')$ and $(a, u, v_0, a', u')$ isometric, and in particular $d(v, u') = \delta - 2i$.

After these preliminaries we can prove the proposition.

Proof. We show that $n = 1$, after which the rest follows directly since if $u$ determines $u'$, then $d(u, v)$ must determine $d(u', v)$.

We have $\delta \geq 5$. We fix $a_1, a_2, a_3$ at mutual distance $\delta$, and fix $i < \delta/2$, to be determined more precisely later.
Take \( u_1, v_1 \in \Gamma_i(a_1) \) with \( d(u_1, v_1) = 2i \), and then correspondingly \( u_2, v_2 \in \Gamma_i(a_2) \), \( u_3, v_3 \in \Gamma_i(a_3) \), with \( u_1, u_2, u_3 \) and \( v_1, v_2, v_3 \) triples of vertices at mutual distance \( \delta \).

Now \( d(u_1, v_3) = \delta - 2i \), and \( d(u_1, u_3) = \delta \), so as usual \( d(u_3, v_3) = 2i \). We now consider the following property of the triple \((u_1, v_1, a_3)\): For \( v_3 \in \Gamma_i(a_3) \) with \( d(v_1, v_3) = \delta \), we have \( d(u_1, v_3) = \delta - 2i \). The triple \((u_1, v_1, a_3)\) is isometric with \((v_3, u_3, a_2)\). It follows that \( d(v_3, u_2) = \delta - 2i \).

This shows that \( i \leq \delta/4 \), so for a contradiction we require

\[
\frac{\delta}{4} < i < \frac{\delta}{2}
\]

and for \( \delta > 4 \) this is possible.

\[\square\]

3. Existence

In the present section we show that all the graphs occurring in the catalog do in fact exist, and are metrically homogeneous. We also give some classification results for the most “sporadic” cases, reserving more general classification results for subsequent sections.

Category I \((\delta \leq 2)\) presents no difficulties. In fact in this case the classification is also complete, by a delicate argument due to Lachlan and Woodrow [LW80]. It is also useful to take note of the following elementary result, characterizing the bipartite graphs which are homogeneous in a suitable language specifying the bipartition. In this context, the class of examples is closed under formation of the bipartite complement, with the same bipartition but a complementary set of edges between its components.

**Fact 3.1 ([GGK96]).** The bipartite graphs which are homogeneous for the language expanding the usual language of graphs by unary predicates for a specific bipartition are the following.

1. An independent set and its bipartite complement, the complete bipartite graph.
2. A perfect matching and its bipartite complement.
3. A generic bipartite graph (universal homogeneous), which is isomorphic to its bipartite complement.

We deal with group II of the catalog in §3.1 following. The treatment of the antipodal bipartite cases under group III is in §3.2 with the exception of \( \Gamma_0^d,2\delta+1;C_0,2\delta+1;S \) left for §3.4. The other cases falling under group III do not require particular attention apart from the cases derived from amalgamation classes: \( \Gamma_0^d,2\delta+1;C_0,2\delta+1;S \) with \( C_0 > 2\delta+2 \) and \( \Gamma_a,n \). The first of these is treated in §3.4, the second in §3.5.

Group IV consists of the graphs \( \Gamma_{K_1,K_2;C,C'};S \) with \( K_1 < \infty \) and \( C' > 2\delta+2 \), in other words leaving aside the bipartite and antipodal cases falling under group III. This is treated in §3.6.
3.1. **Graphs with** $\delta \geq 3$, $\Gamma_1$ **finite or imprimitive.** Under this heading our catalog contains the following graphs.

1. $n$-gons with $n \geq 6$.
2. Antipodal doublings of $C_5$, $L[K_{3,3}]$, or a finite independent set.
3. The tree-like graphs $T_{r,s}$ as described by Macpherson in [Mph82].

**Lemma 3.2.** Let $G$ be a graph such that for all $u, v \in G$ adjacent, there is $w \in G$ adjacent to $v$ and not to $u$. For $v \in G$ let $G_v$ be the graph obtained from $G$ by switching the edges and nonedges between the sets formed by the neighbors of $v$ and the nonneighbors of $v$. Let $\Gamma$ be the antipodal double of $G$.

Then $\Gamma$ is antipodal of diameter 3, and the following are equivalent.

1. $\Gamma$ is metrically homogeneous.
2. $G$ is homogeneous and $G_v \cong G$ for $v \in G$.

**Proof.** Recall that $\Gamma$ has vertex set $G^* \times \mathbb{Z}_2$. Let us identify $G^*$ with $G^* \times \{0\}$ and for $u \in G^*$ write $u' = (u, 1)$, and similarly $G' = G \times \{1\}$. Furthermore let $*$ be the unique vertex in $G^* \setminus G$.

Observe that the metric induced on $G$ is given by

$$d(u, v) = \begin{cases} 1 & \text{if } u \sim v \\ 2 & \text{if } u \not\sim v \end{cases}$$

Indeed, the base point $*$ makes $d(u, v) \leq 2$.

For $u, v \in G$ adjacent there is by hypothesis some $w \in G$ adjacent to $v$ and not to $u$. This shows that $d(u, v') = 2$, and hence easily for general $u, v \in G^*$ we have

$$d(u, v') = 3 - d(u, v)$$

Thus $\Gamma$ is antipodal of diameter 3.

Now evidently $\Gamma$ is metrically homogeneous if and only if the following two conditions are met:

1. With a name for the basepoint $*$ added to the language, $(\Gamma, *)$ is metrically homogeneous.
2. $\Gamma$ is vertex transitive.

The first condition implies that $G \cong \Gamma_1$ is metrically homogeneous. Conversely, if $G$ is metrically homogeneous, we show that $(\Gamma, \ast)$ is homogeneous.

Let $f : A \to B$ be an isometry in $\Gamma$ with $\ast \in A, B$ and $f(\ast) = \ast$. Let $\hat{A} = A \cup A'$, $\hat{B} = B \cup B'$.

Then $f$ extends canonically to an isometry $\hat{f} : \hat{A} \to \hat{B}$. Let $A_0 = \hat{A} \cap G$, $B_0 = \hat{B} \cap G$, $f_0 = \hat{f} \upharpoonright A_0$. Then $f_0$ extends to an automorphism $\alpha$ of $G$ and $\alpha$ extends to an automorphism $\hat{\alpha}$ of $G$. Easily $\hat{\alpha}$ extends $\hat{f}$ and, in particular, $f$.

Now we decode the second condition. $\Gamma$ has the automorphism given by the antipodal map, so $\Gamma$ is vertex transitive if and only if each $v \in G$ is the image of $\ast$ by an automorphism. In particular $G = \Gamma_1(\ast) \cong \Gamma_1(v)$, and
conversely if $\Gamma_1(*) \cong \Gamma_1(v)$ then any isomorphism between these extends first to $\Gamma_1(*) \cup \{v\}$ and then to $\Gamma$.

To complete the proof it suffices to check that $\Gamma_1(v) \cong G_v$ with $*$ in $\Gamma_1(v)$ corresponding to $v$ in $G_v$, and in view of the definition of $\Gamma$ this is immediate. \hfill $\Box$

**Lemma 3.3.** The antipodal graphs of diameter 3 are the antipodal doubles of the graphs $C_5$, $L[K_{3,3}]$, an independent set $I_n$ with $n \leq \infty$, or the Rado graph $G_\infty$.

**Proof.** Let $G = \Gamma_1$, and let $'$ be the antipodal map. Evidently $G' = \Gamma_1$ and $\Gamma$ is the antipodal double of $G$. By the previous lemma, it suffices to identify the homogeneous graphs $G$ for which $G_v \cong G$ for $v \in G$.

By inspection $C_5$ and $L[K_{3,3}]$ have this property. To see that it holds for $G = G_\infty$, just check that the extension properties for $G_v$ follow from the extension properties for $G$.

For $v \in G$ we have $(G')_v \cong (G_v)^c$ so to eliminate the remaining possibilities for $G$ we may consider just the four cases $G \cong K_n$, $G \cong I_n$, $G \cong m \cdot K_n$ ($m \geq 2$) and $G \cong G_n$, a Henson graph. Note that while the first two cases are complementary, the complete graph does not satisfy the hypotheses of the previous lemma, but an independent set $I_n$ does.

In the case $G = I_n$, $G_v = G$ and our condition holds, while with $G = K_n$ the graph $\Gamma$ is disconnected and not relevant (though it is metrically homogeneous).

Suppose $G \cong m \cdot K_n$ with $m \geq 2$. Then the neighbors of $v$ in $G$ are adjacent to every vertex of $G_v$, so the condition $G \cong G_v$ would force $m = 1$, a contradiction.

Now suppose $G \cong G_n$, a Henson graph omitting $K_n$. Then $G$ contains a clique $K \cong K_{n-1}$ and a vertex $v$ nonadjacent to all vertices of $K$. For any neighbor $u$ of $v$ in $G$, the graph induced on $K \cup \{u\}$ in $G_v$ is a clique of order $n$, so $G \not\cong G_v$. \hfill $\Box$

**Lemma 3.4.** The graphs $T_{r,s}$ are metrically homogeneous for $2 \leq r, s \leq \infty$.

**Proof.** We work first with the tree $T = T(r,s)$ such that if $(T_1,T_2)$ is a bipartition of $T$, then the vertices in $T_1$ have degree $r$ and the vertices in $T_2$ have degree $s$.

We claim that $T(r,s)$ is homogeneous as a bipartite graph with unary predicates $T_1, T_2$.

For any finite subset $A$ of an arbitrary tree $T$, we claim first that the metric structure on $A$ induced by $T$ determines the structure of the convex closure $\hat{A}$ of $A$ in $T$, that is the smallest subtree of $T$ containing $A$. Suppose $a, b, c \in A$ and let $u$ be the unique point lying on the three geodesics $[a,b]$, $[a,c]$, $[b,c]$. Then $d(a,b) + d(a,c) + d(b,c) = 2(d(a,u) + d(b,c)$ and therefore $d(a,u)$, $d(b,u)$, $d(c,u)$ are determined by the metric on $(a,b,c)$. For any fourth vertex $x \in A$, it is then easy to determine $d(x,u)$ from the metric on $(a,b,c,x)$. Thus we may suppose that for any three points $a, b, c$ in $A$, the
corresponding point \( u \) is also in \( A \). Then to extend \( A \) to a minimal subtree of \( T \) it suffices to adjoin geodesics \([a, b]\) for which \( a, b \in A \) and no interior points of \([a, b]\) lie in \( A \), and again the metric structure of the extension is uniquely determined.

Now we may check that \( T(r, s) \) is homogeneous. We consider an isometry \( f : A \to B \) with \( A \) finite, preserving the bipartition \((T_1, T_2)\). Without loss of generality \( A \) is connected. Then it is clear that \( f \) extends to an automorphism of \( T(r, s) \).

In particular the induced metric spaces on \( T_1, T_2 \) are homogeneous. The metric on \( T_1 \) is twice the graph metric on \( T_{r,s} \), so the latter is metrically homogeneous. \( \square \)

3.2. Graphs with \( \delta \geq 3 \), \( \Gamma_1 \) infinite imprimitive, and \( \Gamma \) bipartite, antipodal. In this case our catalog contains the following types.

1. \( \delta \) odd, \( \Gamma \) is the antipodal bipartite double cover of a metrically homogeneous graph \( \Gamma' \) of diameter \( \delta' = (\delta - 1)/2 \), with \( \Gamma_\delta' \) complete, of order at least 2.

2. \( \delta = 4 \), and \( B\Gamma \cong K_\infty[I_2] \) and \( \Gamma \) is the bipartite twisted double cover of a generic bipartite graph.

3. \( \Gamma_{\infty,0;2\delta+1}^\delta \).

Theorem 6. Let \( \Gamma' \) be a metrically homogeneous graph of diameter \( \delta' \), such that \( \Gamma'_{\delta'} \) is a complete graph of order at least 2.

The the antipodal bipartite double cover \( \Gamma = (\Gamma')^{ab} \) is a metrically homogeneous bipartite antipodal graph of diameter \( \delta = 2\delta' + 1 \).

Conversely, if \( \Gamma \) is a bipartite antipodal metrically homogeneous graph of odd diameter \( \delta = 2\delta' + 1 \), then \( \Gamma \) is the bipartite double cover of \( \Gamma' = B\Gamma' \), which has diameter \( \delta' \), and \( \Gamma_{\delta'} \) is a complete graph of order at least two.

Proof. First, let \( \Gamma' \) be metrically homogeneous and primitive of diameter \( \delta' \), with no triangle of diameter greater than \( 2\delta + 1 \), and let \( \Gamma \) be its antipodal bipartite double cover. For \( v \in \Gamma' \), let us write \( v_0, v_1 \) for \((v, 0)\) and \((v, 1)\).

We claim that for \( u, v \in \Gamma' \) we have \( d(u_0, v_0) = 2 \) in \( \Gamma \) iff \( u, v \) are adjacent in \( \Gamma' \).

Suppose first that \( u, v \) are adjacent in \( \Gamma' \). By our assumption on \( \Gamma_{\delta'} \) and homogeneity, there is a vertex \( w \in \Gamma' \) with \( d(u, w) = d(v, w) = \delta' \). In \( \Gamma \) we find

\[
 u_0 \sim w_1 \sim v_0
\]

and thus \( d(u_0, v_0) = 2 \).

Conversely, if \( d(u_0, v_0) = 2 \) then there must be \( w_1 \) with \( d(u, w) = d(v, w) = \delta' \) in \( \Gamma' \) and then as noted \( u \sim v \) in \( \Gamma' \).

It then follows easily that the graph metric on \( \Gamma' \times \{0\} \) corresponds to twice the graph metric on \( \Gamma' \). In particular, \( \Gamma' \times \{0\} \) is homogeneous metric space, and the same applies to \( \Gamma' \times \{1\} \).
Now with $\delta = 2\delta' + 1$ we check
\[
d(u_0, v_1) = \delta - d(u_0, v_0)
\]
for $u, v \in \Gamma'$. Let $d(u, v) = \delta' - i$ and choose $w$ with $d(v, w) = i$, $d(u, w) = \delta'$. Then $d(u_0, w_1) = 1$ and $d(v_1, w_1) = 2i$, so $d(u_0, v_1) \leq 1 + 2i = \delta - 2d(u, v) = \delta - d(u_1, v_1)$.

On the other hand, there must be some $w \in \Gamma'$ with $d(u_0, w_1) = 1$, $d(w_1, v_1) = d(u_0, v_1) - 1$. Then $d(u, w) = \delta'$, $2d(w, v) = d(u_0, v_1) - 1$, and $d(u_0, v_1) = 1 + 2d(v, w) \geq 1 + 2[d(u, w) - d(u, v)] = 1 + 2\delta' - d(u, v) = \delta - d(u, v)$.

In particular the relation $"d(x, y) = \delta"$ defines the pairing $v_i' = v_{1-i}$ and $\Gamma$ is antipodal. Furthermore any isometry $A \leftrightarrow B$ in $\Gamma$ extends to $A \cup A' \leftrightarrow B \cup B'$. Since the pairing $v \leftrightarrow v'$ switches the sides of the bipartition of $\Gamma$, to check homogeneity it suffices to work with isometries that preserve each side. The antipodal law then shows that homogeneity of the metric on one side implies homogeneity of $\Gamma$.

Thus for every such graph $\Gamma'$ the antipodal bipartite double cover is metrically homogeneous, as well as antipodal and bipartite of the expected diameter.

Conversely, consider a metrically homogeneous bipartite antipodal graph whose diameter $\delta$ is odd, $\delta = 2\delta' + 1$.

Then $\Gamma' = B\Gamma$ is metrically homogeneous of diameter $\delta'$. The antipodal pairing $v \leftrightarrow v'$ identifies the two halves of $\Gamma$, as $\delta$ is odd, and the edge relation on $\Gamma$ is the one specified in the antipodal bipartite double cover, by the antipodal law.

By Lemma 2.5 any triangle in $\Gamma$ has perimeter at most $2\delta$ and therefore any triangle in $\Gamma'$ has perimeter at most $\delta = 2\delta' + 1$. Hence $\Gamma'_{\delta'}$ is complete.

If $\Gamma'_{\delta'}$ is a singleton, then for $v \in \Gamma$ we have $\Gamma_{2\delta'}(v) = \Gamma_1(v')$ is a singleton, which gives a contradiction. So the properties of $\Gamma'$ have been verified. □

**Lemma 3.5.** The bipartite twisted double cover of a generic bipartite graph is metrically homogeneous of diameter 4, and $B\Gamma \cong K_\infty[I_2]$. Conversely, if $\Gamma$ is a connected, bipartite, and metrically homogeneous graph with $B\Gamma \cong K_\infty[I_m]$ and $2 \leq m \leq \infty$, then $m = 2$, and $\Gamma$ is isomorphic to the bipartite twisted double cover of the generic bipartite graph.

**Proof.** Suppose first that $\Gamma$ is the bipartite twisted double cover of the generic bipartite graph $\Gamma' = (A, B)$. Then evidently the metric on $A \times Z_2$ gives distance 2 except for pairs with the same first coordinate, and thus $B\Gamma \cong K_\infty[I_2]$. It is also clear that the diameter is 4 and the graph is antipodal, and satisfies the usual antipodal law for the metric. Therefore in checking homogeneity we consider only isometries $f : X \to Y$ for which $X$ and $Y$ are closed under the antipodal pairing. Writing $X_0, Y_0$ for $X \cap \Gamma' \times \{0\}, Y \cap \Gamma' \times \{0\}$, we restrict $f$ to $f_0 : X_0 \to Y_0$, and then extend to $f_0 \in \text{Aut}(\Gamma' \times \{0\})$, and again to $\bar{f} \in \text{Aut}(\Gamma)$, which is easily seen to extend $f$. 


Now for the converse, suppose that $\Gamma$ is metrically homogeneous and bipartite, with $B\Gamma \cong K_\infty[I_m]$. Let $A, B$ be the two halves of $\Gamma$.

The relation “$d(x, y)$ is 0 or 4” defines an equivalence relation on $\Gamma$ with classes of size $m$. Call the classes for this relation components of $\Gamma$.

We show now that the diameter of $\Gamma$ is 4. Take $u \in A, v \in B$. There is at most one vertex in the component of $v$ adjacent to $u$, and therefore we may find a vertex $w \in B$ adjacent to $u$ and not in this component. As $d(v, w) = 2$ we have $d(u, v) \leq 3$. Thus the diameter of $\Gamma$ is 4 and the distances realized between $A$ and $B$ are 1 and 3. Now for $u \in A$, some component of $B$ realizes both these distances, and then by homogeneity, every component of $B$ does. That is, every component of $B$ contains a unique neighbor of $u$, and vice versa. That is, the graph induces definable bijections between the components of $A$ and of $B$. Fixing one vertex $u \in A$ and two components $C_1, C_2$ of $B$, we derive an a-definable bijection between $C_1$ and $C_2$ by composing the bijections with the component of $a$. In view of the homogeneity and the metric structure of $C_1 \cup C_2 \cup \{a\}$, it follows that $m = 2$, and $\Gamma$ is antipodal.

Now we claim that $\Gamma$ is isomorphic to the bipartite twisted double cover of the generic bipartite graph; let us call the latter $\Gamma^*$. Fix $u \in A, v \in B$, and let $A_0$ be the set of neighbors of $B$ in $A$, $B_0$ the set of neighbors of $u$ in $B$. Then the induced graph on $(A_0 \cup B_0)$ is homogeneous with respect to the bipartition $(A_0, B_0)$, and bipartite. Furthermore replacing $v$ by the antipodal vertex $v'$ has the effect of taking the complementary graph. Thus the graph on $(A_0, B_0)$ is isomorphic to its complement. By Fact 3.1, this graph is the generic bipartite graph. Evidently its antipodal closure ($\Gamma$ with the components of $u$ and $v$ removed) is the twisted bipartite cover of that graph. It follows that $\Gamma^*$ embeds into $\Gamma$, and conversely that every finite subgraph of $\Gamma$ embeds into $\Gamma^*$. As homogeneous structures are characterized by their finite substructures, we have $\Gamma \cong \Gamma^*$. \hfill \qed

We will give the Fra"issé style construction of $\Gamma^*_\infty,0;2\delta+1$ in §3.4.

3.3. Graphs with $\delta \geq 3$, $\Gamma_1$ infinite imprimitive, and $\Gamma$ primitive, but not both bipartite and antipodal. In this case our catalog has the following entries.

B Bipartite and not Antipodal:
   (a) An infinitely branching regular tree;
   (b) $\delta = 3$: complement of a matching, or generic bipartite;
   (c) $\Gamma^*_\infty,0;C_0,2\delta+1;S$ with $C_0 > 2\delta + 2$.

C Antipodal and not Bipartite:
   (a) $\Gamma^*_a,n$ with $\delta \geq 4$, or $\delta = 3$, $n = \infty$.

The first two items evidently exist and are metrically homogeneous. So we come to constructions involving the direct verification of the amalgamation property. These will involve the classes $A^\delta_{K_1,K_2;C,C'};S$ with admissible parameters and their antipodal variations.
3.4. Bipartite Amalgamation Classes. In such cases we will specify a
class of finite integral metric spaces \( A \) and verify the amalgamation property
by considering a 2-point amalgamation problem \( A_i = A_0 \cup \{a_i\} \ (i = 1, 2) \)
in \( A \). To define a metric on \( A = A_1 \cup A_2 \) extending the given metrics on
\( A_1, A_2 \) it suffices to specify \( d(a_1, a_2) \) appropriately. Two extreme values are
the following.

Definition 3.6. Let \( A_i = A_0 \cup \{a_i\} \ (i = 1, 2) \) with \( a_1, a_2 \) distinct.

1. \( d^+(a_1, a_2) = \min_{x \in A_0} (d_1(a_1, x) + d_2(a_2, x)) \)
2. \( d^-(a_1, a_2) = \max(x \in A_0) |d_1(a_1, x) - d_2(a_2, x)| \)

The following is elementary.

Lemma 3.7. Let \( A_i = A_0 \cup \{a_i\} \ (i = 1, 2) \) be metric spaces with metrics
\( d_1, d_2 \) agreeing on \( A_0 \) and \( a_1 \neq a_2 \). Then \( d^-(a_1, a_2) \leq d^+(a_1, a_2) \), and the
pseudometrics extending \( d_1 \cup d_2 \) on \( A = A_1 \cup A_2 \) are exactly the symmetric
functions \( d \) extending \( d_1 \cup d_2 \) for which the value \( r = d(a_1, a_2) \) satisfies

\[ d^-(a_1, a_2) \leq r \leq d^+(a_1, a_2) \]

Note further that if \( d^-(a_1, a_2) = 0 \) then the corresponding amalgamation
problem can be solved by identifying \( a_1 \) with \( a_2 \), so that in practice only metrics come into consideration.

Of course, these considerations do not apply when \( A_0 \) is empty, but this
case presents no difficulties.

We supplement this by a similar lemma applying to metric spaces with a
bound on the perimeter of triangles.

Lemma 3.8. Let \( A_i = A_0 \cup \{a_i\} \ (i = 1, 2) \) be metric spaces with metrics
\( d_1, d_2 \) agreeing on \( A_0 \) and \( a_1 \neq a_2 \), and suppose that no triangle in \( A_1 \) or \( A_2 \)
has perimeter greater than \( P \). Define

\[ \tilde{d}(a_1, a_2) = \min_{x \in A_0} (P - [d(a_1, x) + d(a_2, x)]) \]

Then \( d^-(a_1, a_2) \leq \tilde{d}(a_1, a_2) \), and the pseudometrics extending \( d_1 \cup d_2 \) to
\( A = A_1 \cup A_2 \) for which the bound on perimeter \( P \) continues to hold are
exactly the symmetric functions \( d \) extending \( d_1 \cup d_2 \) for which the value
\( r = d(a_1, a_2) \) satisfies

\[ d^-(a_1, a_2) \leq r \leq \min(d^+(a_1, a_2), \tilde{d}(a_1, a_2)) \]

Proof. In view of the previous lemma and the definition of \( \tilde{d}(a_1, a_2) \), the
only thing that needs to be checked is the inequality

\[ d^-(a_1, a_2) \leq \tilde{d}(a_1, a_2) \]

For any \( u, v \in A_0 \) the perimeter of the triangle \( (a_1, u, v) \) is at most \( P \), and therefore

\[ d_1(a_1, u) + d_1(a_1, v) + d_2(a_2, v) \leq P + d_2(a_2, v) - d(u, v) \leq P + d_2(a_2, u) \]
Thus
\[ d_1(a_1, u) - d_2(a_2, u) \leq P - (d_1(a_1, v) + d_2(a_2, v)) \]
The same applies with \( a_1, a_2 \) reversed, that is
\[ |d_1(a_1, u) - d_2(a_2, u)| \leq P - (d_1(a_1, v) + d_2(a_2, v)) \]
So \( d^-(a_1, a_2) \leq \tilde{d}(a_1, a_2). \)

Lemma 3.9. Let \( A = A^\delta_{\infty, 2\delta + 1, C_0; S} \) with \( \delta \geq 4 \) and \( C_0 > 2\delta + 2 \) if \( S \) is nonempty. Then \( A \) is an amalgamation class.

Proof. We begin with a 2-point amalgamation problem \( A_i = A_0 \cup \{a_i\} \) \((i = 1, 2)\) in \( A \).
Taking \( P = C_0 - 2 \) we apply the previous lemma and find that for any parameter \( r \) satisfying
\[ d^-(a_1, a_2) \leq r \leq \min(d^+(a_1, a_2), \tilde{d}(a_1, a_2)) \]
we have a pseudometric extending \( d_1 \cup d_2 \) on \( A = A_1 \cup A_2 \) with \( d(a_1, a_2) = r \), and with no triangles of perimeter greater than \( C_0 - 2 \). If \( d^-(a_1, a_2) = 0 \) we amalgamate by identifying \( a_1, a_2 \), so we will suppose \( d^-(a_1, a_2) \geq 1 \).

As \( A \) contains no triangles of odd perimeter, the parity of \( d(a_1, u) \pm d(a_2, u) \) is independent of the choice of \( u \in A_0 \). In particular \( d^-(a_1, a_2), d^+(a_1, a_2), \) and \( \tilde{d}(a_1, a_2) \) all have the same parity, and for any choice of the parameter \( r \) in the specified range, and of the correct parity, the corresponding metric on \( A \) has no triangles of odd perimeter.

If \( S \) is empty nothing more is needed. So suppose that \( S \) is nonempty and that \( \delta \geq 4 \) and \( C_0 > 2\delta + 2 \).

It then suffices to show that the parameter \( r \) can be chosen distinct from 1 and \( \delta \), so that the constraints corresponding to the family \( S \) are not violated. Now \( d^-(a_1, a_2) < \delta \), so we take \( r = d^-(a_1, a_2) \) unless \( d^-(a_1, a_2) = 1 \).

Suppose \( d^-(a_1, a_2) = 1 \). Now \( d^+(a_1, a_2), \tilde{d}(a_1, a_2) \geq 2 \) since \( C \geq 2\delta + 2 \).
Thus \( \min(d^+(a_1, a_2), \tilde{d}(a_1, a_2)) > d^-(a_1, a_2) \), and as these have the same parity, we find that the value \( r = 3 \) meets our constraints. As \( \delta > 3 \) this value is satisfactory.

We may describe these classes, and the associated collection of graphs \( \Gamma^\delta_{\infty, 2\delta + 1, C_0; S} \) more explicitly by using irredundant constraint sets \( S \). Let \( S_n = \{I^\delta_n\} \) for \( n < \infty \), where \( I^\delta_n \) is an independent set of \( n \) vertices with all distances equal to \( \delta \). Let \( S_{\infty} = \emptyset \). Let \( A^\delta_{b, C_0; n} = A^\delta_{\infty, 2\delta + 1, C_0; S_n} \) and let \( \Gamma^\delta_{b, C_0; n} \) be the metrically homogeneous graph associated with the amalgamation class \( A^\delta_{b, C_0; n} \). Then the graphs described in the previous lemma are those of the form:

1. \( \Gamma^\delta_{b, C_0; \infty} \) for \( \delta \) odd;
2. \( \Gamma^\delta_{b, C_0; n} (n \leq \infty) \) for \( \delta \) even.

Here \( 2\delta + 4 \leq C_0 \leq 3\delta + 2 \) with \( C_0 \) even, and \( n \geq 3 \).
3.5. Antipodal Bipartite Amalgamation Classes.

**Lemma 3.10.** Let $A = A_{\infty,0;2\delta+1}^\delta$. Then $A$ is an amalgamation class and the corresponding metrically homogeneous graph $\Gamma_{\infty,0;2\delta+1}^\delta$ is antipodal bipartite.

**Proof.** Consider a 2-point amalgamation problem $A_i = A_0 \cup \{a_i\}$ ($i = 1, 2$) in $A$.

We amalgamate using $d(a_1, a_2) = d^-(a_1, a_2)$. Taking $P = 2\delta$ and $\tilde{d}(a_1, a_2) = \min_{x \in A_0} (P - [d(a_1, x) + d(a_2, x)])$ where as usual this means that $a_1, a_2$ are identified if the distance is 0. we have $d^-(a_1, a_2) \leq \min(d^+(a_1, a_2), \tilde{d}(a_1, a_2))$ and thus this gives an amalgam respecting the bound $P$. Since the parity is correct, the result is again in $A$. \hfill \Box

Now recall that $A_a^\delta$ is the set of finite integral metric spaces in which no triangle has perimeter greater than $2\delta$, and $A_{a,n}^\delta$ is the subclass of $A_a^\delta$ containing no subspaces of the form $I_2^{-1}[K_k, K_\ell]$ ($k + \ell = n + 1$).

We will show that $A_{a,n}^\delta$ is an amalgamation class for $\delta \geq 4$, $n \geq 2$.

**Lemma 3.11.** Let $\delta$ be fixed, and let $A$ be a finite metric space with no triangle of perimeter greater than $\delta$. Then there is a unique “antipodal” extension $\hat{A}$ of $A$, up to isometry, to a metric space satisfying the same condition, in which every vertex is paired with an antipodal vertex at distance $\delta$, and every vertex not in $A$ is antipodal to one in $A$.

If $A$ is in $A_{a,n}^\delta$ then so is $\hat{A}$.

**Proof.** The uniqueness is clear: let $B = \{a \in A : \text{There is no } a' \in A \text{ with } d(a, a') = \delta\}$ and introduce a set of new vertices $B' = \{b' : b \in B\}$. Let $\hat{A} = A \cup B'$ as a set. Then there is a unique symmetric function on $\hat{A}$ extending the metric on $A$, with $d(x, b') = \delta - d(x, b)$ for $x \in \hat{A}, b \in B$.

So the issue is one of existence, and for that we may consider the problem of extending $A$ one vertex at a time, that is to $A \cup \{b'\}$ with $b \in B$, as the rest follows by induction.

We need to show that the canonical extension of the metric on $A$ to a function $d$ on $A \cup \{b'\}$ is in fact a metric, satisfies the antipodal law for $\delta$, and also satisfies the constraints corresponding to $n$.

The triangle inequality for triples $(b', a, c)$ or $(a, b', c)$ corresponds to the ordinary triangle inequality for $(b, a, c)$ or the bound on perimeter for $(a, b, c)$ respectively, and the bound on perimeter for triangles $(a, b', c)$ follows from the triangle inequality for $(a, b, c)$.

Now suppose $n < \infty$ and $b'$ belongs to a configuration $I_2^{-1}[K_k, K_\ell]$ with $k + \ell = n + 1$. We may suppose that $b' \in K_k$: then $K_k \setminus \{b'\} \cup (K_\ell \cup \{b'\})$ provides a copy of $I_2^{-1}[K_{k-1}, K_{\ell+1}]$ in $A$, and a contradiction. \hfill \Box

**Lemma 3.12.** If $\delta \geq 4$ is finite, $2 \leq n \leq \infty$, then $A_{a,n}^\delta$ is an amalgamation class.
Proof. We consider a two-point amalgam with \( A_i = A_0 \cup \{a_i\}, i = 1, 2 \). If \( d(a_2, x) = \delta \) for some \( x \in A_0 \) then there is a canonical amalgam \( A_1 \cup A_2 \) embedded in \( \hat{A}_1 \). So we will suppose \( d(a_i, x) < \delta \) for \( i = 1, 2 \) and \( x \in A_0 \).

We claim that any metric \( d \) on \( A_1 \cup A_2 \) extending the given metrics \( d_i \) on \( A_i \) will satisfy the antipodal law for \( \delta \). So with \( d \) such a metric, consider a triangle of the form \((a_0, a_1, a_2)\) with \( a_i \in A_0, i = 0, 1, 2 \). By the triangle law for \((a_1, a_0, a_2)\) we have

\[
d(a_1, a_2) \leq 2\delta - [d(a_1, a_0) + d(a_2, a_0)]
\]

and this is the desired bound on perimeter.

We know by our general analysis that any value \( r \) for \( d(a_1, a_2) \) with

\[
d^-(a_1, a_2) \leq r \leq d^+(a_1, a_2)
\]

will give us a metric, and in the bipartite case we will want \( r \) to avoid the values \( \delta \) and \( 3\). In this case, we take

\[
r = 3
\]

\( \square \)

3.6. \( \Gamma \) and \( \Gamma_1 \) both primitive, infinite. We deal here with the graphs \( \Gamma_{K_1, K_2; C, C'; S}^\delta \) where we put aside the cases \( K_1 = \infty \), \( C' = 2\delta + 2 \) which fall under group III. This leaves us with the following choices of admissible parameters. We recall the relevant conditions, which vary according as \( C \leq 2\delta + K_1 \) or not.

(1) We require in all cases:
(a) \( \delta \geq 3 \);
(b) \( 1 \leq K_1 \leq K_2 \leq \delta \) (leaving aside the bipartite case);
(c) \( 2\delta + 1 \leq C < C' \leq 3\delta + 2 \) with one of \( C, C' \) even and the other odd;
(d) \( C' > 2\delta + 2 \) (leaving aside the antipodal case);
(e) \( S \) is a set of finite \((1, \delta)\)-spaces of order at least 3.

(2) If \( C \leq 2\delta + K_1 \) we require:
(a) \( C = 2K_1 + 2K + 2 + 1 \);
(b) \( K_1 + K_2 \geq \delta \);
(c) \( K_2 + 2K_2 \leq 2\delta - 1 \);
(d) If \( C' > C + 1 \) then \( K_1 = K_2 \) and \( 3K_2 = 2\delta - 1 \).
(e) If \( K_1 = 1 \) then \( S \) is empty.

(3) If \( C > 2\delta + K_1 \) we require:
(a) \( K_1 + 2K_2 \geq 2\delta - 1 \) and \( 3K_2 \geq 2\delta \);
(b) If \( K_1 + 2K_2 \geq 2\delta - 1 \) then \( C \geq 2\delta + K_1 + 2 \);
(c) If \( C' > C + 1 \) then \( C \geq 2\delta + K_2 \);
(d) If \( K_1 = 1, K_2 = \delta \), and \( C = 2\delta + 2 \), then \( S \) is empty.
We will show that all of the associated classes \( \mathcal{A}_{K_1,K_2;C,C';\delta}^\delta \) are amalgamation classes.

**Lemma 3.13.** Let \( \mathcal{A} = \mathcal{A}_{K_1,K_2}^\delta \) with \( 1 \leq K_1 \leq K_2 \leq \delta \). Let \( A_i = A_0 \cup \{ a_i \} \) \((i=1,2)\) be a 2-point amalgamation problem in \( \mathcal{A} \), \( A = A_1 \cup A_2 \), and \( d_i \) the metric on \( A_i \).

Suppose that

\[
d^+(a_1,a_2) \leq K_2
\]

Let \( d \) be the symmetric extension of \( d_1 \cup d_2 \) to \( A \) defined by

\[
d(a_1,a_2) = d^+(a_1,a_2)
\]

Then \((A,d) \in \mathcal{A}\).

*Proof.* Let \( u \in A_0 \). We must check that the triangle \((a_1,a_2,u)\) is permitted.

Let \( i = d(a_1,a_2) \), \( j = d(a_1,u) \), \( k = d(a_2,u) \), \( P = i + j + k \). We may suppose that \( P \) is odd.

Fix \( v \in A_0 \) with

\[
d(a_1,a_2) = d_1(a_1,v) + d_2(a_2,v)
\]

Set \( i_\ell = d(a_\ell,v) \) for \( \ell = 1,2 \). Then \( i = i_1 + i_2 \) and the perimeter of \((a_1,u,a_2,v)\) is again \( P \), which is odd. Therefore one of the triangles \((a_1,u,v)\) or \((a_2,u,v)\) has odd perimeter, at most \( P \). We may suppose the triangle in question is \((a_1,u,v)\).

The perimeter of \((a_1,u,v)\) is at least \(2K_1+1\), hence \( P \geq 2K_1+1 \). So it suffices to check the inequalities corresponding to \(K_2\):

\[
i + j \leq 2K_2 + k; \quad i + k \leq 2K_2 + j; \quad j + k \leq 2K_2 + i
\]

By assumption \( i \leq K_2 \). Therefore

\[
i + j \leq 2i + k \leq 2K_2 + k
\]

and similarly \( i + k \leq 2K_2 + j \).

As the triangle \((a_1,u,v)\) has odd perimeter, we find

\[
j + d(u,v) \leq 2K_2 + i_1
\]

and therefore

\[
j + k \leq j + d(u,v) + i_2 \leq 2K_2 + i_1 + i_2 = 2K_2 + i
\]

and the final inequality holds as well. \( \square \)

**Lemma 3.14.** Let \( \mathcal{A} = \mathcal{A}_{K_1,K_2;C,C';\delta}^\delta \) with \( 1 \leq K_1 \leq K_2 \leq \delta \), with admissible parameters \( \delta,K_1,K_2,C,C' \), and \( K_1 < \infty \). Let \( A_i = A_0 \cup \{ a_i \} \) \((i=1,2)\) be a 2-point amalgamation problem in \( \mathcal{A} \), \( A = A_1 \cup A_2 \), and \( d_i \) the metric on \( A_i \). Suppose that one of the following holds.

1. \( d^-(a_1,a_2) \geq K_1 \) and \( C \leq 2\delta + K_1 \);
2. \( d^-(a_1,a_2) > K_1 \).
Let \( d \) be the symmetric extension of \( d_1 \cup d_2 \) to \( A \) defined by

\[
d(a_1, a_2) = d^-(a_1, a_2)
\]

Then \((A, d) \in A\).

Proof. As usual we may suppose that \( d(a_1, a_2) > 0 \) so that this gives a metric.

We must show that all resulting triangles \((a_1, a_2, u)\) with \( u \in A_0 \) are permitted. Fix \( u \in A_0 \) and set \( i = d(a_1, a_2), \, j = d(a_1, u), \, k = d(a_2, u), \) \( P = i + j + k \), and \( P_\ell \) the perimeter of \((a_\ell, u, v)\) for \( \ell = 1, 2 \).

We show first that \( P < C' \) where \( C' = 0 \) or \( C' = 1 \) corresponds to the parity of \( P \). As \( d^-(a_1, a) \leq \tilde{d}(a_1, a_2) \) where \( \tilde{d} \) is defined relative to the bound \( C' = C + 1 \) if \( C' = C + 1 \), or \( C' = C \) if \( C' > C + 1 \). Thus we have \( P < C' \) in any case, and \( P < C \) if \( C' = C + 1 \). So we need to show that in the case

\[
P \equiv C \mod 2, \quad C' > C + 1
\]

we have \( P < C \).

Since the parameters are admissible there are two possible cases of this type.

\[
K_1 = K_2, \quad C = 4K_2 + 1 = 2\delta + K_2; \quad \text{or}
\]

\[
C \geq 2\delta + K_2, \quad C > 2\delta + K_1
\]

We have \( P = i_2 - i_1 + j + k \leq i_2 + d(u, v) + k = P_2 \). If \( P_2 < C \) then \( P < C \). If \( P_2 \geq C \) then we have \( P_2 \equiv C' \mod 2 \) and hence \( P_1 \) is odd. Also from \( P_2 \geq C \) we get

\[
2\delta + K_2 \leq C \leq P_2 \leq 2\delta + d(u, v)
\]

and thus

\[
d(u, v) \geq K_2
\]

Now since \( P_1 \) is odd, we have

\[
j + d(u, v) < 2K_2 + i_1
\]

and thus

\[
j \leq d(u, v) - K_2 < K_2 + i_1
\]

Then

\[
P = i_1 - i_1 + j + k < i_1 + K_2 + k \leq 2\delta + K_2 \leq C
\]

and we have the required bound on the perimeter in this case as well.

We still need to show that the triangle \((a_1, a_2, u)\) is in \( A^\delta_{K_1, K_2} \). We may suppose that \( P \) is odd. As \( i \geq K_1 \) it follows that \( P \geq 2K_1 + 1 \). So it remains to check the inequalities

\[
i + j \leq 2K_2 + k; \quad i + k \leq 2K_2 + j; \quad j + k \leq 2K_2 + i
\]

We begin with the last.
If $C \leq 2\delta + K_1$ then $C = 2K_1 + 2K + 2 + 1$ is odd and therefore $P < C$. Thus

$$j + k \leq 2K_1 + 2K_2 - i \leq 2K_2 + i$$

If on the other hand $C > 2\delta + K_1$ then $K_1 + 2K_2 \geq 2\delta - 1$ and we are assuming $i > K_1$, so

$$j + k \leq 2\delta \leq K_1 + 2K_2 + 1 \leq 2K_2 + i$$

The two inequalities which remain to be checked are

$$i + j \leq 2K_2 + k; \quad i + k \leq 2K_2 + j$$

As $P$ is odd, either $P_1$ or $P_2$ is odd.

If $P_2$ is odd, then

$$i + j = i_2 + j - i_1 \leq 2K_2 + d(u, v) - i_1 \leq 2K_2 + j$$

Thus both hold in this case.

Suppose now that $P_1$ is odd

Then we have

$$i + j + d(u, v) \leq i + 2K_2 + i_1$$

$$= 2K_2 + i_2 \leq 2K_2 + d(u, v) + k$$

$$i + j \leq 2K_2 + k$$

We still need to check

$$i + k \leq 2K_2 + j$$

when $P_1$ is odd, and we have several combinations of admissible parameters to consider.

Suppose first that

$$C = 2K_1 + 2K_2 + 1; \quad C' = C + 1$$

Then $P_1 \geq 2K_1 + 1$ and $P_2 \leq 2K_1 + 2K_2$, so we take the difference:

$$(i_2 + k) - (i_1 + j) \leq 2K_2 - 1$$

$$i + k \leq 2K_2 + j$$

Now suppose that

$$K_1 = K_2, \quad C = 4K_2 + 1 = 2\delta + K_2$$

As $P_1 \geq 2K_1 + 1 = 2K_2 + 1$ we have $i_1 + j > K_2$. Hence

$$i + k = i_2 + k - i_1 \leq 2\delta - i_1 = 3K_2 + 1 - i_1 \leq 2K_2 + j$$

Finally, suppose that $C > 2\delta + K_1$ and $K_1 + 2K_2 \geq 2\delta - 1$. As $P_1 \geq 2K_1 + 1$ we find

$$i + k = i_2 - i_1 + k \leq 2\delta - 1 \leq K_1 + 2K_2 + 1 - i_1$$

$$= 2K_2 + (K_1 + 1) - i_1 \leq 2K_2 + (i_1 + j) - i_1 = 2K_2 + j$$
Lemma 3.15. Let \( A = A_{K_1,K_2}^\delta \) with \( 1 \leq K_1 \leq K_2 \leq \delta, C = 2K_1 + 2K_2 + 1 \). Let \( A_i = A_0 \cup \{a_i\} \) \( (i = 1,2) \) be a 2-point amalgamation problem in \( A \), \( A = A_1 \cup A_2 \), and \( d_i \) the metric on \( A_i \). Suppose that
\[
\tilde{d}(a_1,a_2) \leq \min(K_2,d^+(a_1,a_2))
\]
where \( \tilde{d} \) is defined using the bound on perimeter \( P = C - 1 \). Let \( d \) be the symmetric extension of \( d_1 \cup d_2 \) to \( A \) defined by
\[
d(a_1,a_2) = \tilde{d}(a_1,a_2)
\]
Then \((A,d) \in A\).

Proof. We have
\[
d^+(a_1,a_2) - \tilde{d}(a_1,a_2) \leq d^+(a_1,a_2)
\]
by hypothesis and thus \( d \) is at least a pseudometric. As usual we may suppose that \( d(a_1,a_2) > 0 \) and thus we have a metric. In view of the definition of \( \tilde{d} \) our metric respects the bound on perimeters.

We must show that all resulting triangles \((a_1,a_2,u)\) with \( u \in A_0 \) are in \( A \). Fix \( u \in A_0 \) and set \( i = d(a_1,a_2), j = d(a_1,u), k = d(a_2,u), P = i + j + k \). We may suppose that \( P \) is odd.

Choose \( v \in A_0 \) so that
\[
d(a_1,a_2) = C - 1 - [d(a_1,v) + d(a_2,v)]
\]
let \( i_\ell = d(a_\ell,v) \) for \( \ell = 1,2 \). As \( C - 1 \) is even and \( P \) is odd, the perimeter of \((a_1,u,a_2,v)\) is odd. Hence one of the triangles \((a_1,u,v)\) or \((a_2,u,v)\) has odd perimeter. We may suppose that \((a_1,u,v)\) has odd perimeter.

We show first that
\[
P \geq 2K_1 + 1
\]
As \((a_1,u,v)\) has odd perimeter we have the constraint
\[
i_1 + d(u,v) < 2K_2 + j
\]
Hence
\[
(C - 1) - i = i_1 + i_2 \leq i_1 + d(u,v) + k < 2K_2 + j + k
\]
\[
2K_1 + 2K_2 < 2K_2 + i + j + k
\]
\[
2K_1 < P
\]
as required.

Now we deal with the inequalities
\[
i + j \leq 2K_2 + k; \quad i + k \leq 2K_2 + j; \quad j + k \leq 2K_2 + i
\]
Since \( d(a_1,a_2) \leq K_2 \), the first two are immediate as in the proof of Lemma 3.13.

As the triangle \((a_1,u,v)\) has odd perimeter we have
\[
i_1 + j \leq 2K_2 + d(u,v)
\]
Let \( d \) (in particular, \( C \)) amalgamation problem in \( A \) and thus \( j \) that one of the following holds.

\[
(C - i) + j + k = 1 + i_1 + i_2 + j + k \\
\leq 1 + 2K_2 + d(u, v) + i_2 + k \\
\leq 2K_2 + C
\]

and thus \( j + k \leq 2K_2 + i \), as required. \( \square \)

**Lemma 3.16.** Let \( A = \mathcal{A}^{\delta}_{K_1, K_2; C} \) with admissible parameters \( \delta, K_1, K_2, C \) (in particular, \( C' = C + 1 \) here). Let \( A_i = A_0 \cup \{a_i\} \ (i = 1, 2) \) be a 2-point amalgamation problem in \( A \), \( A = A_1 \cup A_2 \), and \( d_i \) the metric on \( A_i \). Suppose that one of the following holds.

1. \( d^+(a_1, a_2) \leq K_2 \) and \( C = 2K_1 + 2K_2 + 1 \);
2. \( d^+(a_1, a_2) \leq K_1 \) and \( C > 2\delta + K_1 \);
3. \( d^+(a_1, a_2) < K_2 \) and \( C \geq 2\delta + K_2 \).

Let \( d \) be the symmetric extension of \( d_1 \cup d_2 \) to \( A \) defined by

\[
d(a_1, a_2) = \min(d^+(a_1, a_2), \tilde{d}(a_1, a_2))
\]

where \( \tilde{d} \) is defined using the bound on perimeter \( P = C - 1 \).

Then \((A, d) \in \mathcal{A} \).

**Proof.** By Lemma 3.13 we have \((A, d^+) \in \mathcal{A}^{\delta}_{K_1, K_2} \).

If \( d^+(a_1, a_2) \leq \tilde{d}(a_1, a_2) \) then \((A, d^+) \in \mathcal{A}^{\delta}_{K_1, K_2; C} \) by Lemma 3.8, and as \( d = d^+ \) in this case our claim follows.

If \( C > 2\delta + d^+(a_1, a_2) \) then indeed \( d^+(a_1, a_2) \leq \tilde{d}(a_1, a_2) \) and our claim follows. This covers the second and third cases, so we need only consider the first case, under the assumption

\[
\tilde{d}(a_1, a_2) < d^+(a_1, a_2)
\]

In this case Lemma 3.15 applies. \( \square \)

**Lemma 3.17.** Let \( A = \mathcal{A}^{\delta}_{K_1, K_2; C} \) with admissible parameters \( \delta, K_1, K_2, C \) and \( C = 2K_1 + 2K_2 + 1 \). Let \( A_i = A_0 \cup \{a_i\} \ (i = 1, 2) \) be a 2-point amalgamation problem in \( A \), \( A = A_1 \cup A_2 \), and \( d_i \) the metric on \( A_i \). Suppose that

\[
K_1 \leq \min(d^+(a_1, a_2), \tilde{d}(a_1, a_2)) \text{ and } d^-(a_1, a_2) \leq K_2
\]

where \( \tilde{d} \) is defined in terms of the bound \( P = C - 1 \) on perimeters.

Then

\[
\max(K_1, d^-(a_1, a_2)) \leq \min(d^+(a_1, a_2), \tilde{d}(a_1, a_2), 2K_2 - d^-(a_1, a_2))
\]

and for any \( i \) between these two bounds, if \( d \) is the symmetric extension of \( d_1 \cup d_2 \) to \( A \) defined by

\[
d(a_1, a_2) = i
\]

then \((A, d) \in \mathcal{A} \).
Proof. In general $d^-(a_1, a_2) \leq \min(d^+(a_1, a_2), \tilde{d}(a_1, a_2))$ so the stated inequality follows from our additional hypotheses.

Suppose now that $i = d(a_1, a_2)$ lies between these bounds. As usual we may suppose that $i > 0$ and thus we have a metric respecting the bound on perimeters. We must show that all resulting triangles $(a_1, a_2, u)$ with $u \in A_0$ are in $\mathcal{A}_{K_1,K_2}^\delta$. Fix $u \in A_0$ and set $j = d(a_1, u)$, $k = d(a_2, u)$, $P = i + j + k$. We may suppose that $P$ is odd.

As $i \geq K_1$ we have $P \geq 2K_1 + 1$. So it suffices to check the inequalities

\[ i + j \leq 2K_2 + k; \quad i + k \leq 2K_2 + j; \quad j + k \leq 2K_2 + i \]

We have $j - k \leq d^-(a_1, a_2) \leq 2K_2 - i$ and hence

\[ i + j \leq 2K_2 + k \]

Similarly $i + k \leq 2K_2 + j$. Finally

\[ i + j + k \leq C - 1 = 2K_2 + 2K_1 \leq 2K_2 + 2i \]

and this concludes the proof. \qed

Lemma 3.18. Let $A = A_{K_1,K_2;C,C'}^\delta$ with $K_1 = K_2$, $C = 4K_2 + 1 = 2\delta + K_2$. Let $A_i = A_0 \cup \{a_i\}$ $(i = 1, 2)$ be a 2-point amalgamation problem in $\mathcal{A}$, $A = A_1 \cup A_2$, and $d_i$ the metric on $A_i$. Suppose that

\[ d^-(a_1, a_2) \leq K_2 \leq d^+(a_1, a_2) \]

Let $d$ be the symmetric extension of $d_1 \cup d_2$ to $A$ defined by

\[ d(a_1, a_2) = \begin{cases} 
K_2 - 1 & \text{if there is } v \in A_0 \text{ with } d(a_1, v) = d(a_2, v) = \delta \\
K_2 & \text{otherwise} 
\end{cases} \]

Then $(A, d) \in \mathcal{A}$.

Proof. We must first verify the condition

\[ d^-(a_1, a_2) \leq d(a_1, a_2) \leq d^+(a_1, a_2) \]

in the case in which $d(a_1, a_2) = K_2 - 1$. So suppose $v \in A_0$ and

\[ d(a_1, v) = d(a_2, v) = \delta \]

By assumption $d^-(a_1, a_2) \leq K_2$ and we claim in this case that $d^-(a_1, a_2) \neq K_2$. Assuming the contrary we may suppose we have $u \in A_0$ with $d(a_2, u) - d(a_1, u) = K_2$. Set $i_\ell = d(a_\ell, u)$ for $\ell = 1, 2$, so that $i_2 = i_1 + K_2$. Then the perimeter $P_2$ of $(a_2, u, v)$ is $\delta + d(u, v) + K_2 + i_1 \geq \delta + K_2 + d(a_1, v) = 2\delta + K_2$, so $P_2 \geq 2\delta + K_2$, and in view of the bound $C = 2\delta + K_2$ it follows that $P_2 \equiv C' \mod 2$. Hence $P_1$ is odd.

Thus

\[ \delta + d(u, v) < 2K_2 + i_1 = K_2 + i_2 \]
\[ 2\delta - i_1 \leq \delta + d(u, v) < K_2 + i_2 \]
\[ 2\delta < i_1 + K_2 + i_2 = 2i_2 \]

a contradiction.
Now consider a triangle \((a_1, a_2, u)\) with \(u \in A_0\). We must show that \((a_1, a_2, u) \in A\). Set \(i = d(a_1, a_2) = K_2 - \epsilon\) with \(\epsilon = 0\) or \(1\). Let \(j = d(a_1, u), k = d(a_2, u)\), \(P = i + j + k\).

Then \(j + k \leq 2\delta - (1 - \epsilon)\) so

\[ P = i + j + k < (2\delta + \epsilon) + i = 2\delta + K_2 \]

So \(P < C\).

We may now suppose that \(P\) is odd. We claim that \(P \geq 2K_1 + 1\). If \(i = K_2\) this is clear. If \(i = K_2 - 1\) then we must consider the possibility

\[ j + k = K_2 \]

We have an element \(v \in A_0\) with \(d(a_1, v) = d(a_2, v) = \delta\), and our assumptions imply that \(K_2\) is odd. Thus the perimeter of \((a_1, u, a_2, v)\) is odd. So we may suppose that the triangle \((a_1, u, v)\) has odd perimeter. But then

\[ d(u, v) + d(a_1, v) \geq d(a_2, v) - k + d(a_1, v) \]

\[ = 2\delta - k <= 2\delta - K_2 + j > 2K_2 + j \]

and the triangle \((a_1, u, v)\) is forbidden, a contradiction. Thus we have \(P \geq 2K_1 + 1\) in all cases.

It remains to check the inequalities

\[ i + j \leq 2K_2 + k; \quad i + k \leq 2K_2 + j; \quad j + k \leq 2K_2 + i \]

Now

\[ i + j \leq 2i + k \leq 2K_2 + k \]

and similarly \(i + k \leq 2K_2 + j\). For the last inequality we have

\[ j + k \leq 2\delta - (1 - \epsilon) = 3K_2 - \epsilon = 2K_2 + i \]

\[ \square \]

**Lemma 3.19.** Let \(A = A^\delta_{K_1, K_2; C}\) with parameters satisfying

1. \(C > 2\delta + K_1\);
2. \(K_1 + 2K_2 \geq 2\delta - 1, 3K_2 \geq 2\delta\);
3. If \(K_1 + 2K_2 = 2\delta - 1\) then \(C > 2\delta + K_1 + 1\).

Let \(A_i = A_0 \cup \{a_i\} \ (i = 1, 2)\) be a 2-point amalgamation problem in \(A\), \(A = A_1 \cup A_2\), and \(d_i\) the metric on \(A_i\). Suppose that

\[ d^-(a_1, a_2) \leq K_1 < d^+(a_1, a_2) \]

Let \(d\) be the symmetric extension of \(d_1 \cup d_2\) to \(A\) defined by

\[ d(a_1, a_2) = \begin{cases} 
K_1 + 1 & \text{if there is } v \in A_0 \text{ with } d(a_1, v) = d(a_2, v) = \delta, \text{ and} \\
k_1 + 2k_2 = 2\delta - 1 & \text{otherwise} \\
K_1 & \text{otherwise}
\end{cases} \]

Then \((A, d) \in A\).
Proof. We must show that for \( u \in A_0 \) the triangle \((a_1, a_2, u)\) belongs to \( \mathcal{A} \).

Let \( i = d(a_1, a_2) = K_1 + \epsilon, j = d(a_1, u), k = d(a_2, u) \), \( P = i + j + k \).

We observe that if \( d(a_1, a_2) = K_1 + 1 \) then as \( K_1 + 2K_2 = 2\delta - 1 \) we have also \( C > 2\delta + K_1 + 1 \) and, since \( 3K_2 \geq 2\delta \), also \( K_2 > K_1 \). In particular \( d(a_1, a_2) \leq K_2 \) in all cases.

Now \( P \geq 2\delta + K_1 + 1 \leq C \), so to violate the bound on perimeter would require \( d(a_1, u) = d(a_2, u) = \delta, d(a_1, a_2) = K_1 + 1 \), in which case as \( C > 2\delta + K_1 + 1 \) we still have \( P < C \).

So we may suppose \( P \) is odd, and since \( i \geq K_1 \) we have \( P \geq 2K_1 + 1 \). So it suffices to check the inequalities

\[
i + j \leq 2K_2 + k; \quad i + k \leq 2K_2 + j; \quad j + k \leq 2K_2 + i
\]

We have

\[
i + j \leq 2i + k \leq 2K_2 + k
\]

and similarly \( i + k \leq 2K_2 + j \). So it suffices to deal with \( j + k \).

We have

\[
j + k \leq 2\delta \leq 2K_2 + K_1 + 1 \leq 2K_2 + i + 1
\]

and hence \( j + k \leq 2K_2 + i \) unless

\[
j + k = 2\delta = 2K_2 + K_1 + 1 = 2K_2 + i + 1
\]

or \( j = k = \delta, K_1 + 2K_2 = 2\delta - 1 \), and \( i = K_1 \); but \( i = K_1 + 1 \) in this case. \( \square \)

**Lemma 3.20.** Let \( \mathcal{A} = \mathcal{A}^\delta_{K_1, K_2; C, C'} \) with parameters satisfying

1. \( C > 2\delta + K_1 \) and \( C \geq 2\delta + K_2 \);
2. \( K_1 + 2K_2 \geq 2\delta - 1 \), \( 3K_2 \geq 2\delta \);
3. If \( 3K_2 = 2\delta \) then \( C > 2\delta + K_2 \).

Let \( A_i = A_0 \cup \{a_i\} \) \((i = 1, 2)\) be a 2-point amalgamation problem in \( \mathcal{A} \), \( A = A_1 \cup A_2 \), and \( d_i \) the metric on \( A_i \). Suppose that

\[
d^+(a_1, a_2) \leq K_1
\]

\[
\min(K_2, C - 2\delta - 1) \leq d^+(a_1, a_2)
\]

Let \( d \) be the symmetric extension of \( d_1 \cup d_2 \) to \( A \) defined by

\[
d(a_1, a_2) = \min(K_2, C - 2\delta - 1)
\]

Then \( (A, d) \in \mathcal{A} \).

**Proof.** Set \( i = \min(K_2, C - 2\delta - 1) \). We have

\[
d^-(a_1, a_1) \leq K_1 \leq i \leq d^+(a_1, a_2)
\]

and thus \( d \) is a pseudometric on \( A \). We may suppose as usual that \( d(a_1, a_2) > 0 \) and thus \( d \) is a metric.

Fix \( u \in A_0 \) we must show that the triangle \((a_1, a_2, u)\) belongs to \( \mathcal{A} \). Let \( j = d(a_1, u), k = d(a_2, u), P = i + j + k \). We have

\[
P \leq 2\delta + i < C
\]
So we may suppose that \( P \) is odd. As \( i \geq K_1 \) we have \( P \geq 2K_1 + 1 \). Thus it suffices to check the inequalities

\[
i + j \leq 2K_2 + k; \quad i + k \leq 2K_2 + j; \quad j + k \leq 2K_2 + i
\]

We have

\[
i + j \leq 2i + k \leq 2K_2 + k
\]

and similarly \( i + k \leq 2K_2 + j \). So it suffices to deal with \( j + k \).

If \( 2K_2 + i \geq 2\delta \) then the inequality \( j + k \leq 2K_2 + i \) is immediate. So we will suppose

\[
2K_2 + i < 2\delta
\]

Then

\[
2\delta - 1 \leq 2K_2 + K_1 \leq 2K_2 + i < 2\delta
\]

and therefore

\[
2K_1 + K_1 = 2\delta - 1, \quad i = K_1
\]

As \( 3K_2 \geq 2\delta \) the first equation implies that \( K_2 > K_1 \) and then the second gives \( C - 2\delta - 1 = K_1 \), in other words

\[
C = 2\delta + K_1 + 1
\]

But \( C \geq 2\delta + K_2 \), so \( K_2 = K_1 + 1 \) and \( C = 2\delta + K_2 \). Also

\[
3K_2 = 2K_2 + K_1 + 1 = 2\delta
\]

and our third condition on the parameters is violated.

\[
\square
\]

Lemma 3.21. Let \( \mathcal{A} = A_0^{K_1, K_2; C', C'} \) with an admissible choice of parameters, and with \( K_1 < \infty \). Then \( \mathcal{A} \) is an amalgamation class.

Furthermore, for any 2-point amalgamation problem \( A_i = A_0 \cup \{a_i\} \) \( (i = 1, 2) \) in \( \mathcal{A} \), a suitable extension \( d \) of \( d_1 \cup d_2 \) to \( A = A_1 \cup A_2 \) is given by the following.

1. If \( C \leq 2\delta + K_1 \):
   a. If \( d^-(a_1, a_2) \geq K_1 \) then take \( d(a_1, a_2) = d^-(a_1, a_2) \).
   Otherwise:
   b. If \( C' = C + 1 \) then:
      i. If \( d^+(a_1, a_2) \leq K_2 \) then take \( d(a_1, a_2) = \min(d^+(a_1, a_2), \tilde{d}(a_1, a_2)) \)
      ii. If \( d^+(a_1, a_2) < K_1 \) and \( K_2 < d^+(a_1, a_2) \) then take \( d(a_1, a_2) = \tilde{d}(a_1, a_2) \) if \( \tilde{d}(a_1, a_2) \leq K_2 \) and \( d(a_1, a_2) = K_1 \) otherwise.
   c. If \( C' > C + 1 \) then:
      i. If \( d^+(a_1, a_2) \leq K_2 \) then take \( d(a_1, a_2) = d^+(a_1, a_2) \);
      ii. If \( d^-(a_1, a_2) \leq K_2 \) then take \( d(a_1, a_2) = \tilde{d}(a_1, a_2) \);

\[
d(a_1, a_2) = \begin{cases} 
  K_2 - 1 & \text{if there is } v \in A_0 \text{ with } d(a_1, v) = d(a_2, v) = \delta \\
  K_2 & \text{otherwise}
\end{cases}
\]

2. If \( C > 2\delta + K_1 \):
   a. If \( d^-(a_1, a_2) > K_1 \) then take \( d(a_1, a_2) = d^-(a_1, a_2) \);
   Otherwise:
   b. If \( C' = C + 1 \) then:
In particular, if \( d^+(a_1, a_2) \leq K_1 \) then take \( d(a_1, a_2) = \min(d^+(a_1, a_2), \tilde{d}(a_1, a_2)) \);
(ii) If \( d^+(a_1, a_2) > K_1 \) then take

\[
d(a_1, a_2) = \begin{cases} 
K_1 + 1 & \text{if there is } v \in A_0 \text{ with } d(a_1, v) = d(a_2, v) = \delta, \text{ and } \\
K_1 & \text{otherwise}
\end{cases}
\]

(c) If \( C' > C + 1 \) then:
(i) If \( d^+(a_1, a_2) < K_2 \) then take \( d(a_1, a_2) = d^+(a_1, a_2) \);
(ii) If \( d^+(a_1, a_2) \geq K_2 \) then take \( d(a_1, a_2) = \min(K_2, C - 2\delta - 1) \).

Proof. Suppose first

\[
C \leq 2\delta + K_1
\]

In this case, admissibility requires

\[
C = 2K_1 + 2K_2 + 1, K_1 + K_2 \geq \delta, K_1 + 2K_2 \leq 2\delta - 1
\]

If \( d^-(a_1, a_2) \geq K_1 \) then by Lemma 3.14 (1) we may take \( d(a_1, a_2) = d^-(a_1, a_2) \). So suppose

\[
d^-(a_1, a_2) < K_1
\]

Suppose \( C' = C + 1 \). Then

\[
A = A_{K_1, K_2; C}^{\delta}
\]

If \( d^+(a_1, a_2) \leq K_2 \) then by Lemma 3.16 (1) we may take \( d(a_1, a_2) = \min(d^+(a_1, a_2), \tilde{d}(a_1, a_2)) \). So suppose \( d^+(a_1, a_2) > K_2 \) then by Lemma 3 we may take \( d(a_1, a_2) = d^+(a_1, a_2) \), while if \( d(a_1, a_2) > K_2 \) then by Lemma 3.17 we may take \( d(a_1, a_2) = \max(K_1, d^-(a_1, a_2)) = K_1 \).

Now suppose \( C' > C + 1 \). Then admissibility requires

\[
K_1 = K_2, 3K_2 = 2\delta - 1
\]

In particular, \( C = 4K_2 + 1 = 2\delta + K_2 \).

If \( d^+(a_1, a_2) < K_2 \) and we take \( d(a_1, a_2) = d^+(a_1, a_2) \), then Lemma 3.13 says that \( (A, d) \in A_{K_1, K_2}^{\delta} \). On the other hand any triangle \((a_1, a_2, u)\) with \( u \in A_0 \) will have perimeter at most \( 2\delta + d(a_1, a_2) < 2\delta + K_2 = C \) and so \((a_1, a_2, u) \in A\).

If \( d^+(a_1, a_2) \geq K_2 \) then Lemma 3.18 applies. This disposes of all cases in which \( C < 2\delta + K_1 \).

Now suppose

\[
C > 2\delta + K_1
\]

If \( d^-(a_1, a_2) > K_1 \) then Lemma 3.14 (2) applies. Suppose therefore

\[
d^-(a_1, a_2) \leq K_1
\]

Suppose \( C' = C + 1 \). If \( d^+(a_1, a_2) \leq K_1 \) then Lemma 3.16 (2) applies. If \( d^+(a_1, a_2) > K_1 \) then Lemma 3.19 applies; the assumptions not given here explicitly are contained in the conditions of admissibility for \( C > 2\delta + K_1 \).
So now suppose $C' > C + 1$, and then by admissibility we have $C \geq 2\delta + K_2$. So if $d^+(a_1, a_2) < K_2$ then the bound on perimeter is respected, and for the rest Lemma 3.13 applies. If $d^+(a_1, a_2) \geq K_2$ then Lemma 3.20 applies.

Now we can prove the first half of Theorem 2.

**Proposition 3.22.** Let $\delta, K_1, K_2, C, C', S$ be an admissible choice of parameters. Then $\mathcal{A}_{K_1,K_2;C,C';S}$ is an amalgamation class.

**Proof.** The bipartite case is covered by Lemma 3.9. So we will suppose $K_1 < \infty$

In the remaining cases we have to show that when $S$ is nonempty we can perform the amalgamation for any 2-point amalgamation problem so as to avoid the distances 1 and $\delta$.

Suppose first that $C \leq 2\delta + K_1$

Recall that in this case $S$ is empty if $K_1 = 1$. So we suppose $K_1 > 1$

Furthermore $K_1 + 2K_2 \leq 2\delta - 1$ and in particular $K_2 < \delta$

If $C = 2\delta + 1$ then all $(1, \delta)$-spaces with at least three points are already forbidden, so we will assume $C > 2\delta + 1$

If $d^-(a_1, a_2) \geq K_1$ then the amalgamation procedure of Lemma 3.21 takes $d(a_1, a_2) = d^-(a_1, a_2) \neq 1, \delta$. So we suppose $d^-(a_1, a_2) < K_1$

Then our amalgamation procedure gives $d(a_1, a_2) \leq K_2 < \delta$ in all cases. So it suffices to show that $d(a_1, a_2) > 1$ in all cases.

The amalgamation procedure under the given constraints takes one of the values $d^+(a_1, a_2), \bar{d}(a_1, a_2), K_1, K_2 - 1, K_2$. The only one of these which might possibly equal 1 is $K_2 - 1$. This would mean that $K_1 = K_2 = 2$ and then by admissibility that $3K_2 = 2\delta - 1$. But then $K_2 \neq 2$. So all cases with $C \leq 2\delta + K_1$ are treated.

Now suppose that $C > 2\delta + K_1$

If $d^-(a_1, a_2) > K_1$ we take $d(a_1, a_2) = d^-(a_1, a_2)$ and this is neither 1 nor $\delta$. So suppose $d^-(a_1, a_2) \leq K_1$

We deal first with the case $C' = C + 1$
Then $\tilde{d}(a_1, a_2) \geq C - 1 - 2\delta \geq K_1$. 
If $d^+(a_1, a_2) \leq K_1$ we take 
\[ d(a_1, a_2) = \min(d^+(a_1, a_2), \tilde{d}(a_1, a_2)) \]
As $d^+(a_1, a_2) \leq K_1$ we have $K_1 \geq 2$ and thus $\tilde{d}(a_1, a_2) \geq 2$. Thus $d(a_1, a_2) \geq 2$ in this case.

If $d(a_1, a_2) = \delta$ then $\delta \leq d^+(a_1, a_2) \leq K_1$ so $K_1 = \delta$. In this case $S$ is empty and we have nothing more to show. So the amalgamation succeeds in this case.

If $d^+(a_1, a_2) > K_1$ then our amalgamation procedure takes $d(a_1, a_2) = K_1$ or $K_1 + 1$. Suppose $d(a_1, a_2) = K_1$. If $K_1 = \delta$ then $S$ is empty and we have no concerns. If $K_1 = 1$ we modify our procedure and take $d(a_1, a_2) = 2$.

We have $C \geq 2\delta + 2$, and if $C = 2\delta + 2$ then $S$ is empty. So suppose $C > 2\delta + 2$. Then with $d(a_1, a_2) = 2$ the bound on perimeter is respected. Let $u \in A_0$ and consider the triangle $(a_1, a_2, u)$. We must show that $(a_1, a_2, u)$ is in $A_{K_2}$. Let $j = d(a_1, u)$, $k = d(a_2, u)$. As $K_1 = 1$ we have $K_2 \geq \delta - 1$.

Therefore the inequality 
\[ j + k \leq 2K_2 + 2 \]
is immediate. We must also check 
\[ 2 + j \leq 2K_2 + k, 2 + k \leq 2K_2 + j \]
But since $\delta \geq 3$ we have 
\[ 2 + \delta \leq 2(\delta - 1) + 1 \]
and both inequalities are immediate. So if our standard amalgamation procedure takes $d(a_1, a_2) = K_1$ after an adjustment in the case $K_1 = 1$ our amalgamation succeeds.

Suppose now that our procedure takes $d(a_1, a_2) = K_1 + 1$. Then $K_1 + 2K_2 = 2\delta - 1$ in this case. Our only concern is with the case $K_1 = \delta - 1$, which is incompatible with this condition. Thus in case $d^+(a_1, a_2) > K_1$, our amalgamation procedure succeeds, after modification in one case.

Now suppose 
\[ C' > C + 1 \]
In this case $C \geq 2\delta + K_2$.

If $d^+(a_1, a_2) < K_2$ we take $d(a_1, a_2) = d^+(a_1, a_2) < K_2$. So $d(a_1, a_2) \neq 1, \delta$.

If $d^+(a_1, a_2) \geq K_2$ we take $d(a_1, a_2) = \min(K_2, C - 2\delta - 1)$. If $d(a_1, a_2) = 1$ then $C = 2\delta + 2$ and $S$ is empty. Suppose now that 
\[ d(a_1, a_2) = \delta \]
Then $K_2 = \delta$ and $C = 3\delta + 1$. But then $C' = C + 1$ after all.

This completes the treatment of amalgamation in all cases. \[ \square \]
4. Classification in Groups I, II

**Theorem 7.** The metrically homogeneous graphs with $\delta \leq 2$ or with $\Gamma_1$ finite or imprimitive are those listed in the catalog:

I. $\delta \leq 2$.
   
   (a) Finite primitive nondefective: $C_5$, $L[K_3,3]$.
   (b) Defective or imprimitive: $m \cdot K_n$, $K_m[J_m]$.
   (c) Infinite primitive, not defective: $G_n$, $G^n$, $G^n_\infty$.

II. $\delta \geq 3$, $\Gamma_1$ finite or imprimitive.

   A disjoint sum of isomorphic connected graphs of one of the following types.
   
   (a) An $n$-gon with $n \geq 6$.
   (b) Antipodal double of one of the graphs $C_5$, $L[K_3,3]$, or a finite independent set.
   (c) A tree-like graph $T_{r,s}$ as described by Macpherson in [Mph82], where $2 \leq r, s \leq \infty$, and if $s = \infty$ then $r \geq 3$.

The classification in Group I is highly nontrivial, but given in [LW80].

**Fact 4.1 ([LW80]).** The homogeneous graphs are as listed above (group I).

So we now deal with Group II, which is a more straightforward case. We use Fact 4.1 to organize the analysis, since it gives a list of possibilities for the structure of $\Gamma_1$.

The finite metrically homogeneous graphs are classified in [Cam80] and the locally finite infinite ones in [Mph82], so we could omit all cases with $\Gamma_1$ finite in what follows.

4.1. $\Gamma_1$ finite primitive, nondefective. The finite primitive homogeneous graphs are on the one hand the defective graphs $K_n$ and $I_n$ ($n < \infty$) which can be taken as homogeneous structures for the empty language, and the nondefective ones, $C_5$ and $L[K_3,3]$.

**Lemma 4.2.** Let $\Gamma$ be a connected metrically homogeneous graph of diameter at least 3, and suppose that $\Gamma_1$ is one of the primitive finite homogeneous nondefective graphs $C_5$ or $L[K_3,3]$. Then $\Gamma$ is the antipodal double of $\Gamma_1$, of diameter 3.

**Proof.** We fix a basepoint $*$ in $\Gamma$ so that $\Gamma_i$ is viewed as a specific subgraph of $\Gamma$ for each $i$. The proof proceeds in two steps.

(1) There is a $*$-definable function from $\Gamma_1$ to $\Gamma_2$.

We will show that for $v \in \Gamma_1$, the vertices of $\Gamma_1$ not adjacent to $v$ have a unique common neighbor $v'$ in $\Gamma_2$.

If $\Gamma_1$ is a 5-cycle then this amounts to the claim that every edge of $\Gamma$ lies in exactly two triangles, and this is clear by inspection of an edge $(*, v)$ with $*$ the basepoint and $v \in \Gamma_1$.

Now suppose $\Gamma_1$ is $L[K_3,3]$. We claim that every induced 4-cycle $C \cong C_4$ in $\Gamma$ has exactly two common neighbors.
Consider \( u, v \in \Gamma_1 \) lying at distance 2, and let \( G_{u,v} \) be the metric space induced on their common neighbors. This is a homogeneous metric space. Since these common neighbors consist of the basepoint \(*\), the two common neighbors \( a, b \) of \( u, v \) in \( \Gamma_1 \), and whatever common neighbors \( u, v \) may have in \( \Gamma_2 \), we see that pairs at distance 1 occur, and the corresponding graph has degree 2 (looking at \(*\)) and is connected (looking at \((a, *, b)\)). So \( G_{u,v} \) is a connected metrically homogeneous graph of degree 2, and furthermore embeds in \( \Gamma_1(u) \cong \Gamma_1 \). So \( G_{u,v} \) is a 4-cycle, and therefore \((u, a, v, b)\) has exactly 2 neighbors, as claimed. This proves (1).

Now let \( f : \Gamma_1 \to \Gamma_2 \) be \(*\)-definable. By homogeneity \( f \) is surjective, and as \( \Gamma_1 \) is primitive, \( f \) is bijective. For \( u, v \in \Gamma_1 \), by homogeneity \( d(u, v) \) determines \( d(f(u), f(v)) \), so \( f \) is either an isomorphism or an anti-isomorphism. Since \( \Gamma_1 \) is isomorphic to its complement, \( \Gamma_1 \cong \Gamma_2 \) in any case. Hence the vertices of \( \Gamma_2 \) have a common neighbor \( v \), and \( v \in \Gamma_3 \). We claim that \( |\Gamma_3| = 1 \).

By homogeneity all pairs \((u, v)\) in \( \Gamma_2 \times \Gamma_3 \) are adjacent. In particular for \( v_1, v_2 \in \Gamma_3 \) we have \( \Gamma_1(v_1) = \Gamma_1(v_2) \) and \( d(v_1, v_2) \leq 2 \). Since we have pairs of vertices \( u_1, u_2 \) in \( \Gamma_1 \) at distance 1 or 2 for which \( \Gamma_1(u_1) \neq \Gamma_1(u_2) \), we find \( |\Gamma_3| = 1 \).

It now follows that \( \Gamma \) is antipodal of diameter 3 and Lemma 3.3 applies. \( \square \)

4.2. \( \Gamma_1 \cong K_m[I_n] \).

**Lemma 4.3.** Let \( \Gamma \) be a metrically homogeneous graph of diameter at least 3, and suppose that \( \Gamma_1 \) is a complete multipartite graph of the form \( K_m[I_n] \) (the complement of \( m \cdot K_n \)) with \( 2 \leq n \leq \infty \). Then \( m = 1 \).

**Proof.** Fix an induced path \((*, u, v)\) of length 2 and let \( \Gamma_i = \Gamma_i(*) \). Let \( A \) be the set of neighbors of \( u \) in \( \Gamma_1 \); if \( m > 1 \), then \( A \) is nonempty. Then \( A \cong K_{m-1}[I_n] \), and the neighbors of \( u \) in \( \Gamma \) include \(*, v, A \). Now “\( d(x, y) > 1 \)” is an equivalence relation on \( \Gamma_1(u) \), and \(*\) is adjacent to \( A \), so \( v \) is adjacent to \( A \). If we replace \( u \) by \( u' \in A \) and argue similarly with respect to \((*, u', v)\), we see that the rest of \( \Gamma_1 \) is also adjacent to \( v \), that is \( \Gamma_1 \subseteq \Gamma_1(v) \). Now switching \( * \) and \( v \), by homogeneity \( \Gamma_1(v) \subseteq \Gamma_1 \). But then the diameter of \( \Gamma \) is 2, a contradiction. Thus \( m = 1 \). \( \square \)

It remains to consider the case \( \Gamma_1 \cong m \cdot K_n \), or rather the two cases arising under the assumption that either \( n \geq 2 \), or \( m < \infty \).

We will treat the case \( \Gamma_1 \cong I_n \) \((n < \infty)\) more broadly, allowing \( n = \infty \) but imposing the following finiteness condition: we require the common neighbors of two vertices at distance two to be finite.

**Lemma 4.4.** Let \( \Gamma \) be a metrically homogeneous graph of diameter at least 2, with \( \Gamma_1 \cong I_n \), \( 3 \leq n \leq \infty \). Suppose that for \( u, v \in \Gamma \) at distance two, the number \( k \) of common neighbors of \( u, v \) is finite. Then either \( k = 1 \), or \( n = k + 1 < \infty \).
There is a $4$-cycle in $\Gamma$, contradicting $k$ at least two neighbors in $\Gamma$.

Therefore $k \leq 2$ or $k = n - 1$, with $n$ finite in the latter case.

The case $k = 2 < n - 1$ is eliminated by a characteristic application of homogeneity. A set of three pairs in $\Gamma_1$ which intersect pairwise may or may not have a common element (once $n \geq 4$), so if we choose $u_1, u_2, u_3$ and $v_1, v_2, v_3$ in $\Gamma_2$ corresponding to these two possibilities for the associated $I_{u_j}$ and $I_{v_j}$, we get isometric configurations $(\ast, u_1, u_2, u_3)$ and $(\ast, v_1, v_2, v_3)$ which lie in distinct orbits of $\text{Aut}(\Gamma)$. So we have $k = 1$ or $n = k + 1$. 

**Lemma 4.5.** Let $\Gamma$ be a connected metrically homogeneous graph of diameter at least $3$, with $\Gamma_1 \cong I_n$, and $3 \leq n \leq \infty$. Suppose any pair of vertices at distance $2$ have $k$ common neighbors, with $k < \infty$. Then one of the following occurs.

1. $n = k + 1$, and $\Gamma$ is the complement of a perfect matching, in other words the antipodal graph of diameter $3$ obtained by doubling $\Gamma_1$.
2. $k = 1$, and $\Gamma$ is an $n$-regular tree.

**Proof.** We fix a basepoint $\ast$ and write $\Gamma_i$ for $\Gamma_i(\ast)$.

Suppose first that $n = k + 1$. Then any two vertices of $\Gamma_2$ lie at distance $2$, and there is a $\ast$-definable function $f : \Gamma_2 \to \Gamma_1$ given by the nonadjacency relation. By homogeneity $f$ is surjective, and as $\Gamma_2$ is primitive, $f$ is bijective. In particular $\Gamma_2 \cong \Gamma_1$ and there is a vertex $v \in \Gamma_3$ adjacent to all vertices of $\Gamma_2$, hence $\Gamma_1(v) = \Gamma_2$. It follows readily that $|\Gamma_3| = 1$ and $\Gamma$ is antipodal of diameter $3$. The rest follows by Lemma 3.3.

Now suppose that $k = 1$. It suffices to show that $\Gamma$ is a tree.

Suppose on the contrary that there is a cycle $C$ in $\Gamma$, which we take to be of minimal diameter $d$. Then the order of $C$ is $2d$ or $2d + 1$.

Suppose the order of $C$ is $2d$. Then for $v \in \Gamma_d$, $v$ has at least two neighbors $u_1, u_2$ in $\Gamma_{d-1}$, whose distance is therefore $2$. Furthermore, in $\Gamma_{d-2}$ there are no edges, and each vertex of $\Gamma_{d-2}$ has a unique neighbor in $\Gamma_{d-3}$, so each vertex of $\Gamma_{d-2}$ has at least two neighbors $u_1, u_2$ in $\Gamma_{d-1}$, whose distance is therefore $2$.

So for $u_1, u_2 \in \Gamma_{d-1}$, there is a common neighbor in $\Gamma_d$, and also in $\Gamma_{d-2}$. This gives a $4$-cycle in $\Gamma$, contradicting $k = 1$.

So the order of $C$ is $2d + 1$. In particular, each $v \in \Gamma_d$ has a unique neighbor in $\Gamma_{d-1}$, and $\Gamma_d$ contains edges. Any vertex $v \in \Gamma_{d-1}$ will have at least two neighbors in $\Gamma_d$: $v$ has a unique neighbor in $\Gamma_{d-2}$, and none in $\Gamma_{d-1}$.

Let $G$ be a connected component of $\Gamma_d$. Suppose $u, v$ in $G$ are at distance $2$ in $G$. Then $u, v$ must have a common neighbor in $\Gamma_{d-1}$ as well as in $G$,
and this contradicts the hypothesis \( k = 1 \). So the connected components of \( \Gamma_d \) are simply edges.

Take \( a \in \Gamma_{d-1} \), \( u_1, u_2 \in \Gamma_d \) adjacent to \( a \), and take \( v_1, v_2 \in \Gamma_d \) adjacent to \( u_1, u_2 \) respectively. By homogeneity there is an automorphism fixing the basepoint \(*\) and interchanging \( u_1 \) with \( v_2 \); this also interchanges \( u_2 \) and \( v_1 \). Hence \( d(u_1, v_2) = 2 \). It follows that \( u_1, v_2 \) have a common neighbor \( b \) in \( \Gamma_{d-1} \). Now \( (a, u_1, v_1, b, v_2, u_2, a) \) is a 6-cycle. Since the minimal cycle length is odd, we have \( |C| = 5 \) and \( d = 2 \).

Furthermore the element \( b \) is determined by \( a \in \Gamma_1 \) and the basepoint \(*\): we take \( u \in \Gamma_2 \) adjacent to \( a \), which determines \( v \in \Gamma_2 \) adjacent to \( u \), and \( b \in \Gamma_1 \) adjacent to \( v \), independent of the choice of \( u \). So the function \( a \mapsto b \) is \(*\)-definable. However \( \Gamma_1 \) is an independent set of order at least 3, so this violates homogeneity.

\[ \Box \]

4.3. **The case** \( \Gamma_1 = m \cdot K_n \), \( n \geq 2 \). We come to the case in which Macpherson’s graphs \( T_{r,s} \) arise: \( \Gamma_1 = m \cdot K_n \). For \( m, n \) finite these are handled in [Mph82] assuming only distance transitivity, by more delicate methods. We work with full metric homogeneity throughout.

Our main goal is to show that any two vertices at distance two have a unique common neighbor. We divide up the analysis into three cases. Observe that in a metrically homogeneous graph with \( \Gamma_1 \cong m \cdot K_n \), the common neighbors of a pair \( a, b \in \Gamma \) at distance 2 will be an independent set, since for \( u_1, u_2 \) adjacent to \( a, b \), and each other, we would have the path \( (a, u_1, b, v_2, u_2, a) \) inside \( \Gamma_1(u_2) \).

**Lemma 4.6.** Let \( \Gamma \) be a metrically homogeneous graph of diameter at least 3 and suppose that \( \Gamma_1 \cong m \cdot K_n \) with \( n \geq 3 \). Then for \( u, v \in \Gamma \) at distance 2, there are at most two vertices adjacent to both.

**Proof.** Supposing the contrary, every induced path of length 2 is contained in two distinct 4-cycles. Fix a basepoint \(*\) \( \in \Gamma \) and let \( \Gamma_i = \Gamma_i(*) \).

Fix \( v_1, v_2 \in \Gamma_1 \) adjacent. For \( i = 1, 2 \), let \( H_i \) be the set of neighbors of \( v_i \) in \( \Gamma_2 \). We claim

\[ H_1 \cap H_2 = \emptyset \]

Otherwise, consider \( v \in H_1 \cap H_2 \) and the path \( (*, v_1, v) \) contained in \( \Gamma_1(v_2) \).

Next, we will find \( u_1 \in H_1 \), and \( u_2, u_2' \in H_2 \) distinct, so that

\[ d(u_1, u_2) = d(u_1, u_2') = 1 \]

Extend the edge \( (v_1, v_2) \) to a 4-cycle \( (v_1, v_2, u_2, u_1) \). Then \( u_1, u_2 \notin \Gamma_1 \cup \{*\} \), so \( u_1 \in H_1 \) and \( u_2 \in H_2 \). By our hypothesis there is a second choice of \( u_2 \) with the same properties.

With the vertices \( u_1, u_2, u_2' \) fixed, let \( A, B, B' \) denote the components of \( H_1 \) and \( H_2 \), respectively, containing the specified vertices. Observe that \( B \) and \( B' \) are distinct: otherwise, the path \( (u_1, u_2, v_2) \) would lie in \( \Gamma_1(u_2') \).
(2) The relation “\(d(x, y) = 1\)” defines a bijection between \(A\) and \(B\). With \(u \in A\) fixed, it suffices to show the existence and uniqueness of the corresponding element of \(B\). The uniqueness amounts to the point just made for \(u_1\), namely that \(B \neq B'\).

For the existence, we may suppose \(u \neq u_1\). Then \(d(u, v_2) = d(u, u_2) = 2\).

We have an isometry

\[ (*, v_1, u, v_2, u_2) \cong (*, v_2, u_2, v_1, u) \]

and hence the triple \((u, u_1, u_2)\) with \(u_1 \in H_1\) corresponds to an isometric triple \((u_2, u', u)\) with \(u' \in H_2\), and \(d(u, u') = 1\).

Thus we have a bijection between \(A\) and \(B\) definable from \((*, v_1, v_2, u_1, u_2)\) and from this we derive a bijection between \(B\) and \(B'\) definable from \((*, v_1, v_2, u_1, u_2, u_2')\).

Using this bijection, we show

\[ n = 2 \]

The graph induced on \(B \cup B'\) is \(2 \cdot K_n\) and any isometry between finite subsets of \(B \cup B'\) containing \(u_2, u_2'\) which fixes \(u_2\) and \(u_2'\) will be induced by \(\text{Aut}(\Gamma)\). So if there is a bijection between \(B\) and \(B'\) invariant under the corresponding automorphism group, we find \(n = 2\). \(\square\)

Lemma 4.7. Let \(\Gamma\) be a metrically homogeneous graph of diameter at least 3 and suppose that \(\Gamma_1 \cong m \cdot K_n\) with \(n \geq 2\). Let \(u, v \in \Gamma\) lie at distance 2, and suppose \(u, v\) have finitely many common neighbors. Then they have a unique common neighbor.

Proof. We fix a basepoint \(*\), and for \(u \in \Gamma_2\) we let \(I_u\) be the set of neighbors of \(u\) in \(\Gamma_1\). Our assumption is that \(k = |I_u|\) is finite. Then any independent subset of \(\Gamma_1\) of cardinality \(k\) occurs as \(I_u\) for some \(u \in \Gamma_2\).

We consider the \(k - 2\) possibilities:

\[ |I_u \cap I_v| = i \quad \text{with} \quad 1 \leq i \leq k - 1 \]

As \(n \geq 2\), all possibilities are realized, whatever the value of \(m\). However in all such cases, \(d(u, v) \leq 2\), so we find \(k - 1 \leq 2\), and \(k \leq 3\).

We claim that for \(u, v \in \Gamma_2\) adjacent, we have \(|I_u \cap I_v| = 1\).

There is a clique \(v, u_1, u_2\) with \(v \in \Gamma_1\) and \(u_1, u_2 \in \Gamma_2\). As \(u_1, u_2\) are adjacent their common neighbors form a complete graph. On the other hand \(I_{u_1}\) and \(I_{u_2}\) are independent sets, so their intersection reduces to a single vertex. By homogeneity the same applies whenever \(u_1, u_2 \in \Gamma_2\) are adjacent, proving our claim.

Now suppose \(k = 3\). Then \(I_u \cap I_v\) can have cardinality 1 or 2, and the case \(|I_u \cap I_v| = 2\) must then correspond to \(d(u, v) = 2\).

Now \(k \leq m\), so we may take pairs \((a_i, b_i)\) for \(i = 1, 2, 3\) lying in distinct components of \(\Gamma_1\). Of the eight triples \(t\) formed by choosing one of the vertices of each of these pairs, there are four in which the vertex \(a_i\) is selected an even number of times. Let a vertex \(v_t \in \Gamma_2\) be taken for each such triple,
adjacent to its vertices. Then the four vertices \( v_t \) form a complete graph \( K_4 \).

It follows that \( K_3 \) embeds \( \Gamma_1 \), that is \( n \geq 3 \). So we may find independent triples \( I_1, I_2, I_3 \) such that \( |I_1 \cap I_2| = |I_2 \cap I_3| = 1 \) while \( I_1 \cap I_3 = \emptyset \). Take \( u_1, u_2, u_3 \in \Gamma_2 \) with \( I_1 = I_{u_1} \) and with \( d(u_1, u_2) = d(u_2, u_3) = 1 \). Then \( I_{u_1} \cap I_{u_3} = \emptyset \), while \( d(u_1, u_2) \leq 2 \), a contradiction. Thus \( k = 2 \).

With \( k = 2 \), suppose \( m > 2 \). For \( u \in \Gamma_2 \), let \( \hat{I}_u \) be the set of components of \( \Gamma_1 \) meeting \( I_u \). We consider the following two properties of a pair \( u, v \in \Gamma_2 \):

\[ |I_u \cap I_v| = 1; |\hat{I}_u \cap \hat{I}_v| = i \ (i = 1 \text{ or } 2) \]

These both occur, and must correspond in some order with the conditions \( d(u, v) = 1 \) or \( 2 \). But just as above we find \( u_1, u_2, u_3 \) with \( d(u_1, u_2) = d(u_2, u_3) = 1 \) and \( I_{u_1} \cap I_{u_3} = \emptyset \), and as \( d(u_1, u_3) \leq 2 \) this is a contradiction.

So we come down to the case \( m = k = 2 \). But then for \( v \in \Gamma_2 \), all components of \( \Gamma_1(v) \) are represented in \( \Gamma_1 \), and hence \( v \) has no neighbors in \( \Gamma_3 \), a contradiction. \( \square \)

**Lemma 4.8.** Let \( \Gamma \) be a metrically homogeneous graph of diameter at least 3 and suppose that \( \Gamma_1 \cong m \cdot K_n \) with \( n \geq 2 \). Let \( u, v \in \Gamma \) lie at distance 2, and suppose \( u, v \) have infinitely many common neighbors. Then \( n = \infty \).

**Proof.** We fix a basepoint *, and for \( u \in \Gamma_2 \) we let \( I_u \) be the set of neighbors of \( u \) in \( \Gamma_1 \). For some finite independent subset of \( \Gamma_1 \) is contained in \( I_u \) for some \( u \in \Gamma_2 \).

For \( u \in \Gamma_2 \), let \( \hat{I}_u \) be the set of components of \( \Gamma_1 \) which meet \( I_u \), and let \( \hat{J}_u \) be the set of components of \( \Gamma_1 \) which do not meet \( I_u \). We show first that \( \hat{J}_u \) is infinite.

Supposing the contrary, let \( k = |\hat{J}_u| < \infty \) for \( u \in \Gamma_2 \). Any set of \( k \) components of \( \Gamma_1 \) will be \( J_u \) for some \( u \in \Gamma_2 \), and the \( k + 1 \) relations on \( \Gamma_2 \) defined by

\[ |\hat{J}_u \cap \hat{J}_v| = i \]

for \( i = 0, 1, \ldots, k \) will be nontrivial and distinct. Furthermore, for any preassigned \( k \) components \( J, \) and any vertex \( a \in \Gamma_1 \) not in the union of \( J, \) there is a vertex \( u \) with \( J_u = J \) and \( a \in I_u \), so our \((k + 1)\) relations are realized by pairs \( u, v \in \Gamma_2 \) with \( I_u \cap I_v \neq \emptyset \), and hence \( d(u, v) \leq 2 \). Hence \( k + 1 \leq 2, \ k \leq 1 \).

Suppose \( k = 1 \) and fix a vertex \( v_0 \in \Gamma_1 \). Then for \( u, v \in \Gamma_2 \) adjacent to \( v_0 \), the two relations \( \hat{J}_u = \hat{J}_v, \hat{J}_u \neq \hat{J}_v \) correspond in some order to the relations \( d(u, v) = 1, d(u, v) = 2 \), and since the first relation is an equivalence relation, they correspond in order.

With \( u \in \Gamma_2, v_0, v_1 \in I_u \) distinct, there are \( u_0, u_1 \) in \( \Gamma_2 \) with \( u_0 \) adjacent to \( u_0 \) and \( v_0 \), and with \( u_1 \) adjacent to \( u_0 \) and \( v_1 \). The neighbors of \( u \) form a graph of type \( \infty \cdot K_n \), so \( d(u_0, u_1) = 2 \). However \( d(u_0, u_0) = d(u_0, u_1) = 1 \) and hence \( \hat{J}_{u_0} = \hat{J}_{u_1} \), a contradiction.

So \( k = 0 \) and for \( u \in \Gamma_2 \), the set \( I_u \) meets every component of \( \Gamma_1 \). That is, \( \Gamma_1(u) \) meets every component of \( \Gamma_1 \), and after switching the roles of \( u \) and
the basepoint *, we conclude $\Gamma_1$ meets every component of $\Gamma_1(u)$, which is incompatible with the condition $\delta \geq 3$. So $\tilde{J}_u$ is infinite for $u \in \Gamma_2$.

Now we claim

For $u, v \in \Gamma_2$ adjacent, $\hat{I}_u \setminus \hat{I}_v$ is infinite.

Supposing the contrary, for all adjacent pairs $u, v \in \Gamma_2$, the sets $\hat{I}_u$ and $\hat{I}_v$ coincide up to a finite difference. We will denote this relation by $\hat{I}_u \approx \hat{I}_v$.

Then also $\tilde{J}_u \approx \tilde{J}_v$.

Take $u \in \Gamma_2$, $v_0, v_1 \in \Gamma_1$ adjacent to $u$, and $u_0, u_1$ adjacent to $u, v_0$ or $u, v_1$ respectively. Then, as above, $d(u_0, u_1) = 2$, while $\tilde{J}_{u_0} \approx \tilde{J}_u \approx \tilde{J}_{u_1}$. Thus for $u, v \in \Gamma_2$ with $d(u, v) \leq 2$ we have $\tilde{J}_u \approx \tilde{J}_v$. Furthermore, the size of the difference $|\tilde{J}_u \setminus \tilde{J}_v|$ is bounded, say by $\ell$. But we can fix $u \in \Gamma_2$ and then find $v \in \Gamma_2$ so that $I_u$ meets $I_v$ but $\hat{I}_v$ picks up more than $\ell$ components of the infinite set $\hat{J}_u$. Since $I_u$ meets $I_v$ we have $d(u, v) \leq 2$, and thus a contradiction. This proves our claim.

Now, finally, suppose $n$ is finite. Take $u, v \in \Gamma_2$, and $u' \in \Gamma_2$ adjacent to $u$ and $v$. Take $A, B$ distinct components of $\Gamma_1$ lying in $I_{u'} \setminus I_u$. For any $a \in A$, $b \in B$ there is $u_{a,b}$ in $\Gamma_2$ adjacent to $u, v, a$, and $b$. This forces $(n - 1)^2 \geq n^2$, a contradiction.

Corollary 4.9. Let $\Gamma$ be a metrically homogeneous graph of diameter at least 3 and suppose that $\Gamma_1 \cong m \cdot K_n$ with $n \geq 2$. Then for $u, v \in \Gamma$ with $d(u, v) = 2$, there is a unique vertex adjacent to both.

Proof. Apply the last three lemmas. If $u, v$ have infinitely many common neighbors, then $n$ is infinite. In particular, $n \geq 3$. But then they have at most two common neighbors. So in fact $u, v$ have finitely many neighbors, and we apply Lemma 4.7.

After this somewhat laborious reduction, we can complete the classification in this case.

Proposition 4.10. Let $\Gamma$ be a connected distance homogeneous graph with $\Gamma_1 \cong m \cdot K_n$, with $n \geq 2$ and $\delta \geq 3$. Then $m \geq 2$, and $\Gamma_1 \cong T_{m, n+1}$.

Proof. If $m = 1$ then evidently $\Gamma$ is complete, contradicting the hypothesis on $\delta$. So $m \geq 2$.

By definition, the blocks of $\Gamma$ are the maximal 2-connected subgraphs. Any edge of $\Gamma$ is contained in a unique clique of order $n + 1$. It suffices to show that these cliques form the blocks of $\Gamma$, or in other words that any cycle in $\Gamma$ is contained in a clique.

Supposing the contrary, let $C$ be a cycle of minimal order in $\Gamma$, not contained in a clique. The cycle $C$ carries two metrics: its metric $d_C$ as a cycle, and the metric $d_\Gamma$ induced by $\Gamma$. We claim these metrics coincide. In any case, $d_\Gamma \leq d_C$. 

If the metrics disagree, let \( u, v \in C \) be chosen at minimal \( \Gamma \)-distance such that \( d_\Gamma(u, v) < d_C(u, v) \), and let \( P = (u, \ldots, v) \) be a geodesic in \( \Gamma \), and let \( Q = (u, \ldots, v) \) be a geodesic in \( C \). Let \( v' \) be the first point of intersection of \( P \) with \( Q \), after \( u \). Then \( P \cup Q \) contains a cycle \( C' \) smaller than \( C \), and containing \( u, v' \). Hence by hypothesis the vertices of \( C' \) form a clique in \( \Gamma \), and in particular \( u, v' \) are adjacent in \( \Gamma \).

If \( u, v' \) are adjacent in \( C \), then \( d_C(v, v') = d_C(u, v) - 1 \), \( d_\Gamma(v, v') = d_\Gamma(u, v) - 1 \), so \( d_\Gamma(v, v') < d_C(v, v') \), and this contradicts the choice of \( u, v \). So \( u, v' \) are not adjacent in \( C \), and the edge \((u, v')\) of \( \Gamma \) belongs to two cycles \( C_1, C_2 \) whose union contains \( C \), each of them shorter than \( C \). So the vertices of \( C_1 \) and \( C_2 \) are cliques. Since \( C \) is not a clique, there are \( u_1 \in C_1 \) and \( u_2 \in C_2 \) nonadjacent in \( \Gamma \). Then \( d(u_1, u_2) = 2 \), and there is a unique vertex adjacent to \( u_1 \) and \( u_2 \); but \( u, v' \) are such, a contradiction.

Thus the embedding of \( C \) into \( \Gamma \) respects the metric. In particular, \( C \) is an induced subgraph of \( \Gamma \).

Let \( d \) be the diameter of \( C \), so that the order of \( C \) is either \( 2d \) or \( 2d + 1 \). Fix a basepoint \( * \in C \), and let \( \Gamma_i = \Gamma_i(*) \).

For \( v \in \Gamma_i \) with \( i < d \), we claim that there is a unique geodesic \([*, v]\) in \( \Gamma \). Otherwise, take \( i < d \) minimal such that \( \Gamma \) contains vertices \( u, v \) at distance \( i \) with two distinct geodesics \( P, Q \) from \( u \) to \( v \). By the minimality, these geodesics are disjoint. Their union forms a cycle smaller than \( C \), hence they form a clique. As they are geodesics, \( i = 1 \) and then in any case the geodesic is unique.

Let \( v \in \Gamma_{d-1} \), and let \( H = \Gamma_1(v) \), a copy of \( m \cdot K_n \). Then \( H \) contains a unique component meeting \( \Gamma_{d-2} \). We claim that no other component of \( H \) meets \( \Gamma_{d-1} \).

Suppose on the contrary that \( u_1, u_2 \) are adjacent to \( v \) with \( u_1 \in \Gamma_{d-1} \), \( u_2 \in \Gamma_{d-2} \), and \( d(u_1, u_2) = 2 \). Then taking \( P_1, P_2 \) to be the unique geodesics from \( u_1 \) or \( u_2 \) to \( * \), \( P_1 \cup P_2 \) contains a cycle shorter than \( C \), and containing the path \((u_1, v, u_2)\), hence not a clique. This is a contradiction.

Therefore \( \Gamma \) contains at least one component of \( H \). In particular there is an edge in \( \Gamma \) whose vertices have a common neighbor in \( \Gamma_{d-1} \). Using this, we eliminate the case \(|C| = 2d + 1 \) as follows.

If \(|C| = 2d + 1 \) then \( C \cap \Gamma \) consists of two adjacent vertices \( v_1, v_2 \), whose other neighbors \( u_1, u_2 \) in \( C \) lie in \( \Gamma_{d-1} \). Furthermore \( v_1, v_2 \) have a common neighbor \( u \) in \( \Gamma_{d-1} \), and \( u \neq u_1, u_2 \). Let \( P, P_1 \) be the unique geodesics in \( \Gamma \) connecting \( * \) with \( u, u_1 \) respectively. Then \( P \cup P_1 \cup \{v_1\} \) contains a cycle shorter than \( C \), which contains the path \((u_1, v, v_1)\), and we have a contradiction.

Thus \(|C| = 2d \), in other words the vertices \( v \in \Gamma \) are connected to the basepoint \( * \) by at least two distinct geodesics, and any two such geodesics will be disjoint.

Take \( u \in \Gamma_d \), and \( u' \in \Gamma_1 \), with \( d(u, u') = d - 2 \). Take \( v' \in \Gamma_1 \) with \( d(u', v') = 2 \). We claim
\[
 d(u, v') = d
\]
Otherwise, with \( P, Q \) the geodesics from \( u \) to \( u' \) and \( v' \) respectively, we have \( |P \cup Q| \leq 2d - 1 < |C| \), and hence \( \langle u', *, v' \rangle \) is contained in a cycle shorter than \( C \), a contradiction since \( u', v' \) are nonadjacent.

Thus \( d(u, v') = d \), and the extension of the geodesic from \( u \) to \( u' \) by the path \( \langle u', *, v' \rangle \) gives a geodesic \( Q \) from \( u \) to \( v' \). There is a second geodesic \( Q' \) from \( u \) to \( v' \), disjoint from \( Q \). Let \( u_1 \) be the unique neighbor of \( u \) in \( \Gamma_{d-2} \); this lies on \( Q \). Let \( u_2' \) be the neighbor of \( u \) in \( Q' \). As the cycle \( Q \cup Q' \) satisfies the same condition as the cycle \( C \), the metric on this cycle agrees with the metric in \( \Gamma \), and in particular \( u_1^* \in \Gamma_d \). Let \( u_2' \) be the following neighbor of \( u_1' \) in \( Q' \). Then \( d(u_1, u_2') = 2 \), with \( u, u_2' \in \Gamma_{d-1} \).

Suppose \( m \geq 3 \). Then for \( u \in \Gamma_{d-2} \) we have as above that \( \Gamma_{d-2} \cup \Gamma_d - 3 \) contains a unique component of \( \Gamma(u) \) and thus \( \Gamma_{d-1} \) contains two neighbors of \( u \) at distance 2. So by homogeneity there is \( u^* \in \Gamma_{d-2} \) adjacent to both \( u \) and \( u_2 \). But the only neighbor of \( u \) in \( \Gamma_{d-2} \) is \( u_1 \), so \( u_1 \) is adjacent to \( u_2 \). In this case, \( u \) and \( u_2' \) have two distinct common neighbors, a contradiction. We conclude

\[ m = 2 \]

Fix an edge \( u_1, u_2 \) in \( \Gamma_{d-1} \). For \( i, j = 1, 2 \) in either order, set

\[ H_{ij} = \{ v \in \Gamma : d(v, u_i) = 1, d(v, u_j) = 2 \} \]

We claim \( H_{ij} \subseteq \Gamma_d \).

As \( u_1, u_2 \) have the same unique neighbor in \( \Gamma_{d-2} \), we have

\[ (H_{12} \cup H_{21}) \cap \Gamma_{d-2} = \emptyset \]

Now suppose \( v \in H_{12} \cap \Gamma_{d-1} \). Then \( d(v, u_2) = 2 \) and \( u_2, v \in \Gamma_{d-1} \). Now since \( |C| = 2d \), there is \( w \in \Gamma_d \) adjacent to \( v \) and \( u_2 \). Then \( u_2, v \) have the two common neighbors \( w \) and \( u_1 \), a contradiction. So \( H_{12}, H_{21} \subseteq \Gamma_d \).

Let \( v_1 \in H_{12}, v_2 \in H_{21} \). We claim that

\[ d(v_1, v_2) = 3 \]

Otherwise, there is a cycle of length at most 5, not contained in a clique. As \( |C| \) is even, it follows that \( C \) is a 4-cycle, in other words vertices at distance 2 have at least two common neighbors, a contradiction. Thus \( d(v_1, v_2) = 3 \), and for any other choice \( v_1', v_2' \in H_{12}, v_2' \in H_{21} \), we have \( \langle *, u_1, u_2, v_1, v_2 \rangle \) isometric with \( \langle *, u_1, u_2, v_1', v_2' \rangle \).

There is a unique element \( u \in \Gamma_1 \) at distance \( d-2 \) from \( u_1 \) and \( u_2 \); namely, the element at distance \( d-3 \) from their common neighbor in \( \Gamma_{d-2} \). On the other hand, for \( v \in \Gamma_d \), if \( I_v \) is the set \( \{ v' \in \Gamma_1 : d(v, v') = d - 1 \} \), then by our hypotheses, \( I_v \) is a pair of representatives for the two components of \( \Gamma_1 \). And if \( v \in H_{12} \) or \( H_{21} \), one of these representatives will be \( u \). Let \( B \) be the component of \( \Gamma_1 \) not containing \( u \). Then the distance from \( u_1 \) or \( u_2 \) to a vertex of \( B \) is \( d \). It follows that all vertices of \( B \) will occur as the second vertex of \( I_v \) for some \( v_1 \in H_{12} \) and for some \( v_2 \in H_{21} \). Therefore, we may choose pairs \( (v_1, v_2) \) and \( (v_1', v_2') \) with \( v_1, v_1', v_2, v_2' \in H_{12}, v_2, v_2' \in H_{21} \), and
\[ I_{v_1} = I_{v_2}, \text{ while } I_{v'_1} \neq I_{v'_2}. \] But as \((*, u_1, u_2, v_1, v_2)\) and \((*, u_1, u_2, v'_1, v'_2)\) are isometric, this contradicts homogeneity. \(\square\)

At this point the proof of Theorem 7 is complete. The case of diameter at most 2 is given by Fact 4.1. When the vertex degree is 2 we get an \(n\)-gon, so we may suppose it is larger. Then in view of Fact 4.1 the possibilities for \(\Gamma_1\) are \(C_5, L[K_{3,3}]\), and graphs of the form \(K_m[I_n]\) or \(m \cdot K_n\). These are covered by Lemmas 4.2, 4.3, 4.4, and 4.10.

5. The structure of \(\Gamma_1\) in the primitive infinite case

**Lemma 5.1.** Let \(\Gamma\) be a metrically homogeneous graph of diameter at least 3 with \(\Gamma_1\) infinite and primitive. Then \(\Gamma_1\) contains an infinite independent set.

**Proof.** Suppose the contrary. Evidently \(\Gamma_1\) contains an independent pair. By the Lachlan/Woodrow classification, if \(n\) is minimal such that \(\Gamma_1\) contains no independent set of order \(n\), then for any finite graph \(G\) which contains no independent set of order \(n\), \(G\) occurs as an induced subgraph of \(\Gamma_1\).

We consider a certain amalgamation diagram involving metric subspaces of \(\Gamma\). Let \(A\) be the metric space with three points \(a, b, x\) constituting a geodesic, with \(d(a, b) = 2, d(b, x) = 1, d(a, x) = 3\). Let \(B\) be the metric space on the points \(a, b\) and a further set \(Y\) of order \(n - 1\) with the metric given by

\[
\begin{align*}
(3) & \quad d(a, y) = d(b, y) = 1, y \in Y \\
(4) & \quad d(y, y') = 2, y, y' \in Y \text{ distinct}
\end{align*}
\]

As the diameter of \(\Gamma\) is at least 3, the geodesic \(A\) occurs as a subspace of \(\Gamma\). On the other hand, the metric space \(B\) embeds into \(\Gamma_1\), and hence into \(\Gamma\). Therefore there is some amalgam \(G = A \cup B\) embedding into \(\Gamma\) as well.

Now for \(y \in Y\), the structure of \((a, x, y)\) forces \(d(x, y) \geq 2\). On the other hand, the element \(b\) forces \(d(x, y) \leq 2\). Thus in \(G\), the set \(Y \cup \{x\}\) is an independent set of order \(n\). Furthermore this set is contained in \(\Gamma_1(b)\), so we arrive at a contradiction. \(\square\)

Thus the remaining possibilities for \(\Gamma_1\) are:

1. An infinite independent set;
2. \(G_n\) generic omitting \(K_n\);
3. \(G_{\infty}\) homogeneous universal (Rado’s graph).

These are all well represented by the graphs \(\Gamma_{K_1, K_2; C; C'; S}^\delta\) depending on whether \(K_1 = 1\) or \(K_1 > 1\), and whether or not \(S\) contains a clique. They also occur in some additional imprimitive graphs \(\Gamma\).
6. Bipartite Metrically Homogeneous Graphs: Exceptional Cases

6.1. Classification. For \( \Gamma \) metrically homogeneous and bipartite we have defined \( B\Gamma \) as the graph induced on either half of \( \Gamma \) by the edge relation

\[
d(x, y) = 2
\]

or in other words the metric space with \( \frac{1}{2} \) of the induced metric. This is again metrically homogeneous, of diameter \( \lfloor \delta \rfloor \), where \( \delta \) is the diameter of \( \Gamma \). However, we generally restrict ourselves to connected metrically homogeneous graphs, so we must notice that the hypothesis is preserved in passing from \( \Gamma \) to \( B\Gamma \).

**Lemma 6.1.** Let \( \Gamma \) be a connected bipartite metrically homogeneous graph. Then \( B\Gamma \) is connected and metrically homogeneous.

**Proof.** Both properties are inherited directly, but we check the connectedness. Let \( u, v \in B\Gamma \), identified with one of the halves of \( \Gamma \). Then any path from \( u \) to \( v \) in \( \Gamma \) will include a path from \( u \) to \( v \) in \( B\Gamma \). \( \square \)

We write \( B\Gamma_1 \) for \((B\Gamma)_1\). This is \( \Gamma_2 \) with the induced metric halved. We consider \( \Gamma \) to be *exceptional bipartite* if \( B\Gamma_1 \) is not the Rado graph. The following gives a classification of the exceptional bipartite metrically homogeneous graphs.

**Theorem 8.** Let \( \Gamma \) be a connected, bipartite, and metrically homogeneous graph, of diameter at least 3, and degree at least 3. Then one of the following occurs, writing \( B\Gamma_1 \) for \((B\Gamma)_1\).

1. \( B\Gamma \cong T_{r,r} \) with \( 2 \leq r \leq \infty \) and \( \Gamma \) is an \( r \)-regular tree;
2. \( B\Gamma \) is a clique, \( \Gamma \) has diameter 3, and \( \Gamma \) is either the complement of a perfect matching, or a generic bipartite graph;
3. \( B\Gamma \cong K_{\infty, \lfloor \frac{3}{2} \rfloor} \), \( \Gamma = \Gamma_{\infty, 0;9}^4 \);
4. \( B\Gamma_1 \) is generic omitting an independent set of order \( n \), for some \( n \geq 3 \). \( \Gamma \cong \Gamma_{\infty, 0;13;S_n}^4 \) with \( S_n = \{I_n^4\} \);
5. \( B\Gamma \) is generic omitting an independent set of order 3. \( \Gamma \) is the antipodal bipartite double cover of \( B\Gamma \), of diameter 5.
6. \( B\Gamma_1 \) is a homogeneous universal graph (Rado’s graph).

We deal with the case in which \( \Gamma_1 \) is finite by consulting the classification (Theorem 7).

**Lemma 6.2.** Let \( \Gamma \) be connected, metrically homogeneous, and bipartite of diameter at least 3 and degree at least 3. If \( \Gamma_1 \) is finite then \( \Gamma \) is one of the following.

1. Finite, and the bipartite complement of a perfect matching.
2. A regular \( r \)-branching tree with \( r < \infty \).

**Proof.** By inspection of (Theorem 7). Note that \( T_{r,2} \) is a regular \( r \)-branching tree. \( \square \)
The remainder of the analysis concerns the case in which $\Gamma_1$ is infinite. Then $\Gamma_1$ represents an infinite clique in $B\Gamma$, and in particular $B\Gamma_1$ contains $K_\infty$.

The next lemma will be applied to $B\Gamma$ to give a list of cases to be analyzed.

**Lemma 6.3.** Let $\Gamma$ be a connected metrically homogeneous graph containing $K_\infty$. Then $\Gamma$ satisfies one of the following.

1. $\Gamma \cong K_\infty[I_n]$ with $n \geq 1$;
2. $\Gamma \cong T_{r,\infty}$ with $2 \leq r \leq \infty$;
3. $\Gamma \cong G^n$ is generic omitting $I_n$, with $3 \leq n < \infty$;
4. $\Gamma \cong G_\infty$ the Rado graph;
5. $\Gamma$ has diameter at least 3 and $\Gamma_1 \cong G_\infty$ the Rado graph.

**Proof.** By assumption $B\Gamma$ contains $K_\infty$, and is connected.

If $\Gamma$ has diameter at most 2 or if $\Gamma_1$ is imprimitive then we consult the classification of Theorem 7.

If $\Gamma$ has diameter $\delta \geq 3$ and $\Gamma_1$ is primitive containing $K_\infty$ then by Lemma 5.1 $\Gamma_1$ also contains an infinite independent set and then by the classification of homogeneous graphs, $\Gamma$ can only be the Rado graph. $\square$

**6.2.** $B\Gamma \cong K_\infty[I_n]$.

**Lemma 6.4.** Let $\Gamma$ be a connected, bipartite, and metrically homogeneous graph, of diameter $\delta \geq 3$. Suppose that $B\Gamma \cong K_\infty[I_n]$ with $n \geq 1$. Then we have one of the following.

1. $n = 1$, $\delta = 3$ and $\Gamma$ is either the complement of a perfect matching, or a generic bipartite graph.
2. $n = 2$, $\delta = 4$, and $\Gamma$ is isomorphic to the bipartite twisted double cover of the generic bipartite graph.

**Proof.** Suppose first

$n = 1$\n
Since $B\Gamma$ has diameter 1, $\Gamma$ has diameter at most 3, hence exactly 3.

Consider the graph $\Gamma$ in the language given by the edge relation and unary predicates for the two halves of $\Gamma$. Evidently any automorphism of $\Gamma$ preserving these relations is an isometry of $\Gamma$. Our claim is then given by Fact 3.1.

Now suppose

$n \geq 2$\n
Then Lemma 3.5 applies: $n = 2$, and $\Gamma$ is as described. $\square$

**6.3.** $B\Gamma$ treelike.

**Lemma 6.5.** Let $\Gamma$ be a connected, bipartite, and metrically homogeneous graph. Suppose that $B\Gamma \cong T_{r,\infty}$ with $2 \leq r \leq \infty$. Then $r = \infty$ and $\Gamma$ is an infinitely branching tree.
Proof. Let $A, B$ be the two halves of $\Gamma$, and identify $A$ with $B\Gamma$. In particular, for $u \in B$, the neighbors of $u$ in $A$ form a clique in the sense of $B\Gamma$.

Suppose that vertices $u, u'$ in $B$ are adjacent to two points $v_1, v_2$ of $A$. These points are contained in a unique clique of $A$, so all the neighbors of $u, u'$ lie in a common clique. Therefore this gives us an equivalence relation on $B$, with $u, u'$ equivalent just in case they have two common neighbors in $A$. But the graph structure on $B$ is also that of $B\Gamma$, which is primitive, so this relation is trivial, and distinct vertices of $B$ correspond to distinct cliques in $A$. In particular, $r = \infty$.

At the same time, every edge in $A$ lies in the clique associated with some vertex in $B$, so the neighbors of the vertices of $B$ are exactly the maximal cliques of $A$. Evidently the edge relation in $B$ corresponds to intersection of cliques in $A$. Thus $B$ is identified with the “dual” of $A$ with vertices corresponding to maximal cliques, and maximal cliques corresponding to vertices. At this point the structure of $\Gamma$ has been recovered uniquely from the structure of $T_{\infty,\infty}$, and must therefore be the infinitely branching tree. □

6.4. $B\Gamma \cong G^e_n$, Diameter 4.

Lemma 6.6. For each $n$ with $3 \leq n < \infty$, there is a unique metrically homogeneous bipartite graph $\Gamma$ of diameter 4 such that $B\Gamma \cong G^e_n$.

Proof. Let $S = \{I_n^4\}$. Then $\Gamma^4_{\infty,0;14,9,S}$ is such a graph, by Lemma 3.9.

We turn to uniqueness. Let $\Delta, \Delta'$ be the two halves of $\Gamma$, each isomorphic to $2B\Gamma$ (that is, $B\Gamma$ with doubled metric). Our claim is that for any finite $A, B \subseteq 2B\Gamma$ and any metric on the disjoint union $A \cup B$ in which all cross-distances between $A$ and $B$ are equal to either 1 or 3, there is an embedding of $A$ into $\Delta$ and $B$ into $\Delta'$ such that the metric of $\Gamma$ induces the specified metric on $A \cup B$.

We will prove this by induction on the order of $k$ of $B$. We may suppose $k > 0$. We now proceed by induction on the number of pairs $u_1, u_2$ in $A$ with

$$d(u_1, u_2) = 4$$

Suppose there is such a pair, and fix one such $(u_1, u_2)$. Take $v \in B$. We may suppose $d(v, u_1) = 3$. We adjoin a vertex $a$ to $A$ as follows:

$$d(a, u_1) = 4, \quad d(a, u') = 2 \text{ for } u \in A, u' \neq u_1$$
$$d(a, v) = 1, \quad d(a, v') = 3 \text{ for } v' \in A, v' \neq v$$

Now the configuration $(A \cup \{a\}, B \setminus \{v\})$ embeds into $\Gamma$ by induction on $|B|$, and the configuration $(A \setminus \{u_1\} \cup \{a\}, B)$ embeds into $\Gamma$ by induction on the number of pairs in $A$ at distance 4. So there is an amalgam $(A \cup \{a\}, B)$ of these two configurations embedding into $\Gamma$, and the metric on this amalgam agrees with the given metric on $(A, B)$ except possibly at the pair $(u_1, v)$. 
But we have $d(u_1, a) = 4$, $d(v, a) = 1$, so $d(u_1, v) \geq 3$, $d(u_1, v)$ is odd, and $d(u_1, v) \leq 4$. Thus $d(u_1, v) = 3$ also in the amalgam.

There remains the case in which there is no such pair, that is $A$ is an independent set in $\Gamma$ of the form $I_{\mathbb{N}}$. In this case, let us first extend $(A, B)$ to a finite configuration $(A, B_1)$ with the following properties.

1. Some vertex $b \in B_1$ is adjacent to all vertices of $A$.
2. No two vertices of $A$ have the same neighbors in $B_1$.

Now consider the configurations $(\{a\}, B_1)$ for $a \in A$. If they all embed into $\Gamma$, then some amalgam does as well, and this amalgam must be isomorphic to $(A, B_1)$ since the vertices of $A$ must remain distinct and the metric is then determined. So it suffices to check that these configurations $(\{a\}, B_1)$ embed into $\Gamma$.

However by symmetry we may equally well consider $(B_1, \{a\})$ in place of $(\{a\}, B_1)$, and hence conclude by induction unless $k = 1$. So suppose $|B| = 1$; by our first reduction, we may also suppose that $A \cong I_{\mathbb{N}}^2$ for some $m$.

Take a basepoint $*$ in $\Gamma$ and a vertex $u$ in $\Gamma_2$. It suffices to show that the set $I_u$ of neighbors of $u$ in $\Gamma_1$ is an infinite and coinfinite subset of $\Gamma_1$. By Theorem 4.5, the set $I_u$ is infinite. Furthermore there are by assumption vertices $u_1, u_2$ in $\Gamma$ with $d(u_1, u_2) = 4$, and we may suppose that the basepoint $*$ lies at distance 2 from both. Then $u_1, u_2 \in \Gamma_2$ and $I_{u_1}, I_{u_2}$ are disjoint. Thus they are coinfinite. This completes the uniqueness proof. \□

6.5. $B\Gamma \cong G_3^{c}$, Diameter 5. Now we turn to diameter 5. The claim in this case is as follows.

Lemma 6.7. Let $\Gamma$ be a metrically homogeneous bipartite graph of diameter 5, and suppose that $B\Gamma_1 \cong G_3^n$ with $3 \leq n < \infty$. Then $n = 3$ and $\Gamma$ is the antipodal bipartite double cover of $B\Gamma$.

We first deal with the case in which, in fact, $n = 3$.

Lemma 6.8. Let $\Gamma$ be a metrically homogeneous bipartite graph of diameter 5, with $B\Gamma \cong G_3^c$. Then $\Gamma$ is antipodal.

Proof. Suppose $|\Gamma_5| \geq 2$. We will show first that there is a triple $(a, b, c)$ in $\Gamma$ with $d(a, b) = 4$, $d(a, c) = 5$, and $d(b, c) > 1$.

Take a pair $u_1, u_2 \in \Gamma_5$. Then $d(u_1, u_2)$ is 2 or 4, and if it is 4 then our triple $(a, b, c)$ can be $(u_1, u_2, \ast)$ with $\ast$ the chosen basepoint. If $d(u_1, u_2) = 2$ then extend $u_1, u_2$ to a geodesic $(u_1, u_2, u_3)$ with $d(u_2, u_3) = 1, d(u_1, u_3) = 3$. As $d(u_2, u_3) = 1$ we find $u_3 \in \Gamma_4$ and therefore the triple $(\ast, u_3, u_1)$ will do.

Now fix a triple $(a, b, c)$ with $d(a, b) = 4$, $d(a, c) = 5$, and $d(b, c) > 3$ or 5.

Take a triple $(b, c, d)$ with $d(c, d) = 1$ and $d(b, d) = 4$; this will be a geodesic of length 4 or 5, and therefore exists in $\Gamma$ by homogeneity.

Now $d(a, b) = d(b, d) = 4$, and consideration of the path $(a, c, d)$ shows that $d(a, d) \geq 4$, and $d(a, b)$ is even, so $d(a, d) = 4$ as well, and we have $I_3^4$ in $\Gamma$, a contradiction. \□
Lemma 6.9. Let $\Gamma$ be a metrically homogeneous bipartite graph of diameter 5, with $B\Gamma \cong G^c_3$. Then $\Gamma$ is the antipodal bipartite double cover of $B\Gamma$; and this graph is in fact metrically homogeneous.

Proof. By the previous lemma, $\Gamma$ is antipodal. By Theorem 6 it suffices to check the following conditions on $G^c_3$:

1. $G^c_3$ is primitive;
2. No triangle in $G^c_3$ has perimeter greater than 5.

This is clear. □

It remains to eliminate the case $n \geq 4$ in diameter 5.

6.6. $B\Gamma \cong G^c_n$, $n \geq 4$, Diameter 5. It remains to show that in an infinite metrically homogeneous graph $\Gamma$ of diameter 5 for which $B\Gamma_1$ contains an independent set of order 3, $B\Gamma_1$ contains arbitrarily large independent sets. We will subdivide this case further according to the structure of $\Gamma_5$.

Lemma 6.10. Let $\Gamma$ be a metrically homogeneous bipartite graph of diameter 5, and suppose that $B\Gamma$ contains $I_{4}^3$. Then $\Gamma_5$ is infinite.

Proof. By Theorem 6, $\Gamma$ cannot be antipodal. On the other hand $\Gamma$ must be infinite. If $\Gamma_5$ is finite, it follows that the transitive closure of the relation

\[ d(x, y) = 0 \text{ or } 5 \]

is a nontrivial equivalence relation with finite classes. By Fact 2.1 this is impossible. □

Lemma 6.11. Suppose that $\Gamma$ is bipartite of diameter 5, and $\Gamma_5 = I_{\infty}^2$. Then $B\Gamma$ is the universal homogeneous graph (Rado’s graph).

Proof. Our claim is that $I_{n}^4$ embeds into $\Gamma$ for all $n$. We proceed by induction. Our assumption implies that $\Gamma$ is not antipodal and thus $I_{3}^3$ embeds into $\Gamma$.

Inductively, suppose $I_{n}^3$ embeds into $\Gamma$, with $n \geq 3$. Let $I \cong I_{n-1}^3$ be a metric subspace of $\Gamma$. We aim to embed subspaces $A = I \cup \{a, u\}$ and $B = I \cup \{b, u\}$ into $\Gamma$, with $I \cup \{a\} \cong I \cup \{b\} \cong I_{n}^4$, and with $u$ chosen so that

\[
\begin{align*}
d(u, a) &= 1 & d(u, b) &= 5 \\
d(u, x) &= 3 & (x \in I)
\end{align*}
\]

Supposing we have this, considering $(a, u, b)$ we see that $d(a, b) \geq 4$ and hence $I \cup \{a, b\} \cong I_{n+1}^4$.

We treat the second factor $I \cup \{b, u\}$ first. Consider the metric space $I \cup \{b, b'\}$ in which $b'$ lies at distance 2 from each point of $I \cup \{b\}$. The corresponding configuration in $B\Gamma$ is a point $b'$ adjacent to an independent set of order $n$, and this we have in $B\Gamma$. Thus the space $I \cup \{b, b'\}$ embeds into $\Gamma$.

By hypothesis, there is also a triple $(u, b, b')$ with $b, b' \in \Gamma_5(u)$. Amalgamate $I \cup \{b, b'\}$ with $(u, b, b')$ over $b, b'$. For $x \in I$, considering $(u, b', x)$, we see that $d(u, x) \geq 3$, and that $d(u, x)$ is odd. Our hypothesis on $\Gamma_5$ implies that $d(u, x) \neq 5$, so $d(u, x) = 3$ for all $x \in I$. Thus $I \cup \{b, u\}$ is as desired.
The construction of the factor $I \cup \{a, u\}$ is more elaborate. Consider the metric spaces $A = I \cup \{a\} \cup J$ and $B = J \cup \{a, u\}$, where $J \cong I_n^2$, and $J$ may be labeled as $\{v^* : v \in I\}$ in such a way that

\[
\begin{align*}
d(a, v^*) &= d(v, v^*) = 4 \quad (v \in I) \\
d(v, w^*) &= 2 \quad v, w \in I \text{ distinct} \\
d(u, v) &= 5 \quad (v \in J)
\end{align*}
\]

Supposing that $A$ and $B$ embed into $\Gamma$, take their amalgam over $J \cup \{a\}$. Then for $v \in I$ the triple $(u, a, v)$ shows that $d(u, v) \geq 3$, and the distance is odd, while the triple $(u, v', v)$ and the hypothesis on $\Gamma_5$ shows that this distance is not 5. Thus the space $I \cup \{a, u\}$ will have the desired metric. It remains to construct $A$ and $B$.

Consider $B = J \cup \{a, u\}$. The graph $\Gamma$ contains an edge $(a, u)$ as well as a copy of $J \cup \{u\}$, the latter by the hypothesis on $\Gamma_5$. Furthermore, in any amalgam of $(a, u)$ with $J \cup \{u\}$, the only possible value for the distance $d(a, v)$, for $v \in J$, is 4. So this disposes of $B$.

Now consider $A = I \cup J \cup \{a\}$, in which all distances are even. So we need to look for the rescaled graph $(1/2)A$ in $B\Gamma$. It suffices to check that the maximal independent sets of vertices in $(1/2)A$ have order at most $n$. This is the case for $I \cup \{a\}$, and any independent set meeting $J$ would have order at most 3. Since $n \geq 3$, we are done. \qed

**Lemma 6.12.** Suppose that $\Gamma$ is bipartite, metrically homogeneous, not antipodal, and of diameter 5. Then $\Gamma_5$ contains a subspace of the form $I_n^2$; in other words, $(1/2)\Gamma_5$ contains an infinite clique.

**Proof.** Supposing the contrary, for each $u \in \Gamma_4$, the set $I_u$ of neighbors of $u$ in $\Gamma_5$ is finite and nonempty, of fixed order $k$. Since any subset of $\Gamma_5$ isomorphic to $I_{k+1}$ would have a common neighbor $u \in \Gamma_4$, it follows that the $I_u$ represent maximal cliques of $(1/2)\Gamma_5$.

As $B\Gamma$ is either generic omitting $I_n$ for some $n \geq 4$, or universal homogeneous, it follows that $(1/2)\Gamma_4$ is primitive and contains both edges and nonedges. Now the relation $I_u = I_v$ defines an equivalence relation on $(1/2)\Gamma_4$ which can only be equality, that is the map $u \mapsto I_u$ is a bijection between $\Gamma_4$ and the maximal cliques of $(1/2)\Gamma_5$. If $(1/2)\Gamma_5 \cong m \cdot K_k$ then by homogeneity $\Gamma_4$ must involve a unique distance, which is a contradiction.

If $|I_u| = 1$ for $u \in \Gamma_4$ then the edge relation is a bijection between $\Gamma_4$ and $\Gamma_5$; but every $v \in \Gamma_5$ has infinitely many neighbors in $\Gamma_4$. So $|I_u| > 1$. On the other hand if $|I_u| \geq 3$ then for $u, u' \in \Gamma_4$ we have the possibilities $|I_u \cap I_{u'}| = 0, 1, 2$ while there are only two distances occurring in $\Gamma_4$. So $|I_u| = 2$ for $u \in \Gamma_4$.

Thus $(1/2)\Gamma_5$ is either $K_2[I_m]$ or generic triangle-free, and the vertices of $\Gamma_4$ correspond to edges of $(1/2)\Gamma_5$. Furthermore when two edges meet in $(1/2)\Gamma_5$ the corresponding vertices in $(1/2)\Gamma_4$ are adjacent. Thus $(1/2)\Gamma_4$ is the line graph of $(1/2)\Gamma_5$. 

If \((1/2)\Gamma_5 \cong K_2[I_\infty]\) then write \((1/2)\Gamma_5 = A \cup B\) with \(A, B \cong I_\infty\) and choose \(a_i \in A, b_j \in B\) for \(1 \leq i \leq 4, 1 \leq j \leq 3\). Consider the edges \(e_1 = (a_1, b_1), e_2 = (a_2, b_1), f_1 = (a_3, b_2), f_2 = (a_4, b_2)\) and \(f'_2 = (a_3, b_3)\). Then in \((1/2)K_4\) the configurations corresponding to \((e_1, e_2, f_1, f_2)\) and \((e_1, e_2, f_1, f'_2)\) are isometric and therefore \((1/2)\Gamma_5\) should have an automorphism fixing \(e_1, e_2, f_1\) and carrying \(f_2\) to \(f'_2\). But to fix \(e_1, e_2\) requires leaving \(A\) and \(B\) invariant, and to carry \((f_1, f_2)\) to \((f_1, f'_2)\) requires switching \(A\) and \(B\), a contradiction.

If \((1/2)\Gamma_5\) is generic triangle-free, consider edges \(e, f\) at distance 2. These are disjoint edges such that at least one edge meets both. The number of such edges may be one or two, As any automorphism of the line graph is induced by an automorphism of the original graph, this graph is not distance transitive. \hfill \Box

**Lemma 6.13.** Suppose that \(\Gamma\) is bipartite, metrically homogeneous, and of diameter 5, and \(\Gamma_5\) contains a pair of vertices at distance 4. Then the relation

\[ d(x, y) = 0 \text{ or } 4 \]

is not an equivalence relation on \(\Gamma_5\).

**Proof.** Supposing the contrary, we have

\[ \Gamma_5 \cong f^2_\infty[I^4_k] \]

for some \(k\) with \(2 \leq k \leq \infty\).

Suppose first \(k \geq 3\). Fix two equivalence classes \(C, C'\) in \(\Gamma_5\), and choose a triple \(u_1, u_2, u_3\) in \(C_1\) and a vertex \(u'_1\) in \(C'\). Choose \(v \in \Gamma\) with \(d(v, u_1) = d(v, u'_1) = 1\), and let \(d_i = d(v, u_i)\) for \(i = 2, 3\). We may then choose \(u'_2, u'_3\) in \(C'\) so that \(d(v, u'_i) = d_i\) for \(i = 2, 3\).

Now the permutation of the \(u_i, u'_i\) which switches \(u'_1\) and \(u'_2\) and fixes the other elements is an isometry, so there is an element \(v'\) with \(d(v', u) = d(v, u)\) for \(u = u_1, u_2, u_3, u'_3\), but with \(d(v', u'_1) = d_2, d(v', u'_2) = 1\).

As \(u_1\) is adjacent to \(v, v'\) we have \(d(v, v') = 2\). Now \(u_3, v, v'\) is isometric with \(u'_3, v, v'\), and the equivalence class of \(u_3\) contains a common neighbor of \(v, v'\); therefore the equivalence class of \(u'_3\) contains a common neighbor of \(v, v'\). But \(v\) can have at most one neighbor in an equivalence class, so this contradicts the choice of \(v'\).

So we are left with the case \(k = 2\):

\[ \Gamma_5 \cong K_\infty[I^4_2] \]

In this case we will consider a specific amalgamation.

Let \(\gamma = (u, v, w)\) be a geodesic with

\[ d(u, v) = 1; d(v, w) = 4; d(u, w) = 5 \]
Let $A = \gamma \cup \{a\}$, $B = \gamma \cup \{b\}$, with the metrics given by

\[
\begin{array}{ccc}
u & v & w \\
a & 4 & 3 & 5 \\
b & 4 & 5 & 5 \\
\end{array}
\]

If $A, B$ embed into $\Gamma$, then their relation to $v$ prevents them from being identified in the amalgam. However $a, b, u \in \Gamma_5(w)$ and $d(a, u) = d(b, u) = 4$. So $d(a, b) = 4$ by our assumption, and this contradicts $k = 2$.

**Lemma 6.14.** Suppose that $\Gamma$ is bipartite, metrically homogeneous, and of diameter 5. Then $\Gamma_i$ is connected with respect to the edge relation given by $d(x, y) = 2$, for $1 \leq i \leq 5$.

**Proof.** This is true for $\Gamma_1$ automatically. It is true for $i = 2$ or 4 in view of the structure of $BI$. It remains to prove it for $i = 3$ or 5.

If $\Gamma_i$ is disconnected with respect to this relation, then for $u \in \Gamma_{i-1}$, the set $I_u$ of neighbors of $u$ in $\Gamma_i$ is contained in one of the equivalence classes of $\Gamma_i$, and there is more than one such class. Thus we have a function from $\Gamma_{i-1}$ to the quotient of $\Gamma_i$. As $i - 1$ is even, in view of the structure of $BI$ we know $\Gamma_{i-1}$ is primitive, so as $\Gamma_i$ contains more than one equivalence class, this function is $1 - 1$. Then the sets $I_u$ for $u \in \Gamma_{i-1}$ must be exactly the equivalence classes of $\Gamma_i$, and $\Gamma_{i-1}$ is in bijection with the quotient. In particular, only one distance occurs in $\Gamma_{i-1}$. But in view of the structure of $BI$, this is not the case. \qed

**Corollary 6.15.** Suppose that $\Gamma$ is bipartite, metrically homogeneous, and of diameter 5, and $\Gamma_5$ contains a pair of vertices at distance 4. Then $\Gamma_5$ is primitive, infinite, and contains a copy of $I^3_{\infty}$.

**Lemma 6.16.** Suppose that $\Gamma$ is bipartite, metrically homogeneous, and of diameter 5, that $\Gamma_5$ contains a pair of vertices at distance 4, and that $BI$ is generic omitting $I_n$. Then $I_{n-2}$ embeds into $\Gamma_5$.

**Proof.** Let $k$ be maximal so that $I_k^4$ embeds into $\Gamma_5$, and suppose $k \leq n - 3$.

Let $I \cong I_{n-2}^4$, and suppose $a, b, u$ are additional vertices with $I \cup \{a\} \cong I \cup \{b\} \cong I_n^4 - 1$, and with $d(a, a) = 1$, $d(u, b) = 5$. If $I \cup \{a\}$ and $I \cup \{b, u\}$ embed into $\Gamma$, then so does an amalgam $I \cup \{a, b, u\}$, and the auxiliary vertex $u$ forces $d(a, b) = 4$, and $I \cup \{a, b\} \cong I_n^4$, a contradiction. So it suffices to embed $I \cup \{a, u\}$ and $I \cup \{b, u\}$ into $\Gamma$.

*Construction of $I \cup \{a, u\}$.*

Introduce a metric space $J = \cup_{v \in I} J_v$ with $J_v \cong I_k^4$ and with $d(x, y) = 2$ for $x \in J_v, y \in J \setminus J_v$. Extend to a metric on $I \cup J$ by taking

\[
d(v, x) = \begin{cases} 
4 & \text{if } x \in J_v \\
2 & \text{if } x \in J \setminus J_v 
\end{cases}
\]

for $v \in I$.

Give $J \cup \{a, u\}$ the metric with $d(a, x) = 4$, $d(u, x) = 5$ for $x \in J$. We claim that $I \cup J \cup \{a\}$ and $J \cup \{a, u\}$ embed into $\Gamma$. Now $BI$ is generic omitting
1. Proof.

Lemma 6.17. \( (1/2)G_5 \) is generic omitting \( I_{k+1} \). Since the space \( I \cup J \cup \{a\} \) does not contain \( I_m^4 \), and all its distances are even, it embeds into \( \Gamma \). Since \( J \) does not contain \( I_{k+1}^4 \), it embeds into \( \Gamma_5 \), so \( I \cup J \cup \{u\} \) embeds into \( \Gamma \). In any amalgam of \( J \cup \{u\} \) with \( \{a, u\} \), we have \( d(a, x) = 4 \) for \( x \in J \), so \( J \cup \{a, u\} \) embeds into \( \Gamma \) as well.

Thus an amalgam of \( I \cup J \cup \{a\} \) and \( J \cup \{a, u\} \) embeds into \( \Gamma \). For \( v \in I \), consideration of \( \{u, a, v\} \) shows that \( d(u, v) \) is 3 or 5, and consideration of \( J \cup \{u, v\} \) shows that \( d(u, v) \) is not 5. Thus we have \( d(u, v) = 3 \) for all \( v \in I \) in our amalgam, and thus \( I \cup \{a, u\} \) embeds isometrically into \( \Gamma \).

Construction of \( I \cup \{b, u\} \).

Let \( J' = \bigcup_{\alpha \in I} J'_{\alpha} \) with \( J'_{\alpha} \cong I_{k+1}^4 \). Let a metric on \( I \cup J' \cup \{b, u\} \) by taking \( d(u, x) = 5 \), \( d(b, x) = 4 \) for \( x \in J' \), while for \( v \in I \) we take \( d(v, x) = 4 \) for \( x \in J' \), and \( d(v, x) = 2 \) for \( x \in J \setminus J' \).

Introduce an auxiliary vertex \( b' \) with \( d(b', u) = 5 \), \( d(b', x) = 2 \) for \( x \in I \cup J' \cup \{b\} \).

We claim that \( I \cup J' \cup \{b, b'\} \) and \( J' \cup \{b, b'\} \) embed isometrically in \( \Gamma \). For \( I \cup J' \cup \{b, b'\} \) we use the structure of \( B \Gamma \), together with the condition \( k + 1 < n \), and for \( J' \cup \{b, b'\} \) we use the structure of \( \Gamma_5 \) to check that \( J \cup \{b, b'\} \) embeds into \( \Gamma_5 \).

Therefore some amalgam \( I \cup J' \cup \{b, b'\} \) embeds into \( \Gamma \). Let \( v \in I \). In the amalgam, the auxiliary vertex \( b' \) ensures that \( d(u, v) \) is 3 or 5. Consideration of \( J' \cup J \cup \{b, v\} \) shows that \( d(u, v) \) is not 5. Thus \( d(u, v) = 3 \) for \( v \in I \), and \( I \cup \{b, u\} \) embeds isometrically in \( \Gamma \).

We will need some additional amalgamation arguments to complete our analysis, beginning with the following preparatory lemma.

Lemma 6.17. Let \( \Gamma \) be bipartite of diameter 5, and not antipodal. Suppose \( B \Gamma \) is generic omitting \( I_n^4 \), with \( n \geq 4 \), and \( \Gamma_5 \) contains a pair of vertices at distance 4. Then the following hold.

1. \( I_{n-1}^4 \) embeds in \( \Gamma_3 \);
2. \( \Gamma_3 \) is primitive.

Proof. 1. \( I_{n-1}^4 \) embeds in \( \Gamma_3 \):

We show inductively that \( I_m^4 \) embeds into \( \Gamma_3 \) for \( m \leq n - 1 \).

Let \( I \cong I_{m-1}^4 \). Form extensions \( I \cup \{u\} \) and \( I \cup \{v\} \) with \( d(u, x) = 5 \), \( d(v, x) = 3 \) for \( x \in I \). Then \( I \cup \{u\} \) embeds into \( \Gamma \) since \( m - 1 \leq n - 2 \), while \( I \cup \{v\} \) embeds into \( \Gamma \) by induction on \( m \). So some amalgam \( I \cup \{u, v\} \) embeds into \( \Gamma \) with \( d(u, v) \) either 2 or 4.

Consider a geodesic \( \{u, v, w\} \) with \( d(u, w) = 1 \), \( d(v, w) = 3 \), and \( d(u, v) \) as specified. There is an amalgam \( I \cup \{u, v, w\} \) of \( I \cup \{u, v\} \) with \( \{u, v, w\} \) over \( u, v, w \), and consideration of \( \{w, x, u\} \) for \( x \in I \) show that \( I \cup \{w\} \cong I_m^4 \).

As \( I \cup \{w\} \subset \Gamma_3(v) \), the induction is complete.

2. \( \Gamma_3 \) is primitive:

By Lemma 6.14 we have \( (1/2)\Gamma_3 \) connected. Suppose now that \( \Gamma_3 \) is disconnected with respect to the edge relation “\( d(x, y) = 4 \).”
Fix two connected components $C, C'$ with respect to this relation. By (1) these have order $n - 1$, and by assumption $n \geq 4$. Fix $u \in C$ and $u_1', u_2' \in C'$, and $v_1 \in \Gamma_2$ adjacent to $u, u_1'$. With $*$ the chosen basepoint for $\Gamma$, consider the isometry of $C \cup C' \cup \{*, v\}$ which interchanges $u_1'$ and $u_2'$ and fixes the remaining vertices. Then this extends to an isometry $C \cup C' \cup \{*, v\} \cong C \cup C' \cup \{*, v', v\}$ for some vertex $v'$. Take $u_3' \in C'$, distinct from $u_1, u_2$. Then the map $(*, u_3', v, v') \mapsto (*, u_3', u_3, v, v')$ is an isometry and therefore extends to $\Gamma$; its extension interchanges $C$ and $C'$ and fixes $v, v'$. However $d(v, x) = d(v', x)$ for $x \in C$, so the same applies to $C'$. But $d(v, u_1') = 1, d(v', u_2') = 1$, and $d(u_1', u_2') = 4$, so this is impossible. \hfill $\square$

Now we can assemble these ingredients.

**Lemma 6.18.** If $\Gamma$ is bipartite of diameter 5 and not antipodal, then $B\Gamma$ is the universal homogeneous graph (Rado’s graph).

**Proof.** The alternative is that $B\Gamma$ is generic omitting $I_n$ for some $n \geq 4$. By Lemma 6.11, we may suppose that $\Gamma_5$ contains a pair of vertices at distance 4, and hence $I_{n-2}^4$ embeds in $\Gamma_5$ by Lemma 6.16.

To get a contradiction, we will aim at an amalgamation of the following form. Let $I \cong I_{n-2}^4$, let $I \cup \{a\} \cong I \cup \{b\} \cong I_{n-1}^4$, and adjoin a vertex $u$ such that

$$d(u, a) = 1; \ d(u, b) = 5; \ d(u, x) = 3 \text{ for } x \in I$$

We will embed $I \cup \{a, u\}$ and $I \cup \{b, u\}$ in $\Gamma$, and then in their amalgam we will have $I \cup \{a, b\} \cong I_n^4$, a contradiction. Each of the factors $I \cup \{a, u\}$ and $I \cup \{b, u\}$ will require its own construction.

The first factor, $I \cup \{a, u\}$. Let $I = I_0 \cup \{c\}$ and introduce a vertex $v$ with

$$d(v, a) = 1, \ d(v, c) = 5, \ d(v, u) = 2, \ d(v, x) = 4 \text{ (} x \in I_0 \cup \{a\} \text{)}$$

If $I_0 \cup \{a, u, v\}$ and $I_0 \cup \{c, u, v\}$ embed into $\Gamma$, then in their amalgam $I_0 \cup \{a, c, u, v\}$ we have $d(a, c) = 4$ and thus the desired metric space $I \cup \{a, u\}$ is embedded into $\Gamma$.

Construction of $I_0 \cup \{a, u, v\}$.

We first embed $I_0 \cup \{a, u\}$ into $\Gamma$. Introduce a vertex $a'$ with

$$d(a', u) = 1; \ d(a', a) = 2; \ d(a', x) = 2 \text{ for } x \in I_0$$

On the one hand, the geodesic $(a, u, a')$ embeds into $\Gamma$; on the other hand, the metric space $I_0 \cup \{a, a'\}$ embeds into $B\Gamma$. So some amalgam $I_0 \cup \{a, a', u\}$ embeds into $\Gamma$, and for $x \in I_0$, consideration of the paths $(u, a, x)$ and $(a, a', x)$ shows that $d(u, x) = 3$, as required. Thus $I_0 \cup \{a, u\}$ embeds into $\Gamma$.

Take $a$ as basepoint. Then $u \in \Gamma_1$, $I_0 \subseteq \Gamma_4$, and $d(u, x) = 3$ for $x \in I_0$. Let $I_1 \subseteq I_0$ be obtained by removing one vertex, so $I_1 \cong I_{n-4}^4$. Consider the sets
\[ A = \{ u \in \Gamma_1 : d(u, x) = 3 \text{ for } x \in I_1 \} \]

\[ B = \{ u \in \Gamma_4 : d(u, x) = 4 \text{ for } x \in I_1 \} \]

The partitioned metric space \((A, B)\) is homogeneous with respect to the metric plus the partition. We consider the structure of \((A, B)\).

We show first that \(A\) is infinite. Assuming the contrary, consider a configuration \(I_1 \cup I_2\) in \(I_2 \cong I^{(2)}_3\), and \(I_1 \cup \{ x \} \cong I^{(4)}_{n-3}\) for each \(x \in I_2\). This configuration embeds into \(\Gamma_4\). There is a pair \(x, y \in I_2\) such that \(I_1 \cup \{ x \}\) and \(I_1 \cup \{ y \}\) have the same vertices at distance 3 in \(\Gamma_1\). Now \(\Gamma_4\) is generic omitting \(I^{(4)}_{n-1}\). By homogeneity it follows easily that any two subsets of \(\Gamma_4\) isomorphic to \(I^{(4)}_{n-3}\) have the same vertices at distance 3 in \(\Gamma_1\). This yields a nonempty subset of \(\Gamma_1\) definable without parameters, and a contradiction. So \(A\) is infinite.

Now \(B\) is generic omitting \(I^{(4)}_3\). In particular \(B\) is primitive. Furthermore, each vertex of \(B\) lies at distance 3 from some vertex of \(A\). By primitivity, this vertex cannot be unique. Take \(c \in B\) and \(u, v \in A\) so that \(d(c, u) = d(c, v) = 3\). Then \(I_0 \cup \{ c, a, u, v \}\) has the desired structure.

Construction of \(I_0 \cup \{ c, u, v \}\).

We introduce a vertex \(d\) with

\[ d(d, u) = d(d, v) = 1; d(d, c) = 4 \]

The relation of \(d\) to \(I_0\) will be determined below.

In any amalgam of \(I_0 \cup \{ c, d, u \}\) with \(I_0 \cup \{ c, d, v \}\) over \(I_0 \cup \{ c, d\}\) we have \(d(u, v) = 2\). It remains to construct \(I_0 \cup \{ c, d, u \}\) and \(I_0 \cup \{ c, d, v \}\).

We claim first that \(I_0 \cup \{ c, u \}\) embeds into \(\Gamma\), in other words that \(I_0 \cup \{ c \}\) embeds into \(\Gamma_3(u)\). This holds by Lemma 6.17. Now we may form \(I_0 \cup \{ c, d, u \}\) by amalgamating \(I_0 \cup \{ c, u \}\) with \(\{ c, d, u \}\) (a geodesic) to determine the metric on \(I_0 \cup \{ d \}\); all distances \(d(d, x)\) will be even for \(x \in I_0\). This amalgamation determines the structure of \(I_0 \cup \{ d \}\) and thereby completes the determination of the second factor \(I \cup \{ c, d, v \}\) as well.

We claim that \(I \cup \{ c, d, v \}\) embeds into \(\Gamma\). Since the distance \(d(c, d) = 4\) is forced in any amalgam of \(I_0 \cup \{ v, c \}\) with \(I_0 \cup \{ v, d \}\), we consider these two metric spaces separately.

Now \(I_0 \cup \{ v, d \} \cong I_0 \cup \{ u, d \}\), so this is not at issue, and we are left only with \(I_0 \cup \{ c, v \}\). This last embeds into the second factor \(I \cup \{ b, u \}\), so we may turn finally to a consideration of this second factor.

The second factor, \(I \cup \{ b, u \}\).

We introduce another vertex \(v\) satisfying

\[ d(v, b) = 1; d(v, u) = 4; d(v, x) = 5 \text{ for } x \in I \]

This will force \(d(b, x) = 4\) for \(x \in I\). So it will suffice to embed \(I \cup \{ u, v \}\) and \(\{ b, u, v \}\) separately into \(\Gamma\). Since \(\{ b, u, v \}\) is a geodesic, we are concerned with \(I \cup \{ u, v \}\).
Introduce a vertex $d$ with
$$d(d, u) = 1; \quad d(d, v) = 5; \quad d(d, x) = 2 \text{ for } x \in I$$
Then amalgamation of $I \cup \{d, u\}$ with $I \cup \{d, v\}$ forces $d(u, v) = 4$. It remains to embed $I \cup \{d, u\}$ and $I \cup \{d, v\}$ into $\Gamma$.

The second of these, $I \cup \{d, v\}$, has a simple structure with $I \cup \{d\} \subseteq \Gamma_5(v)$, and since $I \cup \{d\}$ has order $n$, with all distances even, it embeds into $\Gamma_5$ by Lemma 6.16. So we need only construct $I \cup \{d, u\}$.

Taking $d$ as base point, and $I$ contained in $\Gamma_2$, we are looking for a vertex $u \in \Gamma_1$ at distance 3 from all elements of $I$. For $v \in I$, let $I_v$ be the set of neighbors of $v$ in $\Gamma_1$. Any vertex $u \in \Gamma_1$ which is not in $\bigcup_{v \in I} I_v$ will do. So it remains to be checked that $\bigcup_{v \in I} I_v \neq \Gamma_1$.

The sets $I_v$ for $v \in I$ are pairwise disjoint. Suppose they partition $\Gamma_1$. We may take a second set $J \approx I_n^{4n-2}$ in $\Gamma_2$ overlapping with $I$ so that $|I \cap J| = n-3$, and then the $I_v$ for $v \in J$ will also partition $\Gamma_1$; so the vertices $v_1 \in I \setminus J$ and $v_2 \in J \setminus I$ have the same neighbors in $\Gamma_1$. As $\Gamma_2$ is primitive, it follows that all vertices of $\Gamma_2$ have the same neighbors in $\Gamma_1$, a contradiction. □

At this point, the proof of Theorem 8 is complete. We review the analysis.

Proof. If $\Gamma_1$ is finite, Lemma 6.2 applies. So we suppose $\Gamma_1$ is infinite, and as $B\Gamma$ then contains $K_\infty$, and is connected, Lemma 6.3 gives five possibilities for $B\Gamma$: $K_\infty[I_n]$ with $n \geq 1$, $T_{r,\infty}$ with $2 \leq r \leq \infty$, $G^c_n$ with $3 \leq n < \infty$, the Rado graph, or a graph of diameter at least 3 with $B\Gamma_1$ being the Rado graph.

When $B\Gamma \cong K_\infty[I_n]$ with $n \geq 1$, Lemma 6.4 applies. When $B\Gamma \cong T_{r,\infty}$, Lemma 6.5 applies.

When $B\Gamma \cong G^c_n$ with $3 \leq n \leq \infty$, the diameter $\delta$ of $\Gamma$ is 4 or 5. When $\delta = 4$, Lemma 6.6 applies. If $\delta = 5$ and $n = 3$ then Lemma 6.9 applies. When $4 \leq n < \infty$ and $\delta = 5$, after a lengthy analysis Lemma 6.18 eliminates this case.

In the remaining two cases, $\Gamma_1 \cong G_\infty$. $\Gamma$ may be a homogeneous graph, in which case $\delta = 2$ and $\Gamma \cong G_\infty$, or we may have $\delta \geq 3$ and $\Gamma_1 \cong G_\infty$. In either case, there is nothing more to prove at this stage. □

7. Bipartite Metrically Homogeneous graphs with $B\Gamma = G_\infty$

We continue the analysis of bipartite metrically homogeneous graphs through diameter 5, in other words we deal with all cases in which $B\Gamma$ is a homogeneous graph.

Proposition 7.1. Let $\Gamma$ be a connected bipartite metrically homogeneous graph of diameter at most 5. Then $\Gamma$ is one of the following.

1. Diameter 2: complete bipartite, $B\Gamma \cong K_\infty$;
2. Diameter 3: the bipartite complement of a perfect matching, or generic bipartite, $B\Gamma \cong K_\infty$;
3. Diameter 4:
Theorem 8 we gave this classification with the exception of the cases

\( B \Gamma \cong G_\infty \)

corresponding to one graph of diameter 4 and two of diameter 5.


Lemma 7.2. Let \( \Gamma \) be a bipartite metrically homogeneous graph of diameter 4 with \( B \Gamma \cong G_\infty \). Let \((A, B)\) be a finite bipartite graph of diameter at most 3. Then \((A, B)\) embeds into \( \Gamma \).

Proof. Extending \( B \), we may suppose that \((A, B)\) is the unique amalgam of the graphs \((\{a\}, B)\) \((a \in A)\) and thus reduce to the case \( A = \{a\} \).

Set \( B_i = \{b \in B : d(a, b) = i\} \) for \( i = 1, 3 \). Introduce \( a' \) with

\[
d(a, a') = 2; \ d(a', b) = 1 \quad \text{for} \quad b \in B
\]

Then the amalgamation of \((\{a, a'\}, B_1)\) with \((\{a, c'\}, B_3)\) forces \((\{a\}, B)\). So we must show that these two factors embed into \( \Gamma \).

Now \((\{a, a'\}, B_1)\) consists of a pair \( a, a' \) at distance 2 and some common neighbors, and Lemma 4.5 shows this embeds into \( \Gamma \).

We consider the second factor \((\{a, c'\}, B_3)\). Introduce \( b' \) with

\[
d(b'b) = 2 \quad \text{for} \quad b \in B; \ d(a, b') = d(a', b') = 1
\]

Then amalgamation of \((\{a\}, B \cup \{b'\})\) \((\{a', B \cup \{b'\})\) forces \((\{a, a'\}, B)\).

Now \( a' \) is adjacent to all vertices of \( B \cup \{b'\} \) so \((\{a'\}, B \cup \{b'\})\) embeds into \( \Gamma \).

Consider \((\{a\}, B \cup \{b'\})\). Taking \( b' \) as a basepoint in \( \Gamma \), and \( a \in \Gamma_1 \), we consider

\[
J_a = \{v \in \Gamma_2 : d(a, v) = 3\}
\]

Now \((1/2)\Gamma_2 \cong G_\infty \) and \((1/2)\Gamma_2 \setminus J_a\) is a clique, and it follows easily that \( J_a \) is infinite. \( \square \)

Lemma 7.3. Let \( \Gamma \) be a bipartite metrically homogeneous graph of diameter 4 with \( B \Gamma \cong G_\infty \). Then

\[
\Gamma \cong \Gamma^4_{\infty; 0; 9, 14}
\]

is the generic bipartite graph of diameter 4.
Proof. It suffices to show that any finite bipartite graph \((A,B)\) of diameter at most 4 embeds into \(\Gamma\). We proceed by induction on
\[
\mu_4 = |\{(a,b) \in A \times B : d(a,b) = 4\}|
\]
If \(\mu_4 = 0\) the previous lemma applies, so we may suppose \(\mu_4 > 0\). We suppose that \(A\) contains a pair of vertices at distance 4. Subject to this, take a counterexample \((A,B)\) with \(|A|\) minimized, and then with \(|B|\) minimized.

We claim
\[
A \cong I^k_4 \text{ for some } k
\]
Suppose on the contrary that \(a_1, a_2 \in A\) are at distance 2. Adjoin \(b_1, b_2\) with
\[
\begin{align*}
d(b_i, b) & = 2 \text{ for } b \in B, \quad d(b_1, b_2) = 2 \\
d(b_1, a_1) & = d(b_1, a_2) = 1 \\
d(b_2, a_1) & = 1, d(b_2, a_2) = 3 \\
d(b_i, a) & = 3 \text{ for } a \in A, a \neq a_1, a_2
\end{align*}
\]
Amalgamation of \((A \{a_2\}, B \cup \{b_1, b_2\})\) with \((A \{a_1\}, B \cup \{b_1, b_2\})\) forces \((A,B)\). The two factors \((A \{a_1\}, B \cup \{b_1, b_2\})\) have the same value for \(\mu_4\) as \((A,B)\), and smaller \(|A|\). So we reach a contradiction. Thus under our assumptions \(A \cong I^k_4\) for some \(k\).

If we have \(a \in A, b \in B\) with \(d(a,b) = 1\) then the amalgamation of \((A,B \setminus \{b\})\) with \(\{a\}, B\) forces \((A,B)\). The factor \((A,B \setminus \{b\})\) is in \(\Gamma\) by our initial minimization, while the factor \(\{a\}, B\) has a smaller value of \(\mu_4\). So in this case we have \((A,B)\) in \(\Gamma\).

Therefore we may suppose that
\[
d(a,b) = 3 \text{ for } a \in A \text{ and } b \in B.
\]
Fix vertices \(a_0 \in A, b_0 \in B\), chosen if possible so that \(b_0\) lies at distance 4 from some other element of \(B\). Introduce an element \(b'\) as follows.
\[
\begin{align*}
d(b', b_0) & = 4 \\
d(b', b) & = 2 \text{ for } b \in B, b \neq b_0 \\
d(b', a_0) & = 1 \\
d(b', a) & = 3 \text{ for } a \in A, a \neq a_0
\end{align*}
\]
Then amalgamation of
\[
(A,(B \setminus \{b_0\}) \cup \{b'\}) \text{ with } (A \setminus \{a_0\}, B \cup \{b'\})
\]
forces \((A,B)\) into \(\Gamma\). So we concern ourselves with these two factors.

The factor \((A,(B \setminus \{b_0\}) \cup \{b'\})\) arises by amalgamation of \((A,B \setminus \{b_0\})\) with \(\{a_0\}, (B \setminus \{b_0\}) \cup \{b'\}\). Here we embed \((A,B \setminus \{b_0\})\) by induction on \(|B|\) and the factor \(\{a_0\}, (B \setminus \{b_0\}) \cup \{b'\}\) is available by induction on \(\mu_4\).

This leaves the factor \((A \setminus \{a_0\}, B \cup \{b'\})\) to be constructed.
If \( k \geq 3 \) then this is available by induction on \( \mu_4 \). Suppose

\[
k = 2
\]

Then this factor has the form

\[
\{a\}, B
\]

where \( d(a, b) = 3 \) for \( b \in B \). So it suffices to embed \( B \) into \( \Gamma_3 \).

Now \( B\Gamma = (1/2)\Gamma_1 \cup (1/2)\Gamma_3 \) with \( (1/2)\Gamma_1 \) a clique, and \( B\Gamma \cong G_\infty \). It follows easily that \( (1/2)\Gamma_3 \cong G_\infty \), and our claim follows. \( \square \)

7.2. Diameter 5: Structure of \( \Gamma_5 \). The two graphs arising in diameter 5 may be distinguished by the structure of \( (1/2)\Gamma_5 \).

Lemma 7.4. Let \( \Gamma \) be a bipartite metrically homogeneous graph of diameter 5 with \( B\Gamma \cong G_\infty \). Then

1. For \( u \in \Gamma_4 \), \( u \) has infinitely many neighbors in \( \Gamma_5 \).
2. \( (1/2)\Gamma_5 \) is primitive.

Proof. Suppose that for \( u \in \Gamma_4 \) the set \( I_u = \{v \in \Gamma_5 : d(u, v) = 1\} \) is finite. For \( v \in \Gamma_5 \), all the neighbors of \( v \) are in \( \Gamma_4 \), so for \( u_1, u_2 \) in \( \Gamma_4 \) at distance 2, \( I_{u_1} \) meets \( I_{u_2} \).

Take \( I \cong I_\infty^2 \) in \( \Gamma_4 \). Then \( \{I_u : u \in I\} \) is a collection of finite sets of fixed size. Reducing \( I \), we may suppose that these sets form a \( \Delta \)-system. Then for \( u_1, u_2, u_3 \in I \) we have

\[
I_{u_1} \cap I_{u_2} = I_{u_1} \cap I_{u_3}
\]

For any \( u_1, u_2, u_3 \) with \( d(u_1, u_2) = d(u_2, u_3) = 2 \) we may consider an additional vertex \( v \) with \( d(v, u_i) = 2 \) for \( i = 1, 2, 3 \) and deduce that \( I_u \cap I_{u'} \) is independent of the choice of \( u' \) subject to \( d(u, u') = 2 \), for \( u, u' \in \Gamma_4 \). It follows that \( I_u \subseteq I_{u'} \) by homogeneity, and then that \( I_u = I_{u'} \); in view of the structure of \( \Gamma_4 \), as this holds when \( d(u, u') = 2 \) it holds for all \( u, u' \in \Gamma_4 \), and this is a contradiction.

This proves our first point, and in particular \( K_\infty \) embeds into \( (1/2)\Gamma_5 \).

Now suppose that \( (1/2)\Gamma_5 \) is imprimitive. If \( (1/2)\Gamma_5 \cong m \cdot K_\infty \) with \( m \geq 2 \), then for \( u \in \Gamma_4 \), we have \( I_u \) contained in a single component \( C_u \) of \( (1/2)\Gamma_5 \). By primitivity of \( \Gamma_4 \), the map \( u \mapsto C_u \) is a bijection between \( \Gamma_4 \) and the components of \( (1/2)\Gamma_5 \). It follows that \( \Gamma_4 \) is an indiscernible set, a contradiction.

So we suppose that

\[
(1/2)\Gamma_5 \cong K_\infty[K_n]
\]

with \( n \geq 2 \).

For \( u \in \Gamma_4 \) and \( v \in \Gamma_5 \) there is then \( v' \in I_u \) with \( d(v, v') = 2 \), forcing \( d(u, v) \leq 3 \). So for any component \( C \) of \( (1/2)\Gamma_5 \) there is \( v \in C \) with \( d(u, v) = 3 \). Therefore \( I_u \) meets every component of \( (1/2)\Gamma_5 \).

Taking \( I \subseteq \Gamma_4 \) with \( I \cong I_3^4 \) we find that the sets \( I_u (u \in I) \) are pairwise disjoint, and therefore \( m \geq 3 \).
Let \( u, u' \in \Gamma_4 \) with \( d(u, u') = 2 \) and let \( C \) be a component of \((1/2)\Gamma_5\). As \( m \geq 3 \) there is \( v \in C \) with \( d(u, v) = d(u, v') = 3 \). Now \( I_u \) meets \( I_{u'} \) and hence by homogeneity we get \( I_u \cap C = I_{u'} \cap C \) and thus \( I_u = I_{u'} \), and by the primitivity of \( \Gamma_4 \) we again reach a contradiction.

Thus \((1/2)\Gamma_5\) must be primitive. \( \square \)

The next step is the elimination of the possibility \((1/2)\Gamma_5 \cong G^c_n\) with \( n < \infty \). We prepare for this with the following.

**Lemma 7.5.** Let \( \Gamma \) be a bipartite metrically homogeneous graph of diameter 5 with \( B\Gamma \cong G_\infty \), and suppose that \((1/2)\Gamma_5 \cong G^c_n\) with \( 3 \leq n < \infty \). Then for \( B \cong I^4_{n-2} \) and \( p : B \to \{3, 5\} \) arbitrary, there is an embedding of \( B \) into \( \Gamma_5 \) and an element \( a \in \Gamma_1 \) with

\[ d(a, b) = p(b) \text{ for } b \in B \]

**Proof.** We extend the configuration \( B \cup \{a\} \) by adjoining a vertex \( u \in \Gamma_2 \) with

\[ d(u, a) = 1, \ d(u, b) = p(b) - 1 \text{ for } b \in B \]

Let \( B_5 = \{b \in B : p(b) = 5\} \). Then amalgamation of

\[(\{a, u\}, B_5) \text{ with } (\{u\}, B)\]

with \( a \in \Gamma_1, u \in \Gamma_2, B \subseteq \Gamma_5 \) produces the required embedding of the configuration \( B \cup \{a\} \).

It remains to embed the configurations \( B_5 \cup \{a, u, *\} \) and \( B \cup \{u, *\} \) into \( \Gamma \) where * represents an additional basepoint. The configuration \( B \cup \{u, *\} \) embeds into \( \Gamma \) since \( B\Gamma \cong G_\infty \). So we consider

\[ B_5 \cup \{a, u, *\} \]

Looking at this from the point of view of \( a \) as basepoint, we need to embed \( B_5 \) into \( \Gamma_5 \), and \( u, * \) into \( \Gamma_1 \), so that \( d(x, b) = 4 \) for \( x \in \{u, *\} \) and \( b \in B_5 \).

But this last relation is automatic for \( x \in \Gamma_1 \) and \( b \in \Gamma_5 \). So it suffices to embed \( I_2 \) into \( \Gamma_1 \) and \( I^4_m \) with \( m \leq n - 2 \) into \( \Gamma_5 \), and we have both of these. \( \square \)

**Lemma 7.6.** Let \( \Gamma \) be a bipartite metrically homogeneous graph of diameter 5 with \( B\Gamma \cong G_\infty \). Then \((1/2)\Gamma_5 \cong K_\infty \) or \( G_\infty \).

**Proof.** Supposing the contrary, as \((1/2)\Gamma_5 \) is primitive and contains \( K_\infty \), we find

\[(1/2)\Gamma_5 \cong G^c_n \]

for some \( n \) with \( 3 \leq n < \infty \).

We consider the following amalgamation.
We will determine the relationship of $u_2$ to $B$ in the course of forming the factor $B \cup \{a_2, u_1, u_2\}$.

We form the factor $B \cup \{a_1, u_1, u_2\}$ by amalgamating $B \cup \{a_1, u_1\}$ with $B \cup \{a_1, u_2\}$. To see that $B \cup \{a_1, u_1\}$ embeds in $\Gamma$ it suffices to check that $B \cup \{a_1\}$ embeds in $\Gamma$, and indeed $B \cup \{a_1\} \cong I_{n-1}^4$. The embedding of $B \cup \{a_1, u_2\}$ into $\Gamma$ is the subject of the previous lemma.

So it remains to construct $B \cup \{a_2, u_1, u_2\}$ in such a way that $d(u_2, b) \in \{3, 5\}$ for $b \in B$

If this is not possible, this means we have the following: For $u \in \Gamma_4$ and $B \subseteq \Gamma_5$ with $B \cong I_{n-1}^4$, if there is $b \in B$ with $d(u, b) = 5$, then there is $b' \in B$ with $d(u, b') = 1$.

For any $v_1, v_2 \in \Gamma_5$ with $d(v_1, v_2) = 2$, we can find $B$ in $\Gamma_5$ with $B \cup \{v_1\} \cong I_{n-1}^4$, and then for $b \in B$ we find $B_b \subseteq \Gamma_5$ with $B_b \cup \{b\}, B_b \cup \{v_2\} \cong I_{n-1}^4$. Suppose that $u \in \Gamma_4$ and $d(u, v_1) = d(u, v_2) = 5$. Then for some $b \in B$ we must have $d(u, b) = 1$, and for some $b' \in B_b$ we must also have $d(u, b') = 1$, and then $d(b, b') = 2$, a contradiction. Thus for any $u \in \Gamma_4$, the set

$$J_u = \{v \in \Gamma_5 : d(u, v) = 5\}$$

is of the form $I_k^4$ where in view of the structure of $\Gamma_5$ we have $k < n$.

As the distance 4 occurs in $\Gamma_5$, we find $k \geq 1$. If $k = 1$ then by primitivity of $\Gamma_4$ we have a definable bijection between $\Gamma_4$ and $\Gamma_5$, which must be an isomorphism or anti-isomorphism, which is impossible. So

$$k \geq 2$$

In particular we have a distinct pair $u_1, u_2 \in \Gamma_4$ for which $I_{u_1}$ meets $I_{u_2}$. Let $d(u_1, u_2) = i$ and consider $I \subseteq \Gamma_4$ with $I \cong I_\infty^4$. Then the sets $I_u$ for $u \in I$ intersect pairwise, and reducing $I$, we may suppose they form a $\Delta$-system. It then follows easily that for $u_1, u_2 \in \Gamma_4$ with $d(u_1, u_2) = i$, the intersection $I_{u_1} \cap I_{u_2}$ is determined by $u_1$, and thus $I_{u_1} = I_{u_2}$ for such pairs,
and then by primitivity, for all pairs in $\Gamma_4$, a contradiction. This completes the construction. \qed

7.3. Diameter 5: Identification.

Lemma 7.7. Let $\Gamma$ be a bipartite metrically homogeneous graph of diameter 5 with $B\Gamma \cong G_\infty$. Then $\Gamma \cong \Gamma_{5,0;11,C_0}^5$ with $C_0 = 14$ or 16.

Proof. Let $\Gamma^* = \Gamma_{5,0;11,C_0}^4$ with $C_0 = 14$ if $\Gamma_5 \cong K_\infty$ and $C_0 = 16$ if $\Gamma_5 \cong G_\infty$. We must show that for $(A, B)$ finite, bipartite, of diameter at most 5:

$$(A, B) \text{ embeds into } \Gamma \iff (A, B) \text{ embeds into } \Gamma^*$$

If $(A, B)$ embeds into $\Gamma$ then clearly it embeds into $\Gamma^*$. Conversely, suppose $(A, B)$ embeds into $\Gamma^*$.

If the distance 2 occurs in $A$ then we can reduce $|A|$ at the cost of adjoining elements to $B$ lying at distance at most 3 from any vertex of $A$, so that if $a_1, a_2 \in A$ is a fixed pair at distance 2, then amalgamation of $(A \setminus \{a_2\}, B)$ with $(A \setminus \{a_1\}, B)$ forces an embedding of $(A, B)$. So we may suppose $A \cong I_m^4$

for some $m$.

Suppose $m \geq 2$. If there is $b \in B$ such that for some $a_1, a_2 \in A$ we have

$$d(b, a_1) = 1, d(b, a_2) = 5$$

then we can reduce $|A|$ further. And if there is no such $b \in B$ we can create one, adjoining an element $b'$ to $B$ so that for some pair $a_1, a_2 \in A$.

$$d(b', a_1) = 1, \quad d(b', a_2) = 5$$

$$d(b', a) = 3 \quad \text{ for } a \in A, a \neq a_1, a_2$$

$$d(b', b) = \begin{cases} 2 & \text{if } d(b, a_1) \neq 5 \\ 4 & \text{if } d(b, a_1) = 5 \end{cases}$$

We claim that if there was no triangle of type $(4, 5, 5)$ in $(A, B)$, then there will be no such triangle in $(A, B \cup \{b'\})$. Otherwise, we would have a triangle of the form $(a, b, b')$ with $d(a, b) = d(a, b') = 5$ and $d(b, b') = 4$. This would mean that $d(a_1, b) = 5$ and then the triple $(a_1, a, b)$ contradicts our current assumptions.

Now in the configuration $A \cup B \cup \{b'\}$ the element $b'$ serves to ensure that $d(a_1, a_2) = 4$, and so using the amalgamation of $(A \setminus \{a_2\}, B \cup \{b'\})$ with $(A \setminus \{a_1\}, B \cup \{b'\})$ we reduce $|A|$. So finally we come to the case $|A| = 1$, $A = \{a\}$.

For $i = 1, 3, 5$ let

$$B_i = \{b \in B : d(a, b) = i\}$$

Extending $B_1$, we may suppose that the amalgamation of

$$(\{a\}, B_1 \cup B_5)$$

with $B$
forces \( d(a, b) = 3 \) for \( b \in B_3 \). Thus it suffices to embed \( B \) and \( \{a\}, B_1 \cup B_5 \) into \( \Gamma \).

Now \( B \) embeds into \( \Gamma \) because \( B \Gamma \cong G_\infty \), so we consider \( \{a\}, B_1 \cup B_5 \). The amalgamation of \( \{a\}, B_1 \) with \( \{a\}, B_5 \) forces \( d(b, b') = 4 \) for \( b \in B_1 \), \( b' \in B_5 \) and thus it suffices to embed \( B_1 \) into \( \Gamma_1 \) and \( B_5 \) into \( \Gamma_5 \). The former is clear, and for the latter we either have \( B_5 \cong I_k^2 \) for some \( k \), or \( (1/2)\Gamma_5 \cong G_\infty \), so we have the required embedding in either case. \( \square \)

This completes the proof of Proposition 7.1.

8. Amalgamation classes determined by triangle constraints

All amalgamation classes considered here are classes of finite integral metric spaces of bounded diameter \( \delta \), containing all geodesics of length at most \( \delta \). We will refer to such classes as amalgamation classes of diameter \( \delta \).

In the present section we will only consider amalgamation classes determined by triangle constraints (forbidden triangles), where a triangle is any metric space of order 3. The diameter is fixed as well, that is we include some constraints on subspaces of order 2.

**Theorem 9.** Let \( \mathcal{A} \) be an amalgamation class of diameter \( \delta \) determined by triangle constraints. Then \( \mathcal{A} = \mathcal{A}_{K_1, K_2; C, C'}^\delta \) for some admissible choice of parameters.

Really what needs to be proved is that regardless of the complexity of the constraints for \( \mathcal{A} \), the set of triangles allowed are those in \( \mathcal{A}_{K_1, K_2; C, C'}^\delta \) where the parameters \( K_1, K_2, C, C' \) may be defined for any amalgamation class \( \mathcal{A} \) of finite integral metric spaces as follows.

1. \( K_1 \) is the least \( k \) such that \( \mathcal{A} \) contains a triangle of type \( (1, k, k) \) and \( K_2 \) is the greatest such: with \( K_1 = \infty, K_2 = 0 \) if no such triangles occur.
2. \( C_0 \) is the least even number greater than \( 2\delta \) such that no triangle of perimeter \( C_0 \) is in \( \mathcal{A} \); and \( C_1 \) is the least such odd number
3. \( C = \min(C_0, C_1) \) and \( C' = \max(C_0, C_1) \).

Note that \( C_0, C_1 \) are always well-defined and at most \( 3\delta + 2 \).

**8.1. An inductive lemma.** We will deal systematically with the type of a triangle \( (a, b, c) \), which is the triple of lengths \( d(a, b), d(a, c), d(b, c) \) in any order (most often, nondecreasing), representing its isomorphism type. The type of a triangle is therefore a triple \( (i, j, k) \) with \( i, j, k \in \mathbb{N}, i, j, k \geq 1 \), satisfying the triangle inequality, and most often tacitly assumed to satisfy a bound on diameter \( i, j, k \leq \delta \) as well. When we refer to a triangle type \( (i, j, k) \), in principle this means that the triangle inequality and any relevant bound are assumed. If the necessary inequalities have not been established, we speak of a triple rather than a triangle type.

Our inductive lemma will depend on the following constructions.
Lemma 8.1. Let $\mathcal{A}$ be an amalgamation class of diameter $\delta$ determined by constraints on triangles. Then all triangles of even perimeter $P \leq 2\delta$ are in $\mathcal{A}$.

Proof. We consider triangles of type $(i, j, k)$ with $P = i + j + k \leq 2\delta$ even, and $i \leq j \leq k \leq \delta$. We proceed by induction on $i$.

If the triangle is a geodesic then it lies in $\mathcal{A}$ by hypothesis.

Suppose that the triangle is not a geodesic. In particular $i > 1$. Consider the following amalgamation.

We claim that the factors of this amalgamation are in $\mathcal{A}$. The nongeodesic triangles involved in the diagram have the following triples of lengths:

$$(2, i - 1, i - 1), (i - 1, j - 1, k), \text{ and } (i - 1, j + 1, k)$$

We note that these are indeed triangle types, that is the triangle inequality holds—in the case of the second triple, the inequality

$$k \leq (i - 1) + (j - 1)$$

depends on the assumption that $(i, j, k)$ is not geodesic, and that $P$ is even.

All three triangle types have even perimeter and a lower value of $i$, so we conclude by induction that these are in $\mathcal{A}$. Thus the amalgamation has a completion in $\mathcal{A}$ and the vertices $u_1, u_2$ force $d(a_1, a_2) = j$. Thus the triangle $(a_1, a_2, c)$ has type $(i, j, k)$. □

Lemma 8.2. Let $\mathcal{A}$ be an amalgamation class of diameter $\delta$ determined by triangle constraints. Suppose that $\mathcal{A}$ contains triangles of the types

$$(i - 1, j - 1, k) \text{ and } (i - 1, j + 1, k)$$

where

$$2 \leq i \leq \delta, 2 \leq j < \delta, 1 \leq k \leq \delta$$

Then $\mathcal{A}$ contains a triangle of type $(i, j, k)$. 
Proof. We make use of the following amalgamation.

\[
\begin{align*}
&d(c, u_1) = d(c, u_2) = i - 1. \\
&\text{The nongeodesic triangles involved in this diagram have lengths given by the triples}
\end{align*}
\]

\[(2, i - 1, i - 1), (i - 1, j - 1, k), \text{and} (i - 1, j + 1, k)\]

The first of these triangles is in \(\mathcal{A}\) by the previous lemma, and the other two are in \(\mathcal{A}\) by assumption. So our diagram has a completion in \(\mathcal{A}\), and the vertices \(u_1, u_2\) force \(d(a_1, a_2) = j\), and \((a_1, a_2, c)\) has type \((i, j, k)\). □

Lemma 8.3. Let \(\mathcal{A}\) be an amalgamation class of diameter \(\delta\) determined by triangle constraints. Let \(m, N \in \mathbb{N}\) be fixed with \(m \geq 2\). Suppose that all triangles of perimeter \(N - 2\) and minimal distance at least \(m - 1\) are in \(\mathcal{A}\)

Then the following hold.

1. If some triangle of perimeter \(N\) and minimum distance at least \(m\) is in \(\mathcal{A}\), then all such triangles are in \(\mathcal{A}\);
2. If some triangle of perimeter \(N\) and minimum distance exactly \(m - 1\) is in \(\mathcal{A}\), then any triangle of perimeter \(N\) whose minimum distance is at least \(m - 1\) and with at most one distance equal to \(m - 1\) also belongs to \(\mathcal{A}\).

Proof. If \(N \leq 2\delta\) is even then we know that all such triangles are in \(\mathcal{A}\) and there is nothing to prove. So we may suppose that

\(N\) is odd, or \(N > 2\delta\)

In particular no geodesic triangles come into consideration.

Now suppose \((i, j, k)\) is the type of a triangle with

\[
\begin{align*}
i + j + k &= N \\
m &\leq i, j; \quad m - 1 \leq k \\
j &< \delta \quad k &\leq i + j - 2
\end{align*}
\]
Then \((i - 1, j - 1, k)\) is a triangle type, and by hypothesis \((i - 1, j - 1, k)\) is in \(A\). In view of Lemma 8.2 if there is also a triangle of type \((i - 1, j + 1, k)\) in \(A\) then there is one of type \((i, j, k)\). In symbols
\[
(i - 1, j + 1, k) \implies (i, j, k)
\]
but bearing in mind the side conditions on \(i, j, k\).

Claim 1. Suppose
\[
i + j + k = N \\
i \leq j \leq k \leq i + j \leq i + k - 2 \\
i \geq m - 1, \quad j \geq m, k > m \\
j < \delta
\]
Then there is a triangle of type \((i, j, k)\) in \(A\) if and only if there is one of type \((i, j + 1, k - 1)\) in \(A\).

Take \((i', j', k')\) to be \((j + 1, k - 1, i)\) or \((k, j, i)\) to get respectively
\[
(j, k, i) \implies (j + 1, k - 1, i); (k - 1, j + 1, i) \implies (k, j, i)
\]
But we need to check various side conditions on \(i', j', k'\).

1. \(i' + j' + k' = N\) in both cases.
2. \(m \leq i', j'\) and \(m - 1 \leq k'\).
3. \(j' < \delta\)
4. \(k' \leq i' + j' - 2\).

The first condition is clear. The second comes down to
\[
m \leq j, k - 1; m - 1 \leq i
\]
which was assumed. The third is \(k \leq \delta\) and \(j < \delta\), also assumed. The last comes down in both cases to
\[
i \leq j + k - 2
\]
which holds as \(i \leq j \leq j + k - 2\).

Thus Lemma 8.2 applies.

Claim 2. If \(i + j + k = N, m \leq i \leq j \leq k \leq i + j, i < \delta,\) and \(m + 1 \leq k,\) then there is a triangle of type \((i, j, k)\) in \(A\) if and only if there is one of type \((i + 1, j, k - 1)\) in \(A\).

Take \((i', j', k')\) to be \((i + 1, k - 1, j)\) or \((k, i, j)\) to get respectively
\[
(i, k, j) \implies (i + 1, k - 1, j); (k - 1, i + 1, j) \implies (k, i, j)
\]
Note that the inequality \(j \leq (i+1)+(k-1)-2\) holds since \(j \leq k \leq (i-2)+k\).

Claim 3. For \(\ell = 1, 2\) let \((i, j_\ell, k_\ell)\) be triples satisfying
\[
i + j_\ell + k_\ell = N \\
i \leq j_\ell \leq k_\ell \leq i + j_\ell \\
i \geq m - 1, j_\ell \geq m, k_\ell \geq m + 1
\]
Then there is a triangle of type \((i, j_1, k_1)\) in \(A\) if and only if there is one of type \((i, j_2, k_2)\) in \(A\).
We may suppose that \( j_1 \leq j_2 \) and proceed by induction on \( j_2 - j_1 \). If \( j_1 = j_2 \) then \( k_1 = k_2 \) and our claim holds. So suppose that \( j_1 < j_2 \).

Then \( k_2 < k_1 \) and in particular \( j_1 < k_1 \). So it will suffice to treat the case \( j_2 = j_1 + 1, k_2 = k_1 - 1 \), and then conclude by induction.

We invoke Claim 1 for \((i, j_1, k_1)\). We require

\[
j_1 < \delta, j_1 \leq i + k_1 - 2
\]

We have \( j_1 < j_2 \leq \delta \). As \( j_1 < j_2 \leq k_2 < k_1 \) we have \( j_1 \leq k_1 - 2 \) and thus \( j_1 \leq i + k_1 - 2 \).

Claim 4. For \( \ell = 1, 2 \) let \((i_\ell, j_\ell, k_\ell)\) satisfy

\[
i_\ell + j_\ell + k_\ell = N, m \leq i_\ell \leq j_\ell \leq k_\ell \leq i_\ell + j_\ell
\]

Then there is a triangle of type \((i_1, j_1, k_1)\) in \( \mathcal{A} \) if and only if there is one of type \((i_2, j_2, k_2)\) in \( \mathcal{A} \).

We will suppose \( i_1 \leq i_2 \) and proceed by induction on \( i_2 - i_1 \).

If \( k_1 = i_1 \) then \( N = 3i_1 \) and hence

\[
N = 3i_1 \leq 3i_2 \leq i_2 + j + 2 + k_2 = N
\]

forcing \( i_2 = j_2 = k_2 = i_1 \) and there is nothing to prove. So we will suppose

\[
k_1 > i_1
\]

In particular \( k_1 > m \).

The case \( i_1 = i_2 \) is covered by Claim 3. So suppose

\[
i_1 < i_2
\]

Then Claim 2 applies and a triangle of type \((i_1, j_1, k_1)\) is in \( \mathcal{A} \) if and only if one of type \((i_1 + 1, j_1, k_1 - 1)\) is in \( \mathcal{A} \). So we replace \( i_1 \) by \( i_1 + 1 \) and conclude by induction on \( i_2 - i_1 \).

This completes the proof of the first part of the Lemma.

Now we suppose that some triangle of type \((i, j, k)\) with

\[
i = m - 1 \leq j \leq k \leq i + j, i + j + k = N
\]

is in \( \mathcal{A} \).

If \( N < 3m - 1 \) then the second part of the lemma is vacuous. So suppose

\[
N \geq 3m - 1
\]

We claim that some triangle of type \((i, j, k)\) with \( i = m - 1 \) and \( j, k \geq m \), \( i + j + k = N \) belongs to \( \mathcal{A} \). Supposing the contrary, our triangle of type \((i, j, k)\) with \( i = m - 1, i \leq j \leq k \), and \( i + j + k = N \) must satisfy \( j = m - 1 \) and \( k = N - 2(m - 1) \geq m + 1 \). Consider the triangle type \((m, k - 1, m - 1)\).

We apply Lemma 8.2 with \( i' = m, j' = k - 1, k' = m - 1 \). We need to check that triangles of type \((m - 1, k - 2, m - 1)\) and \((m - 1, k - 1, m - 1)\) are in \( \mathcal{A} \). The first holds by our assumption on \( \mathcal{A} \) after verifying the triangle inequality, and the second holds by our assumptions on \( i, j, k \). So we deduce in this case that a triangle of type \((m - 1, m, k - 1)\) is in \( \mathcal{A} \), and in any case a triangle of the desired type \((i, j, k)\) with \( i \geq m - 1 \) and \( j, k \geq m \) is in \( \mathcal{A} \).
Now we claim that for any triangle types \((i, j, k)\) \((\ell = 1, 2)\) with \(i = m - 1, j \leq k \leq N, \) if we have a triangle of type \((i, j_1, k_1)\) in \(A\) if and only if we have one of type \((i, j_2, k_2)\). This follows as before from Claim 1, taking \(j_1 \leq j_2\) and applying induction on \(j_2 - j_1\).

If \(N < 3m\) then this concludes the analysis, so assume \(N \geq 3m\) In view of the first part of the lemma, it suffices now to find one triangle type \((i, j, k)\) with \(m \leq i \leq j \leq k \leq i + j, i + j + k = N\), represented in \(A\).

We begin with some triangle type \((i, j, k)\) represented in \(A\) with \(i + j + k = N, i = m - 1 < j \leq k \leq i + j, i + j + k = N\). Observe that \(k \geq (N - i)/2 > m\). We consider the triple \((m, j, k - 1)\). This satisfies the triangle inequality. We apply Lemma 8.2 with \(i' = m, j' = k - 1, k' = j\). It suffices to check that \(A\) contains triangles of types \((m - 1, k - 2, j)\) and \((m - 1, k, j)\). The first holds by our assumption on \(A\) and the second by our assumption on \((i, j, k)\). □

8.2. Triangles of even perimeter.

**Lemma 8.4.** Let \(A\) be an amalgamation class of diameter \(\delta\) determined by triangle constraints. Then a triangle of type \((i, j, k)\) with even perimeter \(P = i + j + k\) is in \(A\) if and only if \(P < C_0\).

**Proof.** Let \(C'_0\) be minimal even such that some triangle of perimeter \(C'_0\) does not occur in \(A\). Then \(C'_0 \leq C_0\) and by Lemma 8.1, \(C'_0 > 2\delta\).

If \(P < C'_0\) is even, then by definition all triangles of perimeter \(P\) are in \(A\).

Now suppose \(P = C'_0\), and let \(m = P - 2\delta \geq 2\). Every triangle of perimeter \(P\) has minimal length at least \(m\). Apply Lemma 8.3. Either all such triangles of perimeter \(P\) are in \(A\), or none are. As \(P = C'_0\) the conclusion is that none are, and the \(C'_0 = C_0\). Thus for \(P < C_0\) all such triangles are in \(A\), and for \(P = C_0\) none are.

It remains to treat the case \(P > C_0\). We proceed inductively. So suppose that no triangle of perimeter \(P - 2\) is in \(A\), but that some triangle of type \((i, j, k)\) with \(i + j + k = P\) even is in \(A\). Set \(\ell = \frac{(i + j - k)}{2} - 1\). Then \(1 \leq \ell < \delta\). Let \(i' = i + \ell - 1 = (i - j + k)/2\) and \(j' = j - \ell - 1 = (j - i + k)/2\). Then \(i', j' \geq 2\). As \(P \geq 2\delta + 4\) we also have \(i, j, k \geq 4\). In particular all lengths in the following amalgamation diagram lie in the range \([1, \delta]\).
Notice that the triangle \((c, a_2, u_1)\) is geodesic: \(i' + j' = k\). So the only nongeodesic triangle here is \((c, a_2, u_2)\) of type \((i, j, k)\), which we assume is in \(A\).

Therefore the amalgamation diagram may be completed in \(A\), and the vertices \(u_1, u_2\) force \(d(a_1, a_2) = j - 1\). The triangle \((a_1, a_2, c)\) then has type \((i - 1, j - 1, k)\) and perimeter \(P - 2\), a contradiction. Thus no triangle of even perimeter greater than \(C_0\) is found in \(A\). \(\square\)

**Corollary 8.5.** Let \(A\) be an amalgamation class of diameter \(\delta\) determined by triangle constraints. Suppose \(K_1 = \infty\). Then

\[
A = A_{\infty, 0; C_0, 2\delta + 1}^\delta
\]

with \(C_0 \geq 2\delta + 1\).

**8.3. Triangles of odd perimeter.**

**Lemma 8.6.** Let \(A\) be an amalgamation class of diameter \(\delta\). Assume that some triangle of odd perimeter occurs in \(A\), and let \(P\) be the least odd number which is the perimeter of a triangle in \(A\). Then the following hold.

1. A \(P\)-cycle embeds isometrically in \(\Gamma\).
2. \(\leq 2\delta + 1\).
3. \(P = 2K_1 + 1\)

**Proof.** We introduce some metric space terminology which differs noticeably from graph theoretic terminology. We will call any sequence of vertices \((a_0, \ldots, a_n)\) in \(\Gamma\) a path, and the length of a path is the sum of successive distances \(d(a_i, a_{i+1})\). A path is full if all successive distances are equal to 1. A path is a circuit if \(a_n = a_0\).

By assumption \(\Gamma\) contains a circuit of odd length. Let \(C\) be a circuit of minimal odd length \(P_0\). Then \(P_0 \leq P\). Any path extends to a full path of the same length, so we may suppose that \(C\) is a full path. Then

\[
C = (a_0, \ldots, a_{P_0})
\]
with $a_{P_0} = a_0$.

Apart from the relation $a_0 = a_{P_0}$ the vertices $a_i$ are distinct, by the minimality of $P_0$. We claim that the embedding of $C$ into $\Gamma$ is isometric. If not, we may label $C$ so that for some pair $(a_0, a_m)$ we have

$$d_\Gamma(a_0, a_m) < d_C(a_0, a_m)$$

In particular $m < P_0$. Consider the circuits

$$C' = (a_0, \ldots, a_m, a_0); \quad C'' = (a_0, a_m, \ldots, a_{P_0})$$

Let the lengths of $C'$ and $C''$ be $\ell'$ and $\ell''$ respectively.

By our choice of $a_m$, we have $\ell', \ell'' < P_0$. Furthermore $\ell' + \ell'' = P_0 + 2d(a_0, a_m)$ and thus one of $\ell', \ell''$ is odd. This contradicts the minimality of $P_0$. Thus the embedding of $C$ into $\Gamma$ is an isometry.

With $m = (P_0 - 1)/2$ it follows that the triangle $(a_0, a_m, a_{m+1})$ is of type $(1, m, m)$ and perimeter $P_0$. Thus $P_0 = P$. Our first point is proved.

Since $P = 2m + 1 \leq 2\delta + 1$ our second point follows. And since we have a triangle of type $(1, m, m)$ we have $K_1 \leq m, 2K_1 + 1 \leq P$, hence $2K_1 + 1 = P$ and $m = K_1$.

**Lemma 8.7.** Let $\mathcal{A}$ be an amalgamation class of diameter $\delta$ and associated parameters $K_1, K_2$. Then

$$\mathcal{A} \subseteq \mathcal{A}_{K_1, K_2}$$

**Proof.** We have to show that triangles of types $(i, j, k)$ with $P = i + j + k$ odd and satisfying either of the following constraints do not occur in $\mathcal{A}$.

$$P < 2K_1 + 1$$

$$P > 2K_2 + 2i$$

By Lemma 8.6 (3) we have $P \geq 2K_1 + 1$. So we turn to the second point. Suppose we have a triangle of type $(i, j, k)$ in $\mathcal{A}$ with $P = i + j + k$ odd and

$$P > 2K_2 + 2i$$

Suppose here that $i$ is minimized. If $i = 1$ then $j = k > K_2$ and this contradicts the definition of $K_2$. So $i > 1$.

Consider the following amalgamation.
This has a completion in $\mathcal{A}$. As $i \leq j \leq k$ and $P > 2K_2$ we have $k > K_2$ and hence $d(a,b) \neq k$. So $d(a,b) = k + \epsilon$ with $\epsilon = \pm 1$. Then $\mathcal{A}$ contains a triangle of type

$$(i - 1, j, k + \epsilon)$$

of odd perimeter, and minimal entry $i - 1$, and the perimeter is greater than $2K_2 + 2(i - 1)$. This contradicts the choice of $i$. \hfill \Box

**Lemma 8.8.** Let $\mathcal{A}$ be an amalgamation class of diameter $\delta$ determined by triangle constraints. If $K_1 \leq k \leq K_2$, then a triangle of type $(1,k,k)$ belongs to $\mathcal{A}$.

**Proof.** We may suppose that

$$K_1 < k < K_2$$

Consider the amalgamation

![Diagram](image)

The nongeodesic triangles occurring here have types $(1,K_1,K_1)$ and $(1,K_2,K_2)$. Hence the diagram has a completion in $\mathcal{A}$. Then $d(a_1,a_2) = k$ and the triangle $(a_1,a_2,c)$ has type $(1,k,k)$. \hfill \Box

**Lemma 8.9.** Let $\mathcal{A}$ be an amalgamation class of diameter $\delta$ determined by triangle constraints. Suppose that $(i,j,k)$ is the type of a triangle satisfying the following conditions. $p$

1. $K_1 \leq i \leq K_2$;
2. $P = i + j + k \leq C_0 - 2$;
3. If $C_0 = 2\delta + 2$ then $i < \delta$;
4. If $P$ is odd, then $\min(j,k) < \delta$.

Then there is a triangle of type $(i,j,k)$ in $\mathcal{A}$.

**Proof.** By Lemma 8.4 we may suppose that

$P$ is odd
Our assumptions then rule out the case $j = k = \delta$, so we may suppose $j < \delta$. If $j = 1$ then as $P$ is odd we have $i = k$ and a triangle of type $(i, j, k)$ belongs to $\mathcal{A}$ by the previous lemma. So we suppose $1 < j < \delta$

Consider the following amalgamation.

```
\begin{tikzpicture}
  \node (a1) at (0,0) {$a_1$};
  \node (a2) at (2,0) {$a_2$};
  \node (c) at (1,1) {$c$};
  \node (i) at (0.5,0.5) {$i$};
  \node (k) at (1.5,0.5) {$k$};
  \node (u1) at (0,-1) {$u_1$};
  \node (u2) at (2,-1) {$u_2$};
  \node (a) at (1,-0.5) {$a$};

  \path (a1) edge (a2)
        (a1) edge (c)
        (a2) edge (c)
        (c) edge (i)
        (c) edge (k)
        (i) edge (a)
        (k) edge (a)
        (u1) edge (a)
        (u2) edge (a)
        (u1) edge (u2)
        (u1) edge (i)
        (u2) edge (k);

  \node at (1,0.75) {$d(c, u_1) = d(c, u_2) = i$};
\end{tikzpicture}
```

The nongeodesic triangles involved in this diagram are labeled by the triples

$$(2, i, i), (1, i, i), (i, j - 1, k), \text{ and } (1, j + 1, k)$$

Now $2i + 2 < C_0$ by our hypothesis so a triangle of type $(2, i, i)$ belongs to $\mathcal{A}$. As $K_1 \leq i \leq K_2$, a triangle of type $(1, i, i)$ belongs to $\mathcal{A}$. We consider the remaining two triples

$$(i, j - 1, k), \text{ and } (1, j + 1, k)$$

These satisfy the triangle inequality since $(i, j, k)$ does and $P$ is odd. Thus they represent triangle types, and their perimeters are $P \pm 1 < C_0$, even. Thus triangles of these types occur in $\mathcal{A}$.

Therefore the diagram has a completion in $\mathcal{A}$, and then the triangle $(a_1, a_2, c)$ has type $(i, j, k)$. □

**Lemma 8.10.** Let $\mathcal{A}$ be an amalgamation class of diameter $\delta$ determined by triangle constraints, with associated parameters $K_1, K_2, C_0, C_1$. Then

$$C_1 \geq \min(2\delta + K_2, C_0 - 1)$$

**Proof.** If $C_0 = 2\delta + 2$ then $C_1 \geq C_0 - 1$ and there is nothing to show. So we suppose $C_0 > 2\delta + 2$.

Suppose $C_1 < 2\delta + K_2, C_0 - 1$. Let $K_2 \equiv \epsilon \mod 2$ with $\epsilon = 0$ or $1$. Let $K_2' = K_2 + 1 - \epsilon$. Then $K_2'$ is odd. Set

$$i = K_2, j = \frac{C - K_2'}{2}, k = j + (1 - \epsilon)$$
Then
\[ j < \frac{(2\delta + K_2 - K'_2)/2 \leq \delta}{i + j + k} = K_2 + (C_1 - K'_2) + (1 - \epsilon) = C_1 \]

We claim that \((i, j, k)\) is the type of a triangle. As \(|j - k| \leq 1\) we have
\[ j \leq i + k, k \leq i + j \]

Furthermore \(C_1 \geq 2\delta\) so
\[ i = K_2 < C_1 - K_2 = j + k \]

Thus \((i, j, k)\) is the type of a triangle of perimeter \(C_1\).

Now Lemma 8.9 applies since \(C_1 \leq C_0 - 2, C_0 > 2\delta + 2, \) and \(j < \delta\). Thus there is a triangle of this type in \(A\), contradicting the definition of \(C_1\). \(\Box\)

**Lemma 8.11.** Let \(A\) be an amalgamation class of diameter \(\delta\) determined by triangle constraints, with associated parameter \(C_1\). Then no triangle of odd perimeter \(P \geq C_1\) belongs to \(A\).

**Proof.** Suppose on the contrary that \(P \geq C_1\) is odd and that some triangle of type \((i, j, k)\) with \(i + j + k = P\) belongs to \(A\). Take \(i\) to be minimal. We have \(P > C_1\).

Fix a triangle \((a, b, c)\) in \(A\) with \(d(b, c) = i, d(a, b) = j, d(a, c) = k\). Take \(u\) on a geodesic from \(b\) to \(c\) with
\[ d(b, u) = i - 1, d(u, c) = 1 \]

If \(d(a, u) = k \pm 1\) then \((a, b, u)\) is a triangle of perimeter \(P\) or \(P - 2\) and type \((i - 1, j, k \pm 1)\), violating the minimality of \(i\). Thus
\[ d(a, u) = k \]

So \((a, u, c)\) is a triangle of type \((1, k, k)\). Thus \(k \leq K_2\), and \(P \leq 2\delta + K_2\).

Furthermore \((a, b, u)\) is a triangle of type \((i - 1, j, k)\) and perimeter \(P - 1\), so \(P - 1 < C_0\). Hence \(P \leq C_0 - 1\). Then \(P \leq \min(2\delta + K_2, C_0 - 1)\) and thus \(C_1 < \min(2\delta + K_2, C_0 - 1)\), contradicting the previous lemma. \(\Box\)

**Corollary 8.12.** Let \(A\) be an amalgamation class of diameter \(\delta\) determined by triangle constraints, with associated parameters \(K_1, K_2, C_0, C_1\). Then
\[ A \subseteq A^\delta_{K_1, K_2, C_0, C_1} \]

**Proof.** Lemmas 8.4, 8.7 and 8.11. \(\Box\)

**Lemma 8.13.** Let \(A\) be an amalgamation class of diameter \(\delta\) determined by triangle constraints, with associated parameters \(K_1, K_2\). Then for any triangle type \((i, j, k)\) with \(P = i + j + k\) odd, if
\[ 2K_1 + 1 \leq P \leq 2K_2 + 1 \]
then there is a triangle of type \((i, j, k)\) in \(A\).
Proof. For $P = 2K_1 + 1$ apply Lemma 8.6 (1). So we assume

$$P > 2K_1 + 1$$

and we proceed inductively.

Taking $N = P$ and $m = 2$, Lemma 8.3 implies that any triangle of type $(i', j', k')$ for which

$$i' + j' + k' = P, \ i' \geq 1, \text{ and } j', k' \geq 2$$

belongs to $A$, since our induction hypothesis applies to $P - 2$.

This applies in particular to the triple $(i, j, k)$, taking $i \leq j \leq k$, unless $j = 1$, in which case $k = 1$, $P = 3$, and $2K_1 + 1 < 3$, which is impossible. □

8.4. Identification of $A$.

Proposition 8.14. Let $A$ be an amalgamation class of diameter $\delta$ determined by triangle constraints with associated parameters $K_1, K_2, C_0, C_1$. Then

$$A = A_{K_1, K_2; C_0, C_1}^\delta$$

Proof. We have $A \subseteq A_{K_1, K_2; C_0, C_1}^\delta$ by Corollary 8.12. So we deal with the reverse inclusion.

Let $(i, j, k)$ be a triangle type represented in $A_{K_1, K_2; C_0, C_1}^\delta$. We will show that a triangle of this type belongs to $A$.

Let $P = i + j + k$. If $P$ is even then Lemma 8.4 gives the desired result. So we suppose $P$ is odd, and $i \leq j \leq k$

We have the following conditions.

1. $P \geq 2K_1 + 1$;
2. $P < 2K_2 + 2i$;
3. $P < C_1$.

If $P \leq 2K_2 + 1$ then a triangle of type $(i, j, k)$ belongs to $A$ by Lemma 8.13. So we will suppose

$$P > 2K_2 + 1$$

and proceed by induction on $P$ (odd).

Set

$$m = (P - 2K_2 + 1)/2$$

The condition $P < 2K_2 + 2i$ may be expressed as: $i \geq m$. Thus our claim may be expressed as follows.

If $i + j + k = P$, $\min(i, j, k) \geq m$, then the type $(i, j, k)$ is represented in $A$.

The corresponding claim for $P - 2$ involves $m - 1$, and holds by our inductive hypothesis. By Lemma 8.3, if there is some triangle in $A$ whose type $(i, j, k)$ satisfies the conditions $i + j + k = P$, $\min(i, j, k) \geq m$, then all such types are represented by triangles in $A$. Furthermore, since $A \subseteq A_{K_1, K_2}^\delta$, it suffices to find a triangle of type $(i, j, k)$ in $A$ with $i + j + k = P$, as the other condition then necessarily holds.
If $P > 2\delta$ then since $P < C_1$ there is such a triangle by the definition of $C_1$. So suppose

$$P < 2\delta$$

Consider the triple

$$i = K_2, j = \left\lfloor \frac{P - K_2}{2} \right\rfloor, k = P - (i + j)$$

Then $i \leq j \leq k \leq j + 1$. Thus $(i, j, k)$ is the type of a triangle with $i + j + k = P$. Furthermore

$$j \leq \frac{P - K_2}{2} < \frac{2\delta - K_2}{2} < \delta$$

and thus also $k \leq \delta$.

Apply Lemma 8.9. Since we have $i = K_2$, $P < C_0 - 2$, and $j < \delta$, a triangle of type $(i, j, k)$ is in $A$. This concludes the proof. \qed

This proves a major portion of Theorem 9, leaving open however the question of admissibility of the parameters $K_1, K_2, C_0, C_1$ (or equivalently $K_1, K_2, C, C'$).

8.5. Admissibility: a lower bound for $C$.

Lemma 8.15. Let $A$ be an amalgamation class of diameter $\delta$ determined by triangle constraints. Suppose we have

$$1 < i < k, i + k \leq \delta + 1$$

and triangles of type $(i, k - i + 1, k)$ and $(i, i + k - 1, k)$ are in $A$. Then a triangle of type $(1, k, k)$ is in $A$.

Proof. Consider the following amalgamation.

The nongeodesic triangles involved in the diagram have types

$(i, k - i + 1, k - 1)$, $(i, k - i + 1, k)$, and $(i, i + k - 1, k)$

Of these, all but the first have been assumed to be in $A$. 

For the type \((i, k - i + 1, k - 1)\), one first checks the triangle inequality, which is immediate. The perimeter is \(2k \leq 2\delta\), so this type is represented in \(A\) by Lemma 8.1.

Therefore the diagram has a completion in \(A\) and in this completion the triangle \((a_1, a_2, c)\) has type \((1, k, k)\).  

\[\square\]

**Lemma 8.16.** Let \(A\) be an amalgamation class of diameter \(\delta\) determined by triangle constraints with associated parameters \(K_1, K_2, C_0, C_1\). Suppose that \(i, k\) satisfy
\[
1 < i < k; \\
i + k \leq \min(\delta + 1, (C_0 - 2)/2).
\]
Then \(k \leq K_2\).

**Proof.** By Lemma 8.15 it suffices to check that \(A\) contains triangles of the types
\[
(i, k - i + 1, k) \text{ and } (i, i + k - 1, k).
\]
It is clear that these triples satisfy the triangle inequality.

We apply Lemma 8.9. We have \(i, k < \delta\). It suffices to check that the perimeters are at most \(C_0 - 2\). The two perimeters in question are
\[
2k + 1 \leq 2(i + k) - 1
\]
We have \(2(i + k) - 1 < C_0 - 2\) by hypothesis. This completes the proof.  

\[\square\]

**Lemma 8.17.** Let \(A\) be an amalgamation class of diameter \(\delta\) determined by triangle constraints with associated parameters \(K_1, K_2\). Then one of the following holds.
\[
(1) \ K_1 + K_2 \geq \delta + 1; \\
(2) \ K_1 = 1, \ K_2 = \delta - 1; \\
(3) \ K_1 + K_2 = \delta \text{ and } C_0 = 2\delta + 2.
\]

**Proof.** Suppose first that
\[
K_1 = 1
\]
Then by Lemma 8.9 a triangle of type \((1, \delta - 1, \delta - 1)\) belongs to \(A\), and so \(K_2 \geq \delta - 1\). This leads to Case 1 or 2.

Now suppose that
\[
K_1 > 1
\]
If \(K_1 > \delta/2\) then Case 1 applies, so suppose
\[
K_1 \leq \delta/2
\]
Set \(i = K_1, \ k = \delta + 1 - i\). If \(C_0 > 2\delta + 2\) then by Lemma 8.16 we have \(k \leq K_2\) and \(K_1 + K_2 \geq \delta + 1\).

If \(C_0 = 2\delta + 2\) then we arrive similarly at \(K_1 + K_2 \geq \delta\), and thus Case 1 or 2. If \(K_1 = \delta/2\) this is clear, and if \(K_1 < \delta/2\) we apply Lemma 8.16 with \(k = \delta - i\).  

\[\square\]
The major case division among admissible sets of parameters is between the case $C \leq 2\delta + K_1$ and $C > 2\delta + K_1$. The following is the first step in this direction.

**Lemma 8.18.** Let $\mathcal{A}$ be an amalgamation class of diameter $\delta$ determined by triangle constraints with associated parameters $K_1, K_2, C, C'$. Then

$$C > \min(2\delta + K_1, 2K_1 + 2K_2)$$

**Proof.** We suppose

$$C \leq 2\delta + K_1$$

We will show that

$$C > 2K_1 + 2K_2$$

Set $j = \lfloor \frac{C-K_1}{2} \rfloor$, and $i = (C - K_1) - j$. Then $1 < j \leq i \leq \delta$. Consider the following amalgamation.

The nongeodesic triangles occurring here are of the types

$(2, K_1 - 1, K_1)$, $(2, i - 1, i - 1)$, $(i - 1, j, K_1 - 1)$, and $(i - 1, j, K_1)$

A triangle of type $(2, K_1 - 1, K_1)$ has perimeter $2K_1 + 1$ and belongs to $\mathcal{A}$ by Lemma 8.6. A triangle of type $(2, i - 1, i - 1)$ belongs to $\mathcal{A}$ by Lemma 8.1. If there are also triangles of the other two types

$(i - 1, j, K_1 - 1)$ and $(i - 1, j, K_1)$

then the amalgamation diagram has a completion in $\mathcal{A}$. The vertices $u_1, u_2$ then force $d(a_1, a_2) = K_1$ and thus $\mathcal{A}$ contains a triangle of type $(K_1, i, j)$ and perimeter $C$, a contradiction.

So a triangle of one of the types $(i - 1, j, K_1 - 1)$ or $(i - 1, j, K_1)$ must be forbidden. We should however check the triangle inequality to ensure that these are in fact triangle types.

As $j \leq i \leq j + 1$ the only inequality that needs to be checked is

$$K_1 \leq i - 1 + j$$
But \( i - 1 + j = C - K_1 \) and \( C > 2\delta \), so
\[
K_1 < C - K_1 = i - 1 + j
\]

The perimeters here are \( C - 2 \) and \( C - 1 \) respectively. One of these is even and the corresponding triangle is in \( \mathcal{A} \) by Lemma 8.1. So the one of odd perimeter must be forbidden.

Now we claim
\[
C - 2 \geq 2K_1
\]

We have \( C - 2 \geq 2\delta - 1 \), so this can fail only if
\[
K_1 = \delta, C = 2\delta + 1
\]

But if \( K_1 = \delta \) then \( \mathcal{A} \) contains a triangle of perimeter \( 2\delta + 1 \) and thus \( C > 2\delta + 1 \). Since \( C - 2 \geq 2K_1 \) then writing \( P = C - 2 \) or \( C - 1 \) with \( P \) odd, we have \( P \geq 2K_1 + 1 \). So the forbidden triangle of type \((i - 1, j, K_1 - \epsilon)\) and odd perimeter \( P \) must satisfy
\[
P > 2K_2 + 2\min(i - 1, j, K_1 - \epsilon)
\]

Next we show
\[
C - 1 \leq 2K_2 + 2(i - 1)
\]

We have
\[
j \leq (C - K_1)/2, i \geq (C - K_1)/2
\]

and thus
\[
2K_2 + 2(i - 1) \geq 2K_2 + (C - K_1) - 2 = C + (2K_2 - K_1 - 1) \geq C + K_2 - 2 \geq C - 1
\]

As \( i - 1 \leq j \) we find
\[
P \leq 2K_2 + 2\min(i - 1, j)
\]

and thus we are left with
\[
P > 2K_2 + 2(K_1 - \epsilon)
\]

If \( \epsilon = 0 \) then \( C > P > 2K_2 + 2K_1 \). If \( \epsilon = 1 \) then \( C \geq P + 2 > 2K_2 + 2K_1 \). So we have the required inequality in any case. \( \square \)

8.6. Admissibility: small \( C \).

Lemma 8.19. Let \( \mathcal{A} \) be an amalgamation class of diameter \( \delta \) determined by triangle constraints with associated parameters \( K_1, K_2, C, C' \). Suppose
\[
K_1 < \infty, C_0 \leq 2\delta + K_1
\]

Then \( C_0 = 2K_1 + 2K_2 + 2 \) and \( C_1 \geq C_0 - 1 \).
Proof. We have $C \leq C_0 \leq 2\delta + K_1$, so by the previous lemma we find

$$C > 2K_1 + 2K_2$$

In other words,

$$C_0 \geq 2K_1 + 2K_2 + 2, C_1 \geq 2K_1 + 2K_2 + 1$$

Set $i = K_1$, $j = \lfloor \frac{C_0 - K_1 - 2}{2} \rfloor$, $k = (C_0 - K_1 - 3) - j$. Then $j \geq k \leq j + 1$. We claim

$$i \leq j \leq k < \delta$$

We have

$$j \geq (C_0 - K_1 - 4)/2 \geq (K_1 + 2K_2 - 2)/2$$

As $C_0 \leq 2\delta + K_1$ we find $K_1 \geq 2$ and thus $j \geq K_2 \geq i$. Also

$$k \leq j + 1 \leq (C_0 - K_1 - 2)/2 \leq \delta - 1$$

Since $i \leq j \leq k \leq j + 1$ the triple $(i, j, k)$ satisfies the triangle inequality. The perimeter $P = i + j + k$ is $C_0 - 3$. Lemma 8.9 applies, and there is a triangle of type $(i, j, k)$ in $A$.

As $C_0 - 3$ is odd, and $A \subseteq A_{K_1, K_2}$, we find

$$C_0 - 3 < 2K_2 + 2i = 2K_1 + 2K_2$$

Thus

$$C_0 = 2K_1 + 2K_2 + 2$$

Hence $C_1 \geq C_0 - 1$ as well. \qed

In order to control $C_1$ properly in the context of the preceding lemma we insert the following.

Lemma 8.20. Let $A$ be an amalgamation class of diameter $\delta$ determined by triangle constraints with associated parameters $K_1, K_2, C, C'$. Suppose

$$C \geq 2K_1 + K_2$$

and $C$ is even.

Then

$$C > \min(4K_2, 4\delta - 2K_2 - 2)$$

Proof. Suppose on the contrary that

$$C \leq \min(4K_2, 4\delta - 2K_2 - 2)$$

If $K_1 = 1$ then $2K_2 \geq \delta - 1$ by Lemma 8.17. In this case we have in particular $C \leq 4\delta - 2(\delta - 1) - 2 \leq 2\delta$ which is impossible. So we will suppose

$$K_1 > 1$$

Set $k = \frac{C - 2K_1}{2}, i = \lfloor \frac{C - k}{2} \rfloor$, and $j = (C - k) - i$. Then by our assumptions we have

$$K_1 \leq k \leq K_2$$

We claim also that

$$1 < k < i \leq j \leq \delta$$
The inequality \( k < i \) can be written as

\[
2k \leq C - k - 2
\]

which reduces to

\[
C \leq 6K_2 - 4
\]

This holds since \( C \leq 4K_2 \) and \( K_2 \geq 2 \). Furthermore

\[
2i \leq C - k = (C + 2K_2)/2 \leq (4\delta - 2)/2 = 2\delta - 1
\]

and thus

\[
i \leq \delta - 1, j \leq i + 1 \leq \delta
\]

Now consider the following amalgamation.

The nongeodesic triangles involved here have types

\((2, k - 1, k), (2, i - 1, i - 1), (i - 1, j, k - 1), \) and \((i - 1, j, k)\)

These all satisfy the triangle inequality so they are indeed triangle types.

As \( K_1 \leq k \leq K_2 \) and \( k < \delta \), by Lemma 8.9 a triangle of type \((2, k - 1, k)\) belongs to \( A \). As \( 2i \leq 2\delta \), by Lemma 8.1 a triangle of type \((2, i - 1, i - 1)\) belongs to \( A \). The third triangle has even perimeter \( C - 2 \), so belongs to \( A \) by Lemma 8.4. We claim that there is also a triangle of type

\((i - 1, j, k)\)

in \( A \).

As the perimeter of this triangle is \( C - 1 \) it suffices to check that this triangle belongs to \( A^b_{K_1,K_2} \). This comes down to the inequality

\[
C - 1 \leq 2K_2 + 2k
\]

which corresponds to the definition of \( k \).

Therefore the amalgamation diagram can be completed in \( A \). The distance \( d(a_1, a_2) \) must be \( k - 1 \) or \( k \), and as \( i + j + k = C \) we find

\[
d(a_1, a_2) = k - 1
\]
So the triangle \((a_1, a_2, c)\) has type \((i, j, k - 1)\). As \(i + j + (k - 1) = C - 1\) is odd, we must then have the inequality
\[
C - 1 < 2K_2 + 2(k - 1) = C - 2
\]
and so we arrive at a contradiction. \(\square\)

**Lemma 8.21.** Let \(A\) be an amalgamation class of diameter \(\delta\) determined by triangle constraints with associated parameters \(K_1, K_2, C, C'\). Suppose that \(C_0 \leq 2\delta + K_1 + 1\) and \(K_1 < K_2\).

Then \(C_1 = 2K_1 + 2K_2 + 1\) and \(C_0 = 2K_1 + 2K_2 + 2\).

**Proof.** By Lemma 8.19 we have
\[
C_0 = 2K_1 + 2K_2 + 2
\]
and \(C_1 \geq C_0 - 1\). We claim \(C_1 = C_0 - 1\).

Suppose \(C_1 > C_0\). Then \(C = C_0\) is even, and by the previous lemma we have
\[
2K_1 + 2K_2 + 2 > \min(4K_2, 4\delta - 2K_2 - 2)
\]
As \(K_2 > K_1\) we have
\[
2K_1 + 2K_2 + 2 \leq 4K_2
\]
so our inequality becomes
\[
2K_1 + 2K_2 + 2 \geq 4\delta - K_2
\]
\[
K_1 + 2K_2 \geq 2\delta - 1
\]
\[
C_0 = (K_1 + 2K_2) + (K_1 + 1) \geq 2\delta + K_1 + 1
\]
contradicting our initial hypothesis. \(\square\)

**Lemma 8.22.** Let \(A\) be an amalgamation class of diameter \(\delta\) determined by triangle constraints with associated parameters \(K_1, K_2, C, C'\). Suppose that \(K_2\) is odd and that \(3K_2 \leq 2\delta - 1\).

Then \(C_1 \leq 4K_2 + 1\).

**Proof.** Our assumptions imply
\[
K_2 \geq 3, \delta \geq 5, K_2 \leq \delta - 2, \text{ and } K_1 > 1
\]
where the last follows by Lemma 8.17. Assume toward a contradiction that \(C_1 > 4K_2 + 1\).

If \(C_0 \leq 4K_2\) then \(C = C_0\) and by Lemma 8.18 we find
\[
C_0 \geq \min(2\delta + K_1, 2K_1 + 2K_2)
\]
while \(2\delta + K_1 \geq (3K_2 + 1) + K_1 \geq 2K_1 + 2K_2\), so in fact
\[
C_0 > 2K_1 + 2K_2
\]
Then by Lemma 8.20 we find
\[ C_0 > \min(4K_2, 4\delta - 2K_2 - 2) = 4K_2 \]
and we have a contradiction. Thus in fact
\[ C_0 > 4K_2 \]
Now let \( i = (3K_2 + 1)/2 \). Then \( 1 < i < \delta \). Consider the amalgamation

\[ d(c, u_1) = i - 1, \quad d(c, u_2) = i - 3 \]

Note that \( K_2 + 2 \leq K_2 + (K_1 + 1)/2 = (3K_2 + 1)/2 \leq \delta \). Thus all lengths in this diagram are in the range \([1, \delta]\). There are four nongeodesic triangles involved in this diagram. Two of them have even perimeter:

- \((4, i - 3, i - 1) : \quad P = 3K_2 + 1 \leq 2\delta\)
- \((K_2 + 2, i - 3, i) : \quad P = 4K_2 < C_0\)

Thus these types are represented in \( \mathcal{A} \).

The other two nongeodesic triangles involved have odd perimeter:

- \((4, K_2 + 1, K_2 + 2) : \quad P = 2K_2 + 7 < C_1\)
- \((K_2 + 1, i - 1, i) : \quad P = 4K_2 + 1 < C_1\)

In order to see that triangles of these types are in \( \mathcal{A} \) we must show that they are in \( \mathcal{A}_{K_1,K_2}^\delta \).

Both perimeters are at least \( 2K_2 + 1 \), so it suffices to check the inequalities corresponding to the bound \( K_2 \):

\[ 2K_2 + 7 < 2K_2 + 2 \cdot 4 \]
\[ 4K_2 + 1 < 2K_2 + 2(K_1 + 1) \]

As these are satisfied, our amalgamation diagram has a completion in \( \mathcal{A} \). In this completion, the vertex \( u_1 \) forces
\[ d(a_1, a_2) = K_2 \text{ or } K_2 + 2 \]

Therefore \( \mathcal{A} \) must contain a triangle of one of the following types:

- \((K_2, i, i), (K_2 + 2, K_2 + 2, 3)\)
These have odd perimeter, respectively \(4K_2 + 1\) and \(2K_2 + 7\), so we must then have one of the corresponding inequalities
\[
4K_2 + 1 < 2K_2 + 2 \cdot K_2 \\
2K_2 + 7 < 2K_2 + 2 \cdot 3
\]
and as both fail, we reach a contradiction. \(\square\)

**Lemma 8.23.** Let \(A\) be an amalgamation class of diameter \(\delta\) determined by triangle constraints with associated parameters \(K_1, K_2, C, C'\). Suppose that \(K_2\) is even and
\[
3K_2 \leq 2\delta - 1, \quad C_0 = 4K_2 + 2
\]
Then \(C_1 = 4K_2 + 1\).

**Proof.** We have \(C_1 \geq 4K_2 + 1\) by Lemma 8.10. Suppose \(C_1 > 4K_2 + 1\)
Let \(i = (3/2)K_2 + 1\). Consider the amalgamation

The nongeodesic triangles occurring in this diagram have types
\((2, K_2 - 1, K_2), (K_2, i - 2, i - 1), (K_2 + 1, i - 2, i),\) and \((K_2 - 1, i - 1, i)\)
These have perimeters respectively
\(2K_2 + 1, 4K_2 - 1, 4K_2 + 1,\) and \(4K_2\)
As \(4K_2 < C_0\) the last is realized.
For the other three we apply Proposition 8.14. As the perimeters are odd, at least \(2K_1 + 1\), and less than \(C_1\), it suffices to check the inequalities corresponding to \(K_2\), namely
\[
2K_2 + 1 < 2K_2 + 2 \cdot 2 \\
4K_2 - 1 < 2K_2 + 2 \cdot K_2 \\
4K_2 + 1 < 2K_2 + 2 \cdot (K_2 + 1)
\]
Since these all hold it follows that the amalgamation diagram can be completed in $A$.

The possible values of $d(a_1, a_2)$ are $K_2 - 1$ and $K_2$. So the triangle $(a_1, a_2, c)$ has type

$$(i, i, K_2 - \epsilon)$$

with $\epsilon = 0$ or $1$, and perimeter $4K_2 + 2 - \epsilon$. If $\epsilon = 0$ then this is $C_0$, which is impossible. So $d(a_1, a_2) = K_2 - 1$ and the triangle $(a_1, a_2, c)$ has odd perimeter $4K_2 + 1$, and must therefore satisfy the inequality

$$4K_2 + 1 \leq 2K_2 + 2(K_1 - 1)$$

But this fails, so we have a contradiction. □

Lemma 8.24. Let $A$ be an amalgamation class of diameter $\delta$ determined by triangle constraints with associated parameters $K_1, K_2, C, C'$. Suppose that $K_1 < \infty$ and

$$C_0 \leq 2\delta + K_1$$

Then $C_1 = 2K_1 + 2K_2 + 1$ and $C_0 = 2K_1 + 2K_2 + 2$.

Proof. If $K_1 < K_2$ then Lemma 8.21 applies. So suppose that $K_1 = K_2$.

By Lemma 8.19 we have

$$C_0 = 4K_2 + 2$$

$$C_1 \geq 4K_2 + 1$$

Since $C_0 = 4K_2 + 2 \leq 2\delta + K_1 = 2\delta + K_2$ we have

$$3K_2 \leq 2\delta - 2$$

and one of the previous two lemmas applies. □

Lemma 8.25. Let $A$ be an amalgamation class of diameter $\delta$ determined by triangle constraints with associated parameters $K_1, K_2, C, C'$. Then one of the following holds.

(1) $K_1 + K_2 \geq \delta + 1$;
(2) $K_1 = 1$, $K_2 = \delta - 1$;
(3) $K_1 + K_2 = \delta$, $C_1 = 2\delta + 1$, and $C_0 = 2\delta + 2$.

Proof. This differs from Lemma 8.17 only in Case 3, where we specify the value of $C_1$. So we may suppose

$$K_1 + K_2 = \delta, C_0 = 2\delta + 2, K_1 > 1$$

Then Lemma 8.24 determines $C_1$. □

As a slight variation we have the following.

Lemma 8.26. Let $A$ be an amalgamation class of diameter $\delta$ determined by triangle constraints with associated parameters $K_1, K_2, C, C'$. Suppose that $K_1 < \infty$ and $C_1 = 2\delta + 1$. Then

$$K_1 + K_2 = \delta$$

and $C_0 = 2\delta + 2$.
Proof. By Lemma 8.10 we have

\[ C_1 \geq \min(2\delta + K_2, C_0 - 1) \]

As \( C_1 < 2\delta + K_2 \) it follows that \( C_1 \geq C_0 - 1 \) and thus \( C_0 = 2\delta + 2 \). We still need to check that \( K_1 + K_2 = \delta \).

As \( C_1 = 2\delta + 1 \) we have \( K_2 < \delta \). By Lemma 8.25 if \( K_1 = 1 \) we have our result. So suppose

\[ K_1 \geq 2 \]

By Lemma 8.24 we have

\[ C_1 = 2K_1 + 2K_2 + 1 \]

and thus \( K_1 + K_2 = \delta \). \( \square \)

**Lemma 8.27.** Let \( A \) be an amalgamation class of diameter \( \delta \) determined by triangle constraints with associated parameters \( K_1, K_2, C, C' \). Suppose that

\[ C_0 > 2\delta + K_1 \]

Then

\[ K_1 + 2K_2 \geq 2\delta - 1 \]

Proof. We may suppose that \( K_2 < \delta \).

Let \( K_1 \cong \epsilon \mod 2 \) with \( \epsilon = 0 \) or \( 1 \). Consider the following amalgamation.

![Diagram](image)

Of course the triangle \((a, u, v)\) belongs to \( A \). As the perimeter \( P \) of the triangle \((b, u, v)\) is \( K_1 + 2\delta - \epsilon \), it is even. As \( P < C_0 \), the triangle \((b, u, v)\) also belongs to \( A \). Thus this diagram can be completed in \( A \).

As \( K_2 < \delta \), in the completed diagram we must have \( d(a, b) = \delta - 1 \), so that the triangle \((a, b, u)\) has type

\[ (K_1, \delta - 1, \delta - \epsilon) \]

and in particular, has odd perimeter.

Since this triangle is in \( A \) we must have the inequality

\[ (\delta - 1) + (\delta - \epsilon) \leq 2K_2 + K_1 \]

If \( \epsilon = 0 \) this is our claim.

If \( \epsilon = 1 \) then \( K_1 \) is odd and as \( 2K_2 + K_1 \geq 2\delta - 2 \) in this case, our claim again follows. \( \square \)
Proposition 8.28. Let $\mathcal{A}$ be an amalgamation class of diameter $\delta$ determined by triangle constraints with associated parameters $K_1, K_2, C, C'$. Suppose that

\[ K_1 < \infty \quad C \leq 2\delta + K_1 \]

Then one of the following holds.

1. $C' = C + 1$ and:
   \[ C = 2K_1 + 2K_2 + 1 \geq 2\delta + 1 \text{ and } K_1 + 2K_2 \leq 2\delta - 1 \]

2. $C' > C + 1$ and:
   \[ K_1 = K_2 \text{ and } C = 4K_2 + 1 = 2\delta + K_2 \]

Proof. If $C = 2\delta + 1$, then by Lemma 8.26 we have $K_1 + K_2 = \delta$ and $C_0 = 2\delta + 2$, so $C' = C + 1$ and $c = 2K_1 + 2K_2 + 1$. Also as $C = 2\delta + 1$ we have $K_2 < \delta$ and thus

\[ K_1 + 2K_2 = K_2 + \delta \leq 2\delta - 1 \]

So we arrive at Case 1.

Suppose therefore that

\[ C > 2\delta + 1 \]

If $C_0 \leq 2\delta + K_1$ then by Lemma 8.24 we have

\[ C = 2K_1 + 2K_2 + 1 \text{ and } C' = C + 1 \]

In particular, the condition $C_0 \leq 2\delta + K_1$ becomes

\[ K_1 + 2K_2 + 1 \leq 2\delta \]

and so we have Case 1. So we may suppose

\[ C_0 > 2\delta + K_1 \]

Then our initial hypothesis gives

\[ C_1 \leq 2\delta + K_1 \]

Suppose $C_1 = C_0 - 1$. Then we have

\[ C_1 = 2\delta + K_1, C_0 = 2\delta + K_1 + 1 \]

By Lemma 8.18 we have

\[ 2\delta + K_1 > 2K_1 + 2K_2 \]

\[ K_1 + 2K_2 \leq 2\delta - 1 \]

Then by Lemma 8.27 we have

\[ K_1 + 2K_2 = 2\delta - 1 \]

Thus $C_1 = 2K_1 + 2K_2 + 1$ and we have Case 1.

So we may suppose now that

\[ C_0 > C_1 + 1 \]
Then by Lemma 8.10 we have
\[ C_1 \geq 2\delta + K_2 \]
Hence \( C_1 = 2\delta + K_2 \) and \( K_1 = K_2 \). Again by Lemma 8.18 we have
\[
\begin{align*}
2\delta + K_2 &> 4K_2 \\
3K_2 &\leq 2\delta - 1
\end{align*}
\]
and by Lemma 8.27 we find
\[ 3K_2 = 2\delta - 1 \]
arriving at Case 2. □

8.7. **Admissibility: Large \( C \).**

**Lemma 8.29.** Let \( \mathcal{A} \) be an amalgamation class of diameter \( \delta \) determined by triangle constraints with associated parameters \( K_1, K_2, C, C' \). Suppose that
\[ C > 2\delta + K_1 \]
Then \( 3K_2 \geq 2\delta \).

*Proof.* By Lemma 8.27 we have
\[ K_1 + 2K_2 \geq 2\delta - 1 \]
If \( 3K_2 < 2\delta \) we find \( K_1 = K_2 \) and \( 3K_2 = 2\delta - 1 \). Then by Lemma 8.22 we have
\[ C_1 \leq 4K_2 + 1 = 2\delta + K_2 = 2\delta + K_1 \]
contradicting our hypothesis. □

**Lemma 8.30.** Let \( \mathcal{A} \) be an amalgamation class of diameter \( \delta \) determined by triangle constraints with associated parameters \( K_1, K_2, C, C' \). Suppose that \( C_1 > C_0 > 2\delta + K_1 \) and \( 3K_2 = 2\delta \). Then
\[ C_0 > 2\delta + K_2 \]

*Proof.* Suppose on the contrary that \( C_0 \leq 2\delta + K_2 \). Then \( K_1 < K_2 \). By Lemma 8.27 we find
\[ 3K_2 \geq 2K_2 + (K_1 + 1) \geq 2\delta \]
As \( 3K_2 = 2\delta \) we conclude
\[ K_2 = K_1 + 1, C_0 = 2\delta + K_2 \]
Consider the following amalgamation.
The nongeodesic triangles involved in this diagram have the types
\((1, K_2, K_2), (K_2, \delta - 1, \delta - 1), (K_2 - 1, \delta - 1, \delta),\) and \((K_2, \delta - 1, \delta)\)
The first of these is in \(\mathcal{A}\) by definition. The second and third have perimeter equal to \(C_0 - 2\) and are in \(\mathcal{A}\) by Lemma 8.4. So consider the last type:
\((K_2, \delta - 1, \delta)\)
of perimeter \(P = C_0 - 1 < C_1\). It suffices to check that triangles of this type are in \(\mathcal{A}_{K_1,K_2}^\delta\).
As \(P \geq 2K_1 + 1\) it suffices to check the inequality
\[C_0 - 1 \leq 2K_2 + 2 \cdot K_2 = 2\delta + K_2\]
which holds.

The amalgamation diagram can be completed in \(\mathcal{A}\) with \(d(a_1,a_2) = K_2 - 1\)
or \(K_2\). Thus the triangle \((a_1,a_2,c)\) has type \((K_2 - 1, \delta, \delta)\) or \((K_2, \delta, \delta)\), and in the second case the perimeter is \(C_0\), which is impossible. So \(\mathcal{A}\) must contain a triangle of type \((K_2 - 1, \delta, \delta)\) and odd perimeter \(C_0 - 1\), yielding the inequality
\[C_0 - 1 \leq 2K_2 + 2(K_2 - 1) = 2\delta + K_2 - 2\]
which is a contradiction. \(\square\)

**Lemma 8.31.** Let \(\mathcal{A}\) be an amalgamation class of diameter \(\delta\) determined by triangle constraints with associated parameters \(K_1, K_2, C, C'\). If \(K_2 \geq \delta - 1\)
then either \(C' = C + 1\) or \(C = 3\delta - 1\).

**Proof.** If \(C \geq 3\delta - 1\) then our claim follows easily. So suppose
\[C < 3\delta - 1\text{ and }C' > C + 1\]

If \(C\) is odd then by Lemma 8.10 we find
\[C \geq 2\delta + K_2 \geq 3\delta - 1\]
a contradiction. So
\[ C = C_0 \] is even

Now let \((i, j, k)\) be the type of a triangle in \(\mathcal{A}\) with perimeter
\[ P = i + j + k = C + 1 \]

and with \(i \leq j \leq k\). Then
\[ i < \delta \text{ and } k \geq 3 \]

If \(C \leq 2\delta + K_1\) then by Proposition 8.28 we have
\[ C = 2\delta + K_2 \geq 3\delta - 1 \]
a contradiction. Hence
\[ C > 2\delta + K_1 \]

Therefore
\[ i > K_1 \]

Now consider the following amalgamation.

![Diagram](image)

The nongeodesic triangles involved in this diagram have types
\[(2, i - 1, i), (1, i, i), (i - 1, j, k - 2), \text{ and } (i, j, k).\]

The last of these is in \(\mathcal{A}\) by hypothesis. As \(K_1 \leq i \leq K_2\) there is a triangle of type \((1, i, i)\) in \(\mathcal{A}\). Lemma 8.9 shows that a triangle of type \((2, i - 1, i)\) is in \(\mathcal{A}\). We claim that there is also a triangle of type
\[(i - 1, j, k - 2)\]
in \(\mathcal{A}\).

First we check the triangle inequality for this triple. As \(i \leq j \leq k \leq i + j\)
this comes down to the inequality
\[ j \leq (i - 1) + (k - 2) \]
As \( i + j + k = C_0 + 1 \geq 2\delta + 3 \) we have \( i \geq 3 \) and the inequality follows. Now the perimeter \( P = (i - 1) + j + (k - 2) \) is \( C_0 - 2 \), so there is a triangle of this type in \( A \) by Lemma 8.4.

The amalgamation diagram has a completion in \( A \). Then \( d(a_1, a_2) = k - 1 \), so the triangle \((a_1, a_2, c)\) has type \((i, j, k - 1)\) and perimeter \( C \), a contradiction.

\textbf{Lemma 8.32.} Let \( A \) be an amalgamation class of diameter \( \delta \) determined by triangle constraints with associated parameters \( K_1, K_2, C, C' \). If 
\[ 2K_2 + 2 < C < \delta + 2K_2 \]

then \( C' = C + 1 \).

\textit{Proof.} Let \( k = C - 2K_2 \). Then
\[ 3 \leq k < \delta \]

Consider the following amalgamation.

\[ d(c, u_1) = d(c, u_2) = K_2 \]

If this diagram has a completion in \( A \) then we have \( d(a_1, a_2) = k \) and the triangle \((a_1, a_2, c)\) has type \((k, K_2, K_2)\) and perimeter \( C \), a contradiction. So one of the nongeodesic triangles involved must be a forbidden triangle.

The types in question are
\[ (2, K_2, K_2), (1, K_2, K_2), (k \pm 1, K_2, K_2) \]

As \( 2K_2 + 2 < C \) is even, there is a triangle of the first type in \( A \), and by definition there is one of the second type. So one of the triangle types
\[ (k \pm 1, K_2, K_2) \]

must be forbidden.

Let us check that these are in fact triangle types. The triangle inequality takes the form
\[ k + 1 \leq 2K_2 \]
\[ C \leq 4K_2 - 1 \]
Now \( 2K_2 \geq K_1 + K_2 \geq \delta \) and \( C < \delta + 2K_2 \leq 4K_2 \) so this holds.

These triangles have perimeter \( C \pm 1 \). Now suppose toward a contradiction that \( C' > C + 1 \). If \( C \) is odd then both triangle types are realized, so suppose that \( C \) is even. We claim that these triangle types are in \( A \) also in this case. We require the following inequalities.

\[
\begin{align*}
C - 1 & \geq 2K_1 + 1 \\
C + 1 & \leq 2K_2 + 2 \cdot K_2 \\
C \pm 1 & < 2K_2 + 2(k \pm 1)
\end{align*}
\]

Now by assumption

\[
C - 1 > 2K_2 + 1 \geq 2K_1 + 1
\]

and we showed above that \( C \leq 4K_2 - 1 \). Furthermore

\[
C \pm 1 = 2K_2 + k \pm 1 < 2K_2 + 2(k \pm 1)
\]

since \( k > 1 \). So all of these triangle types are in \( A \) and we arrive at a contradiction.

Thus \( C' = C + 1 \). \( \square \)

**Lemma 8.33.** Let \( A \) be an amalgamation class of diameter \( \delta \) determined by triangle constraints with associated parameters \( K_1, K_2, C, C' \). If \( K_2 = \delta \) and \( C > 2\delta + K_1 \) then \( C' = C + 1 \).

**Proof.** If \( C > 2\delta + 2 \) then by Lemma 8.32 either \( C \geq 3\delta \) or \( C' = C + 1 \). If \( C \geq 3\delta \) then in any case \( C' = C + 1 \). So we may suppose

\[
C \leq 2\delta + 2
\]

As \( C > 2\delta + K_1 \) we find

\[
K_1 = 1, C = 2\delta + 2
\]

Consider the following amalgamation.

![Diagram](image-url)
The nongeodesic triangles involved here have types

\[(2, \delta - 1, \delta), (1, \delta, \delta), \text{ and } (3, \delta, \delta)\]

and the first two are certainly in \(\mathcal{A}\). If \(C' > C\) then a triangle of the third type is also in \(\mathcal{A}\), and the diagram can be completed in \(\mathcal{A}\), with \(d(a_1, a_2) = 2\). So the triangle \((a_1, a_2, c)\) has type \((2, \delta, \delta)\), contradicting \(C = 2\delta + 2\).

\[\square\]

**Lemma 8.34.** Let \(\mathcal{A}\) be an amalgamation class of diameter \(\delta\) determined by triangle constraints with associated parameters \(K_1, K_2, C, C'\). If \(C' > 2\delta + K_2\) then \(C \geq 2\delta + K_2\).

**Proof.** Suppose on the contrary \(C < 2\delta + K_2 < C'\).

Then by the previous lemma we have \(K_2 < \delta\).

Let \(C' - K_2 \equiv \epsilon \mod 2\) with \(\epsilon = 0\) or \(1\). Consider the following amalgamation.

The triangle \((a, u, v)\) has type \((\delta - K_2, K_2, \delta - 1)\) and perimeter \(2\delta - 1 \geq 2K_1 + 1\). So to see that this is realized in \(\mathcal{A}\) it suffices to check the inequality \((\delta - 1) + K_2 \leq 2K_2 + (\delta - K_2)\), which is evident.

The triangle \((b, u, v)\) has type \((K_2, \delta, \delta - \epsilon)\) and perimeter \(P = 2\delta + K_2 - \epsilon \equiv C' \mod 2\). As \(P < C'\), to see that this is represented in \(\mathcal{A}\) it suffices to check the inequalities

\[
\begin{align*}
P & \geq 2K_1 + 1 \\
P & \leq 2K_2 + 2
\end{align*}
\]

The first is clear, and the second reduces to \(\epsilon \leq K_2\) which is also clear.

Therefore the diagram has a completion in \(\mathcal{A}\). Let \(i = d(a, b)\) in this completion. Then \(i \geq K_2\).

We claim that the perimeter of \((a, b, u)\) is at least \(C\). If \(\epsilon = 0\) then \(i + (\delta - 1) + (\delta - \epsilon) \geq 2\delta + K_2 - 1 \geq C\). If \(\epsilon = 1\) then \(C \equiv K_2 \mod 2\) and \(C < 2\delta + K_2\), so \(C \leq 2\delta + K_2 - 2\) and the inequality still holds.
Since the perimeter $2\delta + i - 1 - \epsilon \geq C$ we must have

$$i + \epsilon + 1 \equiv C' \mod 2$$
$$i \equiv K_2 + 1$$

So the perimeter of the triangle $(a, b, v)$ is odd. This gives the inequality

$$\delta + i \leq 2K_2 + 2(\delta - K_2)$$
$$i \leq K_2$$

which contradicts the conditions

$$i \geq K_2 \text{ and } i \equiv K_2 + 1 \mod 2$$

□

**Lemma 8.35.** Let $A$ be an amalgamation class of diameter $\delta$ determined by triangle constraints with associated parameters $K_1, K_2, C, C'$. Suppose that

1. $C > 2\delta + K_1$;
2. $K_1 + 2K_2 = 2\delta - 1$.

Then $C \geq 2\delta + K_1 + 2$.

**Proof.** Consider the following amalgamation.

The nongeodesic triangles involved in this diagram have the types

$(1, K_1, K_1), (K_1, \delta - 1, \delta - 1), (K_1 + 1, \delta - 1, \delta)$, and $(K_1, \delta - 1, \delta)$

A triangle of the first type is in $A$ by definition. The last type has even perimeter

$$2\delta + K_1 - 1 < C$$

and thus is also in $A$.

The other two types have odd perimeters

$$2\delta + K_1 - 2, 2\delta + K_1 < C$$
As these are both at least $2K_1 + 1$, to see that triangles of these types are in $\mathcal{A}$ it suffices to check the following inequalities.

\[
\begin{align*}
2\delta - 2 & \leq 2K_2 + K_1 \\
2\delta - 1 & \leq 2K_2 + K_1 + 1
\end{align*}
\]

both of which hold by hypothesis.

Therefore the amalgamation diagram has a completion in $\mathcal{A}$, with

\[d(a_1, a_2) = K_1 \text{ or } K_1 + 1\]

A triangle of type $(K_1, \delta, \delta)$ has odd perimeter, and

\[\delta + \delta = K_1 + 2K_2 + 1 > 2K_2 + K_1\]

so this triangle type does not occur in $\mathcal{A}$. Thus we have

\[d(a_1, a_2) = K_1 + 1\]

Then the triangle $(a_1, a_2, c)$ has perimeter $2\delta + K_1 + 1$ and thus $C \neq 2\delta + K_1 + 1$.

**Lemma 8.36.** Let $\mathcal{A}$ be an amalgamation class of diameter $\delta$ determined by triangle constraints with associated parameters $K_1, K_2, C, C'$. Suppose that

1. $2\delta + K_1 < C < 2\delta + K_2$;
2. $K_2 \leq \delta - 2$.

Then $C' = C + 1$.

**Proof.** Let $k = C - 2\delta$. Then $k < \delta - 2$. Consider the following amalgamation.

The triangle $(a, u, v)$ has type $(2, k + 1, k + 2)$ and perimeter $2k + 5$. We claim that $2k + 5 < C$, or equivalently $C < 4\delta - 5$. Indeed $C < 2\delta + K_2 \leq 3\delta - 2 \leq 4\delta - 5$. To see that the triangle $(a, u, v)$ is in $\mathcal{A}$ it suffices to check the following inequalities.

\[
\begin{align*}
2k + 5 & \geq 2K_1 + 1 \\
2k - 1 & \leq 2K_2 + 2 \cdot 2
\end{align*}
\]

As $K_1 < k < K_2$, both hold. Thus the triangle $(a, u, v)$ belongs to $\mathcal{A}$.

The triangle $(b, u, v)$ has type $(k + 1, \delta, \delta)$ and perimeter $C + 1$. Suppose toward a contradiction that

\[C + 1 < C'\]
We claim that the triangle \((b, u, v)\) is in \(\mathcal{A}\). It suffices to check the inequalities
\[
C + 1 \geq 2K_1 + 1 \\
C + 1 \leq 2K_2 + 2(k + 1)
\]
As \(C > 2\delta\) the first is clear. The second reduces to
\[
C + 2K_2 + 1 \geq 4\delta
\]
and indeed, we have
\[
C + 2K_2 + 1 > (2\delta + K_1) + (2K_2 + 1) \geq 2\delta + (2\delta - 1) + 1 = 4\delta
\]
Thus the diagram can be completed in \(\mathcal{A}\), with \(d(a, b) \geq \delta - 2\). If \(d(a, b) = \delta - 2\) or \(\delta\), then the triangle \((a, b, u)\) has perimeter \(C\) or \(C + 2\), a contradiction. Thus
\[
d(a, b) = \delta - 1
\]
So the type of the triangle \((a, b, v)\) is \((2, \delta - 1, \delta)\), which yields the inequality
\[
(\delta - 1) + \delta \leq 2K_2 + 2 \\
K_2 \geq \delta - 1
\]
contrary to our hypothesis. \(\square\)

**Proposition 8.37.** Let \(\mathcal{A}\) be an amalgamation class of diameter \(\delta\) determined by triangle constraints with associated parameters \(K_1, K_2, C, C'\). Suppose that
\[
K_1 < \infty \\
C > 2\delta + K_1
\]
Then the following hold.
1. \(K_1 + 2K_2 \geq 2\delta - 1\);
2. \(3K_2 \geq 2\delta\);
3. If \(K_1 + 2K_2 = 2\delta - 1\), then \(C \geq 2\delta + K_1 + 2\);
4. If \(C' > C + 1\) then \(C \geq 2\delta + K_2\).

**Proof.** The first three points hold by Lemmas 8.27, 8.29, and 8.35 respectively. Now suppose
\[
C' > C + 1
\]
We must show that \(C \geq 2\delta + K_2\).
If \(C = C_1\) then \(C_1 < C_0 - 1\) and by Lemma 8.10 we find \(C \geq 2\delta + K_2\). So suppose
\[
C = 0
\]
If \(K_2 \geq \delta - 1\) then by Lemmas 8.31 and 8.33 we find
\[
K_2 = \delta - 1 \text{ and } C = 3\delta - 1 = 2\delta + K_2
\]
and our claim holds.
On the other hand if \(K_2 \leq \delta - 2\) then Lemma 8.36 gives the result. \(\square\)

At this point, we have our main theorem.
Theorem (9). Let $A$ be an amalgamation class of diameter $\delta$ determined by triangle constraints. Then $A = A_{\delta, K_1, K_2; C, C'}^\delta$ for some admissible choice of parameters.

Proof. By Proposition 8.14, we have

$$A = A_{\delta, K_1, K_2; C, C'}^\delta$$

for some choice of parameters. It remains to show that the parameters must be admissible.

If $K_1 = \infty$ then there are no odd triangles by Lemma 8.6 and thus $C = 2\delta + 1$. So we may suppose

$$K_1 < \infty$$

We then apply Propositions 8.28 and 8.37 to conclude. □
References


