System Reliability Sensitivity Measures

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Abstract

System reliability sensitivity measures are proposed to assist designers and reliability analysts in prioritizing reliability improvement and testing activities. In complex system designs, a common goal is to improve system reliability by increasing the reliability of the components to be used within the system. Component reliability is generally estimated from field failure data or test data. Unfortunately, these data are generally insufficient for highly precise reliability estimation. Thus, it is necessary to study the variability of the component and system reliability estimates and the propagation of uncertainty. This paper explores the underlying relationships between a lower-bound on system reliability, and individual component reliability estimates and the variance of those estimates. Three sensitivity indices are presented to analyze the system-level impact of component-level reliability data. These indices provide rich information useful to assess system reliability improvement strategies.

Keywords:
Confidence Intervals, Sensitivity Measures, Binomial Data

1. Introduction

A complex system design requires the prediction or estimation of system reliability based on component-level information and a system reliability model. In most cases, there is insufficient failure and survival data at the component level, particularly with high reliability components offered by many new technologies. This is particularly true if the system is new or scheduling dedicated life testing is very expensive. Even when the component reliability estimates are unbiased, variability of the estimates may be too high. It is therefore important to consider the variability of the system reliability estimate as well. For a system with risk-averse designers or users, minimization of the variance of the system reliability estimate is a key consideration. As the market becomes more and more global and competitive, it will be even more difficult for manufacturers to allocate long test times and a large number of units to collect the failure data.

In Section 2 of this paper, a lower confidence bound for system reliability is described based on an assumption of a log-normal distribution for the system reliability estimate. Section 3 presents three sensitivity measures that link the component reliability and variance to the system reliability confidence limit, $R_{1\text{-}\alpha}(t)$. Section 4 provides an illustrative example for the sensitivity analysis.

There have been many procedures developed to compute confidence intervals for system reliability, but these existing models are usually computational difficult or limited to exponential failure times. Most existing research assumes the exponential distribution due to its mathematical convenience. However, this assumption is not always appropriate for many system design problems or component failure time distributions in practice.

Gertsbakh [1, 2] describes methods to compute confidence intervals for series, parallel, series-parallel and k-out-of-n:G system. In all instances, it was assumed that component failure time distributions are exponential and components have high reliability. Gnedenko [3] proposes a maximization procedure, and Pavlov [4] develops a substitution process to determine the system reliability confidence intervals. Ushakov [5] described methods for the computation of confidence intervals for series, parallel, series-parallel and complex system configurations. All of these require the exponential distribution assumption.

Easterling [6] presents a method based on the asymptotic variance of the system reliability estimate for a series system. Mann, et al. [7] provides a comprehensive review and description of different techniques to determine both exact and approximate system reliability confidence intervals.

Gandini [8] proposes a heuristic-based generalized perturbation theory (GPT) methodology by introducing the concept of importance of a state in the Markov transition process. The propagation of uncertainties from component parameters to system reliability and variance can be estimated by GPT. Melo et al. [9] presented sensitivity indices with respect to the equipment failure and repair rates. Instead of seeking derivative information as Gandini does, they use the incremental technique for the interested system parameters, thus simplifying the computational complexity. Both papers assume constant failure and repair rates in their models.

For practical applications, many of the required assumptions are too restrictive, and thus, it is very hard to implement these models in practice. Coit [10] uses reliability information from component levels to develop a more general procedure to estimate a system reliability lower bound (at mission time *t*). There is no underlying assumption for component failure time distribution. The only information the model requires is the estimated reliability and the estimate variance for each individual component. This model is practically attractive and applicable for a greater range of problems. The following approach uses this method as a basis to develop the three sensitivity measures.

2. Reliability Confidence Interval

2. 1 Introduction

Many procedures have been developed for the estimation of system reliability confidence bounds. Most methods assume that system or components follow some parametric distribution. The most widely used distribution is the exponential because of its mathematical simplicity. However, it is not always applicable in many real situations. The approach used here is to determine a confidence interval for system reliability based on empirical estimates of component reliability. It is a distribution-free method since there is no underlying assumption on the failure distribution for each component. This method only involves the calculation of the variance of the reliability estimate for each component, and the propagation of component-level variances to the system-level.

The procedure consists of three steps. First the reliability for each component is estimated based upon the field failure data or experimental data. Then, the variance of the component reliability is computed. The examples presented assume that failure data are binomial, although more general methods are presented in Coit [10]. Finally, the system reliability and the variance of system reliability are estimated by aggregating the component reliability and variance information. These steps are explained in the following paragraphs.

2.2 Computing System Reliability and Variance

Component reliability and the associated variance estimate are estimated from binomial data. For the i^{th} component used in the system, N_i components are each put on test for t mission hours. Then F_i failures are observed during the testing period. The status of each component (survival/failure) is considered as an independent Bernoulli trial with parameter $r_i(t)$. An unbiased estimate of $r_i(t)$ and an approximation of the variance of the estimate are determined as following

$$\hat{r}_i(t) = 1 - F_i / N_i \tag{1}$$

$$Var\left(\hat{r}_i(t)\right) = \frac{r_i(t)(1 - r_i(t))}{N_i} \tag{2}$$

$$V\hat{a}r\left(\hat{r}_i(t)\right) = \frac{\left(1 - F_i / N_i\right) F_i / N_i}{N_i} \tag{3}$$

A subsystem is defined as any collection of components in a strictly series or parallel configuration. Variance of subsystem reliability can be estimated as a function of component reliability and estimate variance, from Equations (1) and (3). For any defined system, the variance of the reliability estimate can be obtained using a decomposition methodology. It is assumed that each system can be decomposed into series or parallel subsystem configurations, and component reliability estimates are statistically independent. This is applicable to many systems in practice. The following are expressions for the variance of the reliability estimate for series and parallel system or subsystem configurations.

$$V\hat{a}r\left(\hat{R}_{series}(t)\right) = \prod_i \left(\hat{r}_i(t)^2 + V\hat{a}r\left(\hat{r}_i(t)\right)\right) - \prod_i \hat{r}_i(t)^2 \tag{4}$$

$$V\hat{a}r\left(\hat{R}_{parallel}(t)\right) = \prod_i \left(\left(1 - \hat{r}_i(t)\right)^2 + V\hat{a}r\left(\hat{r}_i(t)\right)\right) - \prod_i \left(1 - \hat{r}_i(t)\right)^2 \tag{5}$$

More complete derivation of above formulas are presented in Coit [10]. Following the decomposition procedure, variance estimates for series-parallel, parallel-series or more complex systems can be determined. In this paper, only series-parallel systems will be presented. The method described here can be applied to parallel-series systems with minor changes. The variance for the series-parallel system is given by,

$$V\hat{a}r\left(\hat{R}(t)\right)=\prod_{i=1}^{m}\left(\left(1-\prod_{j=1}^{s_i}\left(1-\hat{r}_{ij}(t)\right)\right)^2+\prod_{j=1}^{s_i}\left(\left(1-\hat{r}_{ij}(t)\right)^2+\frac{\hat{r}_{ij}(t)\left(1-\hat{r}_{ij}(t)\right)}{n_{ij}}\right)-\prod_{j=1}^{S_i}\left(1-\hat{r}_{ij}(t)\right)^2\right)-\prod_{i=1}^{m}\left(1-\prod_{j=1}^{s_i}\left(1-\hat{r}_{ij}(t)\right)\right)^2 \quad (6)$$

where

$r_{ij}(t)$ = reliability of j^{th} component in i^{th} subsystem at time t.

n_{ij} = number of units tested for j^{th} component in subsystem i.

m = total number of subsystems.

s_i = total number of components in subsystem i, where i = 1, 2, 3, ..., m

2.3 Confidence Interval Estimation

A confidence interval or a one-sided lower-bound on system reliability is derived based on an assumption that the system reliability estimate is log-normally distributed. The log-normal distribution parameters are then estimated from the system reliability estimates and variance estimates. For large systems, the log-normal assumption is very accurate because there are many subsystems which collective form a series system. Coit [10] showed that both the series-parallel and the parallel-series configuration can be analyzed by invoking the central limit theory when dealing with sufficient large complex systems. Therefore the lognormal distribution is a sound and convenient tool for the derivation of the confidence intervals of complex system.

For a series configuration of subsystems, the expected value and the variance of a log-normal distribution as a function of distribution parameters μ and σ^2 are given as follows

$$\hat{R}(t) \sim LN(\mu, \sigma^2) \quad (7)$$

$$E\left(\hat{R}(t)\right)=e^{\mu+\frac{1}{2}\sigma^2}, \quad Var\left(\hat{R}(t)\right)=e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right) \quad (8)$$

μ and σ^2 are unknown parameters. The log-normal distribution parameter σ^2 can be estimated as,

$$\hat{\sigma}^2=\ln\left[1+\frac{V\hat{a}r\left(\hat{R}(t)\right)}{\hat{R}(t)^2}\right] \quad (9)$$

A (1-α)$\times$100% confidence interval for $R(t)$ can be estimated using the approximation in Coit [10]. A lower, single-sided (1-α)$\times$100% confidence limit for $R(t)$ is computed as,

$$R_{1-\alpha}(t)=\hat{R}(t)\exp\left[\sigma\left(\tfrac{1}{2}\sigma-z_\alpha\right)\right] \quad (10)$$

$$R_{1-\alpha}(t)\approx\hat{R}(t)\exp\left[\hat{\sigma}\left(\tfrac{1}{2}\hat{\sigma}-z_\alpha\right)\right] \quad (11)$$

3. Sensitivity Analysis

Three sensitivity measures are presented. Used collectively, these metrics can be used to prioritize reliability improvement and testing activities. For some components, the measures may indicate that higher reliability will readily improve system reliability. For other components, the existing reliability estimate may seem satisfactory, but the variance estimate must be decreased by additional testing to increase $R_{1-\alpha}(t)$. Alternatively, the sensitivity measures may reveal that neither reliability improvement nor additional testing is required for some components because they will not be particularly effective or efficient in increasing $R_{1-\alpha}(t)$.

One of the objectives in system design is to achieve high system reliability. Since reliability is usually hard to estimate or predict at the system-level, system reliability estimates are computed based on information at the component-level. Because component reliability is also estimated, the variability from the component levels potentially has a strong impact on the precision of system reliability estimation.

For a complex system such as series-parallel or parallel-series system, the system-level impact of changes at the component-level are not well understood or particularly obvious. Some components will have stronger impact on the system reliability than others. Thus improving the system reliability effectively and economically is not necessarily achieved by upgrading each component reliability within the system evenly or by attempting to increase only the lowest reliability components. Due to the complexity of the system configuration, improving the reliability of some components will have more effect on system reliability estimation variability than others. It becomes necessary to intelligently decide which components require reliability improvement and which components require additional testing to decrease the estimate variance.

3.1 Value of Perfect Reliability (*VPR*)

For a specific component, VPR_{ij} is defined to be the improvement in a lower-bound for system reliability by assuming that that the reliability of that particular component is 1.0 with a variance equal to zero. Expressed mathematically,

$$VPR_{ij} = R_{1-\alpha}(t\,; r_{ij} = 1, \sigma_{ij} = 0) - R_{1-\alpha}(t) \tag{12}$$

The VPR_{ij} can be regarded as the upper limit on the system reliability improvement by setting a certain specific component reliability to 1.0. VPR_{ij} indicates the reliability improvement *potential* for a particular component. If VPR_{ij} is relatively low, then reliability improvement for that particular component may not be a priority.

3.2 Value of Perfect Information (*VPI*)

The value of perfect information is defined to measure the impact of the component estimate variance on a lower-bound on system reliability. For a specific component, the VPI_{ij} is the system reliability improvement by assuming that the component variance is zero. A zero variance means that component reliability is known explicitly or that there is no estimation variability, i.e., perfect information. In mathematical form, VPI_{ij} is as follows,

$$VPI_{ij} = R_{1-\alpha}(t; \sigma_{ij} = 0) - R_{1-\alpha}(t) \tag{13}$$

Differing from VPR_{ij}, the VPI_{ij} does not assume the component reliability is 1.0. In practice, we always can decrease the reliability variance by allocating more testing units. VPI_{ij} indicates which component variance has more significant effect on a lower-bound for system reliability.

3.3 Value of Testing One Additional Unit (*VTAU*)

VPI_{ij} illustrates the potential benefits and need for reducing the reliability estimate variance, but it does not indicate how efficiently that can be done. $VTAU_{ij}$ is computed by assuming that data are binomial and then observing the increase in $R_{1\text{-}\alpha}(t)$ resulting from testing one additional unit without changing the component reliability estimate. The mathematical form of $VTAU_{ij}$ for a specific component is given by

$$VTAU_{ij} = \left[\frac{\partial}{\partial \sigma_{ij}} R_{1-\alpha}(t)\right]\left[\sqrt{\hat{r}_{ij}(t)(1-\hat{r}_{ij}(t))}\left(\frac{1}{\sqrt{n_{ij}}} - \frac{1}{\sqrt{n_{ij}+1}}\right)\right] \tag{14}$$

where

n_{ij} = sample size (for reliability and variance estimates from a binomial data set).

If VPI_{ij} and $VTAU_{ij}$ are both high, this means that it is important to reduce the estimation variability and this can be achieved efficiently. If VPI_{ij} is relatively high but $VTAU_{ij}$ is low, it is important to reduce the reliability estimation variability, but it may be difficult to achieve. If VPI_{ij} is low, then $VTAU_{ij}$ will almost certainly also be very low and it will not be particularly informative.

4. Illustrative Example

This complex system consists of five subsystems with a series configuration. Each subsystem has two to four parallel-configured components. Subsystems A, D and E each are *s*-independent parallel (IP) configurations and subsystem B and C each are *s*-dependent configurations (DP). The independent configuration means that component reliability estimates within the subsystem are statistically independent. The dependent configuration means that component estimates within the subsystem are statistically dependent. For instance, in subsystem A, the reliability and variance estimates of components 1, 2 and 3 are from three different samples. In subsystem B, however, two of the same component type are used. The reliability and variance estimates of two components (component 4) are derived from the same sample. Small errors or variability with these estimates will likely have a higher impact at the system-level because the estimate is used more than once.

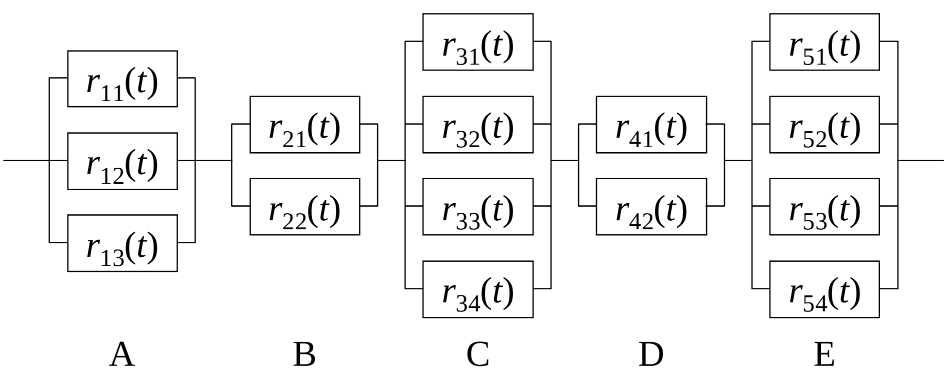

Figure1. Series-Parallel Configuration System

Table 1 lists in detail each component reliability and variance. Based on these parameters, we have calculated the subsystem reliabilities, variances and sensitivity measures.

Table1. System Parameters and Sensitivity Values

sub no.	structure	subsystem reliability	subsystem variance	component no.	component (i, j)	reliability estimate	variance estimate	VPR_{ij}	VPI_{ij}	$VTAU_{ij}$
A	IP	0.9972	5.56E-6	1	(1,1)	0.86	0.001	0.00372	0.00011	8.70E-07
				2	(1,2)	0.90	0.003	0.00372	0.00052	1.63E-05
				3	(1,3)	0.80	0.01	0.00372	0.00045	2.59E-05
B	DP	0.9991	2.475E-6	4	(2,1) (2,2)	0.98	4.9E-4	0.00209	0.00170	2.10E-05
C	DP	0.9987	4.436E-6	5	(3,1) (3,2) (3,3) (3,4)	0.84	3.84E-3	0.00140	0.00075	1.77E-06
D	IP	0.9948	5.468E-6	6	(4,1)	0.96	0.00012	0.00609	0.00038	1.52E-06
				7	(4,2)	0.87	0.002	0.00609	0.00058	9.78E-06
E	IP	0.9999	7.936E-9	8	(5,1)	0.81	0.001	0.00007	5.40E-08	3.49E-10
				9	(5,2)	0.89	0.01	0.00007	9.06E-07	8.60E-08
				10	(5,3)	0.92	0.003	0.00007	6.39E-07	2.53E-08
				11	(5,4)	0.96	0.00001	0.00007	1.24E-08	3.24E-12

Figures 2, 3 and 4 present *VPR*, *VPI* and *VTAU* for each component type in the example system. As can be noticed from the figures, components can have a variety of combinations of these measures. These three measures can then be used collectively to make assessments about the system reliability improvements efforts.

Notice from the figures that component no. 6 has a high *VPR* but a low *VPI*. For this component, additional reliability testing is not needed to increase $R_{1-\alpha}(t)$. However, use of a more reliable component needs to be investigated for use here. It may be possible that there is no feasible method to improve the reliability of this component. Nevertheless, it remains a component where reliability improvement is desirable.

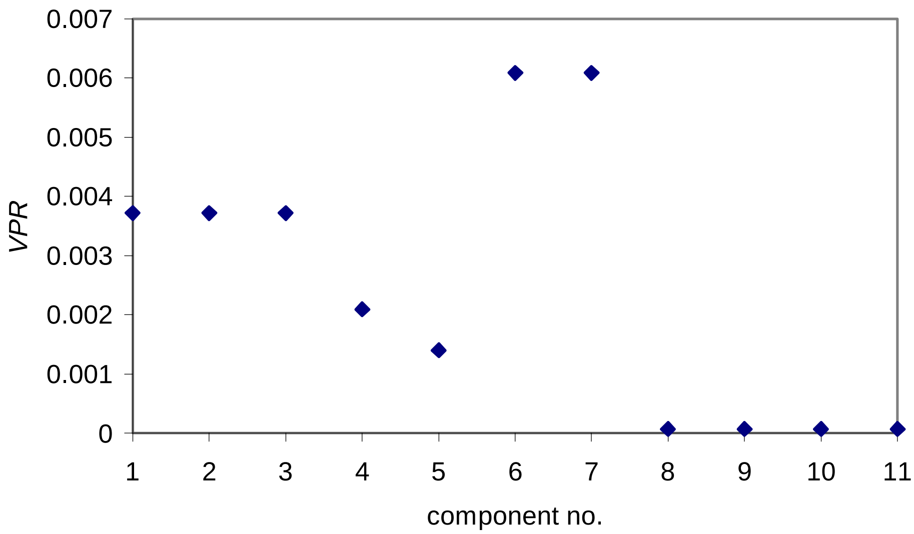

Figure 2. *VPR* for Different Components

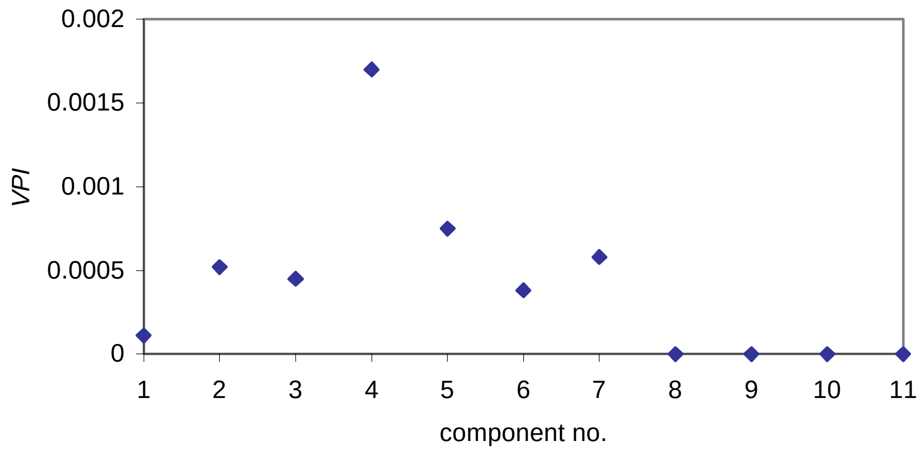

Figure 3. *VPI* for Different Components

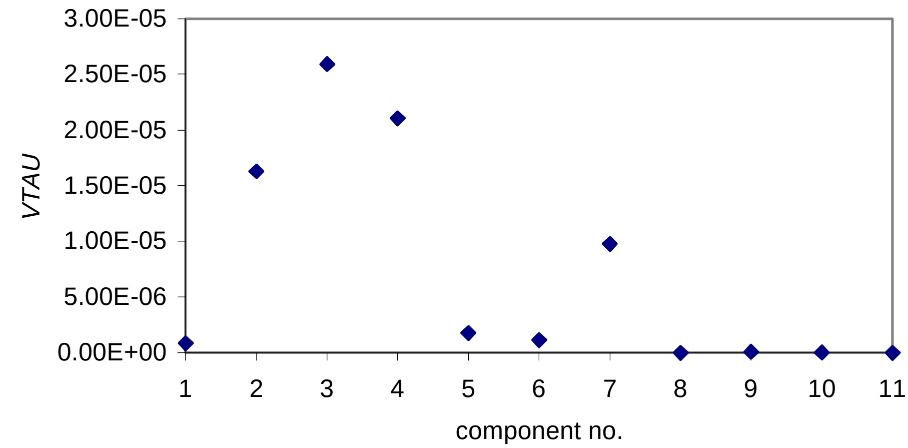

Figure 4. *VTAU* for Different Components

Alternatively, consider component no. 4 from the figures. For this component, *VPI* is high but *VPR* is relatively low. Interpretation of these results are that the unbiased estimate of reliability is satisfactory. That is, improving the reliability of this component has relatively less effect on the lower-bound on system reliability. However, decrease in the estimation variability has greater potential to increase $R_{1-\alpha}(t)$ compared to the other components. It is also noticed that *VTAU* is relatively high for this component. Therefore, there is the potential for the implementation of an efficient testing program.

It is also interesting to note the relationship between the component reliability estimate and VPR_{ij}. Initially, an analyst may think that lower component reliability values would indicate that those components have a greater potential to increase $R_{1-\alpha}(t)$. For this example, the component with the lowest reliability (i=5, j=1, $r_{ij}(t)$=.81) actually has the lowest VPR_{ij}.

Figures 5 presents a graph of *VTAU* vs. *VPI* while Figure 6 presents a graph of *VPI* vs. *VPR*. A graph similar to Figure 6 is useful to identify "problem" components, to prioritize reliability improvement efforts and to prioritize testing. If a component has both high *VPI* and *VPR*, then successful improvement of reliability <u>and</u> the development of an estimate with minimal variability, depends critically on this component. A graph similar to Figure 5 indicates the relative efficiency of the testing activities. A high *VTAU* indicates that the component reliability variance can be reduced with fewer test items. If both *VPI* and *VTAU* are high, then additional testing of the component should be considered a priority.

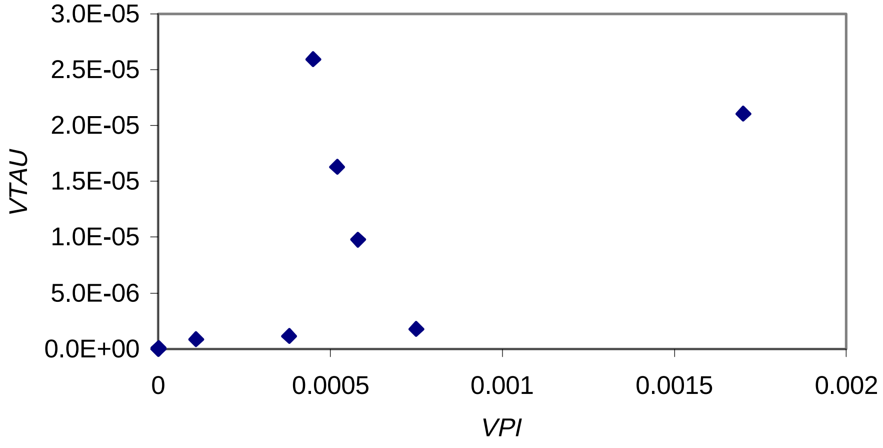

Figure 5. *VPI* and *VTAU* for Example Problem

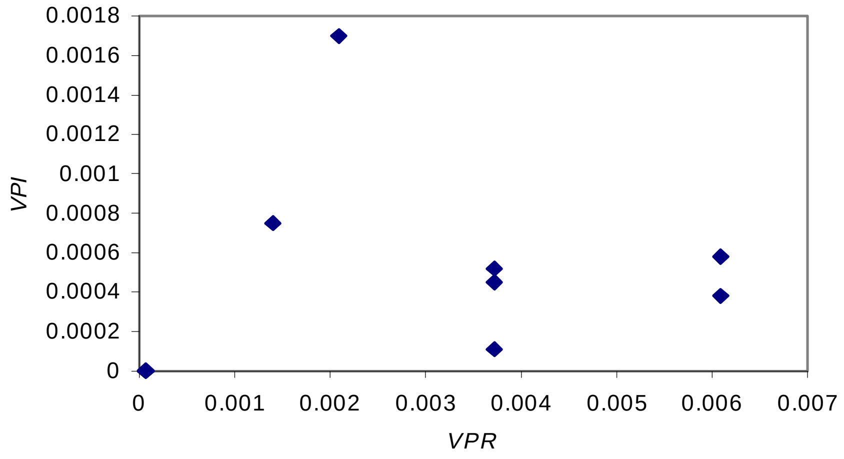

Figure 6. *VPR* vs. *VPI* for Example Problem

5. Conclusions

In a complex system, it is always desirable to improve the system reliability and decrease the reliability estimation variability at the component levels. Due to the complexity of the system configuration, it can be quite hard to find which component has the more significant impact on system reliability. Because testing resources and budgets are always limited for the design of a certain system, the designer always faces the problem of which components to select for additional testing or for devoting additional effort for reliability improvement.

In this paper, three sensitivity measures are presented for the system reliability analysis. VPR_{ij} is a component reliability sensitivity measure, while the VPI_{ij} and $VTAU_{ij}$ can be categorized as component variance sensitivity measures. VPR_{ij} defines the upper limit of the system reliability improvement given that specific component has perfect reliability. Reliability improvement is indicated by the change in a lower-bound estimate on system reliability. VPI_{ij} defines the maximum improvement of the system reliability lower-bound when the component reliability estimate is known with certainty, i.e., zero variance. In practice, this situation is unlikely because the variance approaches zero as the testing sample size approaches infinity. However, the VPI_{ij} index does give provide information for the improvement of system reliability by decreasing the component variability by allocating more testing units. The $VTAU_{ij}$ then gives the system reliability improvement when one more test unit is allocated for that a specific component.

References

1. Gertsbakh, I. B., *Statistical Reliability Theory*, 1989, Marcel Dekker.
2. Gertsbakh, I. B., "Confidence Limits for Highly Reliable Coherent Systems with Exponential Distributed Lifetimes," *Journal of American Statistical Association*, vol. 77, 1982, pp. 673-678.
3. Gnedenko, B. V., *Mathematical Models of Reliability Theory*, 1969, Academic Press.
4. Pavlov, I. V., "Interval Estimation of Quasi-Convex Functions in Reliability Problem," *Engineering Cybernetics*, vol. 17, no. 3, 1979.
5. Ushakov, I. A. (ed.), *Handbook of Reliability Engineering*, 1996, John Wiley & Sons.
6. Easterling, Robert G., "Approximate Confidence Limits for System Reliability of Series System," *Journal of the American Statistical Association*, vol. 67, 1972, pp. 220-222.
7. Mann, R., Schafer, R. E., Singpurwalla, N. D., *Methods for Statistical Analysis of Reliability & Life Data*, 1970, John Wiley & Sons.
8. Gandini, A., "Importance and Sensitivity Analysis in Assessing System Reliability," *IEEE Transaction on Reliability*, vol. 39, no. 1, Apr 1990, pp. 61-70.
9. Melo, A. G., Pereira, M. F., "Sensitivity Analysis of Reliability Indices with Respect to Equipment Failure and Repair Rates," *IEEE Transaction on Power Systems*, vol. 10, no. 2, May 1995, pp. 1014-1021.
10. Coit, D. W. "System-Reliability Confidence Intervals for Complex Systems With Estimated Component Reliability," *IEEE Transactions on Reliability*, vol. 46, no. 4, December 1997, pp. 487-493.