MULTI-CRITERIA OPTIMIZATION: MAXIMIZATION OF A SYSTEM RELIABILITY ESTIMATE AND MINIMIZATION OF THE ESTIMATE VARIANCE

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ABSTRACT

Problem formulations and descriptions of optimization approaches are presented when there are multiple design objectives; (1) to maximize an estimate of system reliability, and (2) to minimize the variance of the reliability estimate. This is a sound and risk-averse formulation that reflects the actual needs of system designers and users. Designs with the highest estimate of system reliability may actually be a poor choice for a risk-averse applications because of an associated high variance. More realistic approaches to the problem are presented to incorporate the risk associated with the reliability estimate. System design optimization then becomes a multi-criteria optimization problem. An example is solved using nonlinear integer programming methods. The results indicate that very different designs are obtained when the formulation is changed to incorporate minimization of reliability estimation variance as a design objective.

1 INTRODUCTION

Selection of an engineering system design generally involves the selection of a design architecture to maximize system reliability while satisfying constraints on system weight, cost, performance and others. Estimation of system reliability is generally based on a system model and component or lower-level estimates of reliability. However, the component reliability estimates are often uncertain, particularly for new and evolving designs, and thus, the associated system reliability estimates are uncertain as well. For the design of modern electronic (and other) systems, considering the multiple objectives of maximization of a system reliability estimate and minimization of the variance of that estimate is an accurate and realistic formulation.

The very recent development of many high-technology components and materials precludes extensive testing that would be required to support deterministic knowledge of component reliability. The problem is further complicated because system designers and users are generally risk-averse. They generally prefer a component or system design with a slightly lower reliability estimate if the variance of that estimate is low. Thus, maximization of the estimated system reliability by itself is not necessarily an effective design strategy, although this has generally been the approach previously taken. System designers and users desire a system design with a high reliability estimate, but also one associated with low estimation variability. To properly consider these objectives, the system reliability estimate should be considered as a random variable with an associated distribution.

There are many insights to be realized from analysis of a general reliability model and optimization methodology to maximize a system reliability estimate and to minimize the estimate variance. Many commonly practiced design rules for component selection and the use of redundancy do not consider uncertainty and can be demonstrated to be inefficient for some applications. That is, they lead to solutions that are dominated by other solutions with similar estimated reliability but lower variance. For example, designers often use identical component types in parallel as a convenient design practice, but use the same reliability estimate (and associated error) every time the component is used. This leads to an increased variance compared to the use of different, but functionally equivalent components in parallel. This decreases the variance under certain conditions. It is the logic used by software designers when implementing N-version programming for safety critical applications.

Painton & Campbell [1, 2] have solved a formulation of the reliability optimization problem that incorporates risk. In their work, component time-to-failure is distributed according to an exponential distribution, but the distribution parameter is a random variable. In their formulation, there are defined reliability improvement levels that can be selected for the components within the system. They used a genetic algorithm (GA) to maximize the $\alpha\times100\%$ lower percentile of the mean time between failure (MTBF) given component repair assumptions. The percentile level, α, can be thought of as a user-defined risk level. For risk-neutral applications, a percentile of .50 would be appropriate. For risk-averse applications, lower levels of α would be selected. Coit & Smith [3] presented a similar formulation to maximize a lower percentile of the system time-to-failure distribution.

Rubinstein, et al [4] also use a GA to solve a redundancy allocation problem with uncertain component properties. They maximize the expected values of the random system reliability. Maximization of an expected value is a common practice for the solution of difficult stochastic programming problems, but it does not sufficiently account for decision-makers' aversion for risk regarding system reliability estimation.

2 RELIABILITY OPTIMIZATION APPROACHES CONSIDERING ESTIMATION UNCERTAINTY

The redundancy allocation problem pertains to a system of s subsystems in series. For each subsystem, there are m_i functionally equivalent components, with different levels of cost, weight, reliability and other characteristics, which may be selected. There is an unlimited supply of each of the m_i choices. A minimum of one component must be chosen for each subsystem, but it is often advantageous to add redundant components. The use of redundancy improves system reliability but adds to system cost, weight, etc. There are system-level constraints and the problem is to select the design configuration that maximizes some stated objective function. The multi-criteria optimization problem can be stated generally as follows:

Problem P1:

$$\max R(\mathbf{x}), \quad \min Var[R(\mathbf{x})]$$

$$\text{s.t.} \quad g_k(\mathbf{x}) \le b_k \quad \text{for } k = 1, \ldots, n$$

where, $R(\mathbf{x})$ = system reliability, for design vector $\mathbf{x}$

$Var[R(\mathbf{x})]$ = variance of the system reliability estimate, for design vector $\mathbf{x}$

$g_k(\mathbf{x})$ = k^{th} constrained system property (weight, cost, fuel consumption, etc.)

n = number of constraints

b_k = k^{th} constraint limit

$\mathbf{x}$ = $(x_{11}, x_{12}, ..., x_{1,m_1}, x_{21}, x_{22}, ..., x_{2,m_2}, x_{31}, ..., x_{s,m_s})^{\text{T}}$

x_{ij} = quantity of the j^{th} available component used in subsystem i

m_i = number of available components for subsystem i

s = number of subsystems

The preceding definition of the $\mathbf{x}$ vector is specifically intended for redundancy allocation for a series-parallel system. However, the problem could be formulated more generally to accommodate other system designs as well.

The first objective is to maximize $R(\mathbf{x})$. However, in this formulation, $R(\mathbf{x})$ is a random variable described by a probability distribution function. The intent then is to maximize the expected value or an unbiased estimate of $R(\mathbf{x})$. If the underlying distribution (or prior distributions) of the component reliability values are known or assumed to be known, then the objective is to maximize the expected value of $R(\mathbf{x})$, or $E[R(\mathbf{x})]$. Alternatively, if only an unbiased empirical estimate of the component reliability values are available, the objective is to maximize the system reliability estimate, $\hat{R}(\mathbf{x})$. In either case, the second objective is to minimize the variance; either of the random reliability, $R(\mathbf{x})$, or the estimate, $\hat{R}(\mathbf{x})$. Coit [5] demonstrated the propagation of component reliability estimation uncertainty to the system-level.

Different criteria and/or optimization methodologies can be considered to select a recommended design when considering the multi-criteria formulation. Possible solution strategies and/or associated optimality criteria include goal programming and Pareto optimization criteria. Other approaches to the problem involve mathematically combining the multiple objectives into a single objective function so that traditional mathematical programming or a heuristic search can be applied. Possible combined objective functions include weighted additive objectives, utility theory measures, coefficient of variation (CV), or a lower-bound on system reliability, $R_{1-\alpha}(\mathbf{x})$.

Utility theory measures, $U(\mathbf{x})$, could also be applied to the problem where $\mathbf{x}$ is a solution vector that defines the design. Utility theory depends on a customer's subjective utility function. Utility functions characterize a decision-makers inclination for risk-aversion. If the utility function is linear, then the decision maker is risk-neutral and maximization of $U(\mathbf{x})$ is the same as maximizing the expected value of the system reliability estimate. When there are severe implications of poor reliability (e.g., safety), it is unlikely that the utility function is linear. When system designers and program managers are risk-averse, maximization of $U(\mathbf{x})$ can lead to a distinct solution that does account for risk.

Utility function for a prospective design solution is as follows:

$$U(\mathbf{x}) = \int_0^1 u(r)\, f_R(r \mid \mathbf{x})\, dr \qquad (1)$$

where: $u(r)$ = user-specified utility function

$f_R(r|\mathbf{x})$ = system reliability probability density function, for design solution $\mathbf{x}$

The utility function approach is intuitively appealing but difficult to implement for several reasons. $u(r)$ is defined as a user-specified continuous function. It is subjective, and in practice, difficult to determine. More critically, $f_R(r|\mathbf{x})$ will be a complex, non-standard distribution, even if the constituent component distribution are well-understood. Additionally, for many actual design problems, only component and system-level variance estimates can be made and $f_R(r|\mathbf{x})$ is unknown.

Maximization of a lower-bound of the system reliability estimate is an appealing option, similar to the approach of Painton & Campbell [1, 2]. $R_{1-\alpha}(\mathbf{x})$ is defined as a $(1-\alpha)\times 100\%$ lower bound on system reliability where α is the distribution percentile. If $R_{1-\alpha}(\mathbf{x})$ is used as an objective function, α can be changed to incorporate the user's risk. An α of 0.50 approximates a risk-neutral scenario and lower values of α ($\alpha < 0.10$) are appropriate for risk-averse design problems. Conceptually, maximization of $R_{1-\alpha}(t,\mathbf{x})$ is attractive because it incorporates risk (via α) and will provide different solutions depending on different risk profiles. However, determination of $R_{1-\alpha}(\mathbf{x})$ can be difficult. There is no general closed-form expression and other options, such as simulation or approximations, must be considered.

3 MULTI-CRITERIA OPTIMIZATION

Two approaches to multi-criteria problems are the use of weighted objectives and goal programming. Multi-attribute decision making often involves the determination of subjective, but quantitative "weights" to indicate the relative importance of the various criteria. For this problem, the criteria are the reliability estimate and the variance of that estimate. Of course, the approach could be further extended to consider cost as a third criteria. These subjective weights allow trade-offs among different objectives and design solutions so they can be incorporated into a singular measure. Thus, designs can be compared. For this problem, the weighted attribute objective function to be maximized would be:

$$z = w_1 \mathrm{E}[R(\mathbf{x})] - w_2 Var[R(\mathbf{x})], \quad w_1 + w_2 = 1 \qquad (2)$$

Goal programming is a similar approach where defined goals are established for each criteria. Then, deviation from the goals is penalized and the objective is to select the design solution that minimizes the weighted sum of the penalties. For the reliability optimization problem, the "goals" are a reliability of 1.0 and a variance of 0.0. The goal programming objective function can be stated as,

$$z = w_1 |\mathrm{E}[R(\mathbf{x})] - 1.0| + w_2 |Var[R(\mathbf{x})] - 0.0|, \quad w_1 + w_2 = 1 \qquad (3)$$

$$z = w_1 (1.0 - \mathrm{E}[R(\mathbf{x})]) + w_2\, Var[R(\mathbf{x})]$$

$$z = w_1 - w_1\mathrm{E}[R(\mathbf{x})]) + w_2\, Var[R(\mathbf{x})]$$

By comparing the objective functions for weighted objectives and goal programming, it can be observed that maximization of the weighted objective function (equation 2) is functionally the same as minimization of the goal programming objective function (equation 3).

It may also be advisable to identify a set of non-dominated solutions instead of identifying a singular "optimal" solution. The problem could be formulated to identify solutions that simultaneously maximize the expected system reliability and minimize the variance of the system reliability estimate. Since it is doubtful that a unique solution will optimize both objectives, a set of Pareto optimal solutions will be identified. Dominance criteria is as follows:

Solution *a* dominates solution *b* if:

- $E[R(\mathbf{x}_a)] > E[R(\mathbf{x}_b)]$ and $Var(R(\mathbf{x}_a)) \leq \mathrm{Var}(R(\mathbf{x}_b))$, or,
- $E[R(\mathbf{x}_a)] \geq E[R(\mathbf{x}_b)]$ and $Var(R(\mathbf{x}_a)) < \mathrm{Var}(R(\mathbf{x}_b))$

Instead of identifying a single solution, a set of all dominated solutions is the final product of the optimization process. A recommended solution could then be chosen by considering the set of all Pareto optimal solutions.

Other researchers who have considered multi-objective reliability optimization problems include Busacca, Marseguerra, and Zio [9]. They propose a Multiple Objective Genetic Algorithm (MOGA) based on identifying Pareto optimal solutions. The objective function associated with each solution is the "rank." All Pareto optimal solutions have rank one. When they are removed from the GA population, the Pareto optimal solutions from the remaining population have rank of two, and so on. They demonstrate their methodology on the system design of a standby safety system of a nuclear power plant.

4 ILLUSTRATIVE PROBLEM

An example reliability optimization problem has been formulated and solved using weighted objectives and Pareto optimality. These were the only applicable methods given the available problem data and the stated objective to minimize risk. The example has three subsystems and there are five component choices for each subsystem. Component data including expected reliability, variance of the reliability estimate, cost and weight are presented in Table 1. System-level constraints are a maximum cost of 40 and a maximum weight of 55. Solutions to this problem, using other criteria, were originally presented by Coit [6].

To solve the multi-criteria optimization problem, it was assumed that the component reliability estimates were independent, and the methods presented by Coit [5] were used to determine the system reliability estimate variance. If the random component reliability estimates had been repeated wherever the same component type was used, then they would not have been independent, and the results from Jin & Coit [7] could be used to compute variance. A general-purpose nonlinear integer programming algorithm, based on branch-and-bound, was used to generate solutions.

The problem was solved repeatedly by varying the values of w_1 and w_2. As w_1 was varied from .001 to 1, four different solutions maximized the objective function depending on the range of w_1 and w_2 (remembering that $w_1 + w_2 = 1$). These four solutions are presented as Figures 1 through 4. The absolute values of the w_i values itself have little meaning because reliability and variance are scaled differently. The important observation is that as w_1 increases, it becomes more important to maximize reliability and less important to minimize the variance, and the design changes accordingly.

Table 1: Test Set Problem Parameters

	subsystem 1				subsystem 2				subsystem 3			
j	$E[r_{ij}]$	σ_{ij}^2	c_{ij}	w_{ij}	$E[r_{ij}]$	σ_{ij}^2	c_{ij}	w_{ij}	$E[r_{ij}]$	σ_{ij}^2	c_{ij}	w_{ij}
1	.94	.003	7	4	.97	.001	6	5	.96	.002	7	6
2	.91	.010	6	6	.86	.004	3	7	.89	.022	6	8
3	.89	.004	6	8	.70	.052	2	4	.72	.003	4	5
4	.75	.016	3	7	.66	.014	2	7	.71	.015	3	4
5	.72	.013	3	8	.65	.170	2	6	.67	.130	2	4

As the system diagrams indicate, the "optimal" solution changes as the weights vary. From a mathematical programming perspective, this is hardly surprising because the objective function changed. However, from a system reliability perspective these are important results. If estimation variance is ignored (the traditional approach), then the solution in Figure 4 will be determined to be optimal. In practice, the solution in Figure 4 may not be desirable even though it maximizes reliability. It also has a much larger variance, largely through the use of component 5 for subsystem 3.

The problem was then re-solved using Pareto optimality criteria. The problem was now solved using the solution space reduction methodology presented by Sung & Cho [7]. Figure 5 presents a graph of all solutions with system reliability greater than .965. The non-dominated solutions in the lower-right of the graph are the Pareto optimal sloutions. These solutions would then be examined by a decision-makers, considering possibly additional non-quantitative criteria to select a preferred solution.

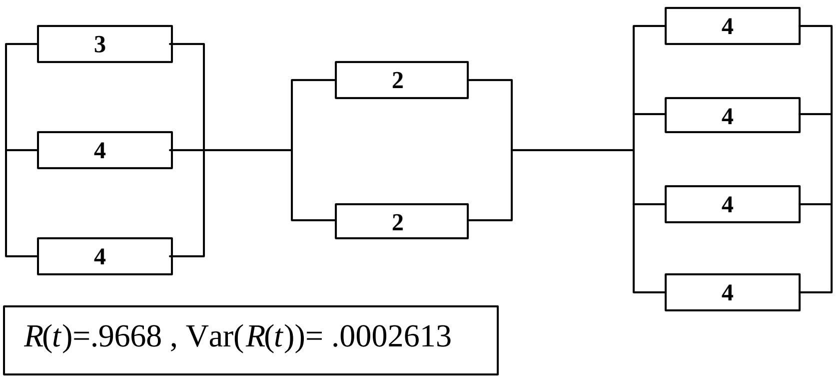
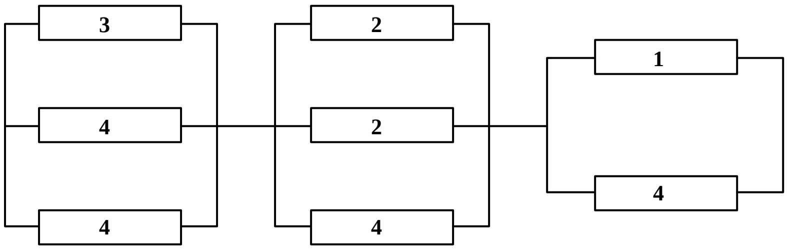
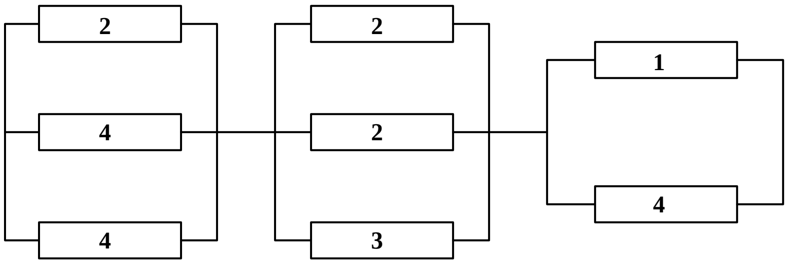


Figure 1: Optimal Solution (.0011 ≤ w_1 ≤ .0039)



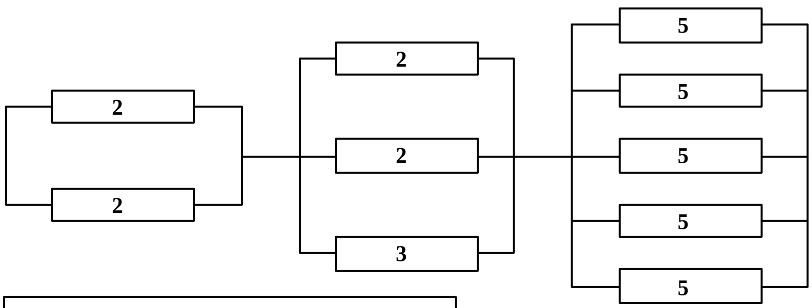
$R(t)$=.9751 , Var($R(t)$)= .0002931

Figure 2: Optimal Solution (.0039 < $w_1 \leq$ .0221)



$R(t)$=.9771, Var($R(t)$)=.0003373

Figure 3: Optimal Solution (.0221 < $w_1 \leq$ .120)



$R(t)$=.9822, Var($R(t)$)=.001042

Figure 4: Optimal Solution (.120 < $w_1 \leq$ 1.0)

5 CONCLUSIONS

This paper presents methods to simultaneously consider the many components in a system with varying levels of reliability estimation uncertainty. The redundancy allocation problem was formulated as a multi-criteria problem to maximize a system reliability estimate (or expected value) and to minimize the variance of that estimate. The availability of reliability analysis tools which reflect variability in component reliability estimates can provide better, more reliable designs and provide important insights to engineering design strategies involving components with uncertain reliability. The methods were demonstrated on a sample problem, and as expected, the optimal design solution varies depending on the relative influence of the two criteria.

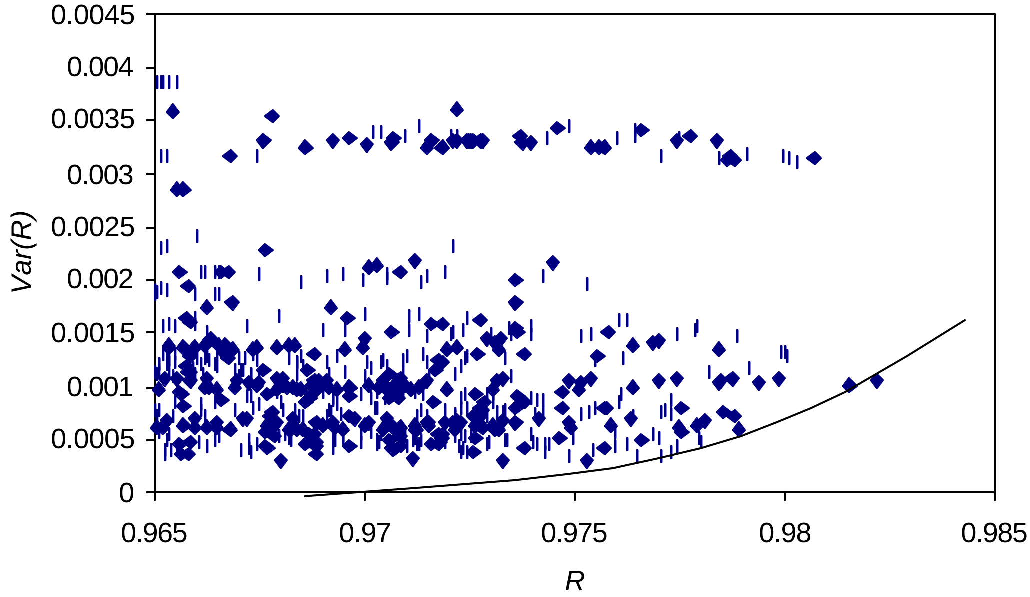


Figure 5: Set of Pareto Optimal Solutions

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