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TEST PLAN ALLOCATION TO MINIMIZE SYSTEM RELIABILITY ESTIMATION VARIABILITY

JOSE E. RAMIREZ-MARQUEZ

Department of Systems Engineering and Engineering Management,
Stevens Institute of Technology, Hoboken, NJ, USA

DAVID W. COIT

Department of Industrial & Systems Engineering,
Rutgers University, Piscataway, NJ, USA
coit@rci.rutgers.edu

TONGDAN JIN

Teradyne, Inc., Boston, MA, USA

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A new methodology is presented to allocate testing units to the different components within a system when the system configuration is fixed and there are budgetary constraints limiting the amount of testing. The objective is to allocate additional testing units so that the variance of the system reliability estimate, at the conclusion of testing, will be minimized. Testing at the component-level decreases the variance of the component reliability estimate, which then decreases the system reliability estimate variance. The difficulty is to decide which components to test given the system-level implications of component reliability estimation. The results are enlightening because the components that most directly affect the system reliability estimation variance are often not those components with the highest initial uncertainty. The approach presented here can be applied to any system structure that can be decomposed into a series-parallel or parallel-series system with independent component reliability estimates. It is demonstrated using a series-parallel system as an example. The planned testing is to be allocated and conducted iteratively in distinct sequential testing runs so that the component and system reliability estimates improve as the overall testing progresses. For each run, a nonlinear programming problem must be solved based on the results of all previous runs. The testing allocation process is demonstrated on two examples.

Keywords: Optimal test allocation; system reliability variance; series-parallel system.

1. Introduction

System reliability estimation is generally based on a system model and reliability estimates at the subsystem and/or component levels. In most cases, these reliability estimates are based on limited component data, and it is desirable to increase the amount of available data to yield reliability estimates with lower variability at both the component-level and system-level. It is desirable, but often unrealistic to allocate numerous testing units or plan a lengthy testing schedule. This is particularly true for a new system or product. For systems where a new design or architecture has already been determined, deciding the number of components to be tested generally involves trade-offs and compromises. A strategy to efficiently allocate testing units based on system-level effects is required.

Component and system-level reliability testing are conducted for a variety of reasons:

(1) To identify design flaws and to improve the design reliability
(2) To evaluate the system reliability compared to a quantitative requirement
(3) To provide an accurate measure of reliability that can be later used for maintenance planning, warranty assessments, safety hazards analyses, etc.

The methodology presented in this paper pertains to the third reason. From a general perspective, this methodology solves the problem of deciding, for a specific fixed system architecture, which and how many components should be tested in order to reduce the variance at the system level. The methodology addresses the difficulties in allocating component test sample sizes once a system design has been determined and a tentative decision to produce a system has been made.

If reduced variability of the estimate of system reliability is of primary importance, then one strategy is to simply wait until the system is designed and manufactured and conduct system-level testing. However, this is not always possible or efficient, particularly for large systems where the various subsystems and components are designed by different teams and perhaps even in different geographical locations. Additionally, component-level testing can be started earlier, thereby offering greater potential efficiencies.

At the system-level, reliability estimation variability is associated with the propagation of component-level uncertainties. The propagation of variability depends on

(1) the system structure (e.g., series-parallel),
(2) the component reliability estimates, and
(3) the variance of those reliability estimates.

It is beneficial to allocate testing units in accordance with the relative impact at the system-level if an accurate system reliability estimate is required.

In general, most existing test allocation strategies pertain to one of two areas:

- Component-level testing without regard to system-level implications.
- Component-level testing used to demonstrate system-level quantitative reliability requirements.

If component-level tests are conducted to demonstrate system reliability, then there are constraints that must be satisfied for Type I and Type II error, but the system-reliability estimation variance is not directly considered. These plans are preferable and recommended in cases where a vendor or contractor must demonstrate system reliability to satisfy a customer's specifications (internal or external), or in cases where there is an explicit and meaningful reliability level that must be achieved for the system to be successful.

The problem addressed here is fundamentally different; it is of interest to provide a sound estimate of system reliability through component testing. In this paper, the methodology focuses on presenting an optimal component-testing plan that considers how the variability at the component level influences the variability at the system level. The objective of testing is to provide a system reliability estimate with minimum estimation variability.

2. Component Test Plan Research

Easterling *et al.*[1] designed a test plan for a series system of independent components based on the binomial component data. The objective was to minimize the testing cost subject to a reliability constraint. They evaluate the test plan by considering the system reliability operating characteristic (OC) envelope by pre-assigning an acceptable reliability threshold. For this case, an equal numbers of test units for each component yields the minimum cost.

An extension of Easterling *et al.*,[1] presented by Sando and Fujii[2] propose an optimal life test with Type I censoring. Based on a Bayesian approach, they derived an optimal test plan to allocate the number of test items and the censoring time. To extrapolate based on lifetime distributions, they assume the Weibull distribution with known shape parameter, and the scale parameter is estimated from the testing data.

Willits *et al.*[3] describes series system reliability estimation based on component-level binomial data. They used a Bayesian approach to estimate the posterior reliability distribution of small binomial data sets through the preassumed beta prior reliability distribution. A comparison between the Bayes method and the classical Lindstrom-Madden was conducted by Monte Carlo simulation. It was concluded that the Bayes method has no strong advantage if the sample data set is small.

There has been significant literature describing component level test plans and sample size determination without regard to system-level implications. A summary of these is provided by Nelson.[4]

There has only been limited research pertaining to the propagation of component-level reliability estimation uncertainty to the system-level. Coit[5] derived an approximate confidence interval for system reliability applicable to systems that can be decomposed to an equivalent series-parallel system. The system reliability confidence interval is derived based on component reliability estimates and the estimate variances. This model is general and has many practical advantages. Jin and Coit[6] extended this work to yield the system reliability estimate variance when there are arbitrarily repeated components used within the system, and thus, the subsystem reliability estimates may not be independent. Coit and Jin[7] and

Coit[8] present a method that prioritizes system-reliability prediction. These studies, present methods that can be used to allocate limited testing and analysis resources efficiently. An index is defined to provide a relative ranking of components based on their potential for improving the accuracy of a system-level reliability prediction by decreasing the variance of the system-reliability estimate. The ranking is based on whether a decrease of the component-reliability estimate variance meaningfully decreases the system-reliability estimate variance.

The development of test plans to minimize system reliability estimation variance has not been adequately considered, although this is a common system design problem in practice. When there are not clearly defined or meaningful system quantitative requirements, then a logical objective is to minimize the system reliability estimate variance. The algorithm presented in this paper addresses this need and allows for the development of efficient testing programs.

3. Variance of System Reliability Estimate

An exact expression for system reliability estimate variance can be determined if the component reliability estimates are independent and the system can be decomposed into an equivalent series-parallel system.[5] An approximation of the variance can be computed when available data is used to estimate component reliability values.

Estimation of component reliability and variance are often based on binomial data. For the ith component used in the system, consider that n_i components were each put on test for t hours and f_i failures are observed. The status of each component (survival/failure) is considered as an independent Bernoulli trial with parameter $r_i(t)$. An unbiased estimate of $r_i(t)$ and an approximation of the variance of the estimate are determined from the binomial distribution using the following well-known equations.

$$\hat{r}_i(t) = 1 - \frac{f_i}{n_i}$$

$$\mathrm{Var}(\hat{r}_i(t)) = \frac{r_i(t)(1 - r_i(t))}{n_i}$$

$$\hat{\mathrm{Var}}(\hat{r}_i(t)) = \frac{(1 - \frac{f_i}{n_i})\frac{f_i}{n_i}}{n_i}$$

where,

n_i = testing sample size for the ith component.
f_i = number of failures for the ith component.
$r_i(t)$ = reliability of ith component (an unknown constant)
$\hat{r}_i(t)$ = estimate of reliability of ith component
$\mathrm{Var}(\hat{r}_i(t))$ = variance of the reliability estimate for ith component
$\hat{\mathrm{Var}}(\hat{r}_i(t))$ = estimate of $\mathrm{Var}(\hat{r}_i(t))$

For any system that can be decomposed into an equivalent series-parallel system, the variance of the system reliability estimate can be obtained using a decomposition methodology. The algorithm presented in this paper pertains to series-parallel systems, but it can be readily adapted to other system configurations as well. The variance of the system reliability estimate is given below for series-parallel (s-p) systems. The first expression is directly from Coit[5] and the second expression is simplified.

$$\begin{aligned}\mathrm{Var}(\hat{R}_{s\text{-}p}(t)) &= \prod_{i=1}^{m}\left(\left(1-\prod_{j=1}^{s_i}(1-r_{ij}(t))\right)^2 + \prod_{j=1}^{s_i}((1-r_{ij}(t))^2+\mathrm{Var}(\hat{r}_{ij}(t)))\right.\\ &\quad \left. - \prod_{j=1}^{s_i}(1-r_{ij}(t))^2\right) - \prod_{i=1}^{m}\left(1-\prod_{j=1}^{s_i}(1-r_{ij}(t))\right)^2\\ \mathrm{Var}(\hat{R}_{s\text{-}p}(t)) &= \prod_{i=1}^{m}\left(1-2\prod_{j=1}^{s_i}q_{ij}(t)+\prod_{j=1}^{s_i}(q_{ij}(t)^2+\mathrm{Var}(\hat{r}_{ij}(t)))\right)\\ &\quad - \prod_{i=1}^{m}\left(1-\prod_{j=1}^{s_i}q_{ij}(t)\right)^2 \qquad (1)\end{aligned}$$

where,

$r_{ij}(t)$ = reliability of jth component in ith subsystem at time t
$q_{ij}(t)$ = $1 - r_{ij}(t)$
m = total number of subsystems
s_i = total number of components in subsystem i, where $i = 1, 2, \ldots, m$

If the reliability of each component is estimated from binomial data, then the variance expression becomes,

$$\begin{aligned}\mathrm{Var}(\hat{R}_{s\text{-}p}(t)) &= \prod_{i=1}^{m}\left(1-2\prod_{j=1}^{s_i}q_{ij}(t)+\prod_{j=1}^{s_i}\left(q_{ij}(t)^2+\frac{r_{ij}(t)q_{ij}(t)}{n_{ij}}\right)\right)\\ &\quad - \prod_{i=1}^{m}\left(1-\prod_{j=1}^{s_i}q_{ij}(t)\right)^2 \qquad (2)\end{aligned}$$

where, n_{ij} = units tested for jth component in subsystem i

4. Problem Formulation

The problem is to iteratively allocate the number of test units for each component, in a sequence of testing runs, to minimize the system reliability estimate variance.

For each testing run, tests are conducted, failures are observed, and component and system reliability estimates are updated. Thus, each sequential testing run relies on improved reliability information.

The problem is formulated with an overall cost constraint or testing budget of C. The testing is to be conducted with $g+1$ distinct runs, and the testing budget, C, is allocated into budgets for the individual runs, C_k, such that $C_0+C_1+C_2+\cdots+C_g = C$. There is a cost associated with testing each component. The cost of testing multiple units of a particular component is proportional to the number of units tested. The unit test cost may be different for different components because of the needed test fixtures, the component cost and the test facility cost.

4.1. *Determination of testing budget for initial testing run*

C_0 represents the budget allocated to conduct testing at the preliminary run. For some components in the system design, there may be plentiful data already available, but for others, there may be very little or even no data available. Therefore, the initial budget, C_0, is devoted to obtaining initial component reliability estimates that can be used in the subsequent algorithm steps to reduce system reliability variance. If sound reliability estimates are not already available, then the budget, C_0, needs to represent a meaningful amount of the overall testing budget, C, so that reasonable and sound initial estimates are available. It is recommended that 40 to 50% of the initial budget be allocated to run 0. This can be reduced greatly if there are already some initial data available to estimate component reliability. The remaining budget $(C - C_0)$ can be equally allocated among the g testing runs.

To conduct the component testing in the preliminary run, within the testing budget of C_0, there are four general alternatives to select component test sample sizes.

(1) Allocate sample sizes equally to each component until the budget is relinquished.
(2) Allocate an equal testing cost to each component (C_0 divided by the total number of components).
(3) Determine initial allocation by adjusting equal allocation results (1 or 2 above) based on all available quantitative or qualitative data indicating the anticipated reliability of the components.
(4) Determine allocation based on assumed Bayesian prior distributions of the component reliability estimates.

If the component reliability values are completely unknown, then options 1 or 2 above are recommended. However, normally this is not the case because there is usually some data, even judgmental or qualitative data, to indicate the relative reliability values of the components. It is suggested that a larger proportion of the initial C_0 be allocated to the components with higher anticipated reliability because it is more difficult to estimate reliability when the likelihood of observing a failure is less. Also, if there is plentiful previous data or valid expert opinion, then Bayesian prior distributions for component reliability can be determined using an assumed beta distribution. Then, the C_0 allocation can be made based on minimizing the expected variance of the posterior distribution of system reliability. For the remainder of the paper, it is assumed that options 1, 2 or 3 are selected.

Once the initial allocations are made, testing is conducted and failures are observed. After the preliminary run testing is conducted, component reliability estimates are obtained. However, it must be recognized that these estimates are subject to error. By testing in runs, the variability of the estimates is iteratively reduced, enhancing both the reliability estimates of the components and the allocations for the next run.

4.2. *System reliability variance reduction problem*

Once the preliminary estimates are made, a test allocation methodology is sequentially applied. As the actual testing of components is performed, the reliability estimates are updated and improved. For those components where no additional test units are assigned after the preliminary or some later run, it can be concluded that the system reliability estimate variance is not sensitive to uncertainty at the component level.

A general formulation for the kth run of the testing allocation problem is presented below as Problem SPK. The decision variables, x_{ijk}, are limited to non-negative integers.

<u>Problem SPK:</u>

$$\min_{\mathbf{x}_k} \hat{\mathrm{Var}}(\hat{R}_{s\text{-}p}(t)) = \prod_{i=1}^{m}\left(1 - 2\prod_{j=1}^{s_i}\hat{q}_{ijk}(t) + \prod_{j=1}^{s_i}\left(\hat{q}_{ijk}(t)^2 + \frac{\hat{r}_{ijk}(t)\hat{q}_{ijk}(t)}{\sum_{l=0}^{k-1} x_{ijl} + x_{ijk}}\right)\right) - \prod_{i=1}^{m}\left(1 - \prod_{j=1}^{s_i}\hat{q}_{ijk}(t)\right)^2$$

subject to:

$$\sum_{i=1}^{m}\sum_{j=1}^{s_i} c_{ij}x_{ijk} \le C_k \quad x_{ijk} \in \{0, 1, 2, \ldots,\} \forall\ i, j, k$$

where

$\mathbf{x}_k$ $= (x_{11k}, x_{12k}, \ldots, x_{1s_1k}, x_{21k}, x_{22k}, \ldots, x_{2s_2k}, x_{31k}, x_{32k}, \ldots, x_{ms_mk})$

x_{ijk} = number of units to test of the jth component in subsystem i in run k

$\hat{r}_{ijk}(t) = 1 - \sum_{l=0}^{k-1} f_{ijl} / \sum_{l=0}^{k-1} x_{ijl}$
$\hat{q}_{ijk}(t) = 1 - \hat{r}_{ijk}$
c_{ij} = unit testing cost for jth component in subsystem i
C_k = maximum allowable testing cost for run k.

In the Problem SPK objective function, $\hat{r}_{ijk}(t)$ and $\hat{q}_{ijk}(t)$ are used to approximate $r_{ij}(t)$ and $q_{ij}(t)$. An approximation for the system reliability estimate variance is then used as a surrogate in the problem formulation.

4.3. *Problem transformation*

Direct solution of Problem SPK would be preferable, but it is generally not practical or efficient to solve directly. When the product terms in the Problem SPK objective function are expanded, the number of terms can present a problem even for relatively small systems. However, the problem can be rewritten in a simpler, equivalent form.

Consider the objective function for Problem SPK. The last term in Eq. (1) is subtracted from the remainder, and it does not include any decision variables (x_{ijk}). A modified equivalent objective function can be written as,

$$\gamma_k(\mathbf{x}_k) = \prod_{i=1}^{m}(1 - 2Q_{ik} + I_{ik})$$

where,

$$I_{ik} = \prod_{j=1}^{s_i}\left(\hat{q}_{ijk}(t)^2 + \frac{\hat{r}_{ijk}(t)\hat{q}_{ijk}(t)}{\sum_{l=0}^{k} x_{ijl}}\right)$$

$$Q_{ik} = \prod_{j=1}^{s_i}\hat{q}_{ijk}(t)$$

k = index corresponding to run k.

When expanded, the modified objective function still contains terms that can be discarded, for optimization purposes, because they are additive terms that do not depend on the decision variables. Let $\Lambda_m(\mathbf{x}_k)$ represent all terms necessary for optimization purposes. Consider the following cases:

Case $m = 1$:

$$\gamma_k(\mathbf{x}_k) = (1 - 2Q_{1k} + I_{1k}),$$

then

$$\Lambda_1(\mathbf{x}_k) = I_{1k}.$$

Case $m = 2$:

$$\gamma_k(\mathbf{x}_k) = (1 - 2Q_{1k} + I_{1k})(1 - 2Q_{2k} + I_{2k}),$$

then,

$$\Lambda_2(\mathbf{x}_k) = I_{1k}I_{2k} + (I_{1k} + I_{2k}) - 2(Q_{1k}I_{2k} + Q_{2k}I_{1k})\,.$$

A general form of the expression for $\Lambda_m(\mathbf{x}_k)$ can serve as the objective function in a nonlinear programming optimization problem. A general form follows for the case with $m = l$.

Case $m = l$:

$$\gamma_k(\mathbf{x}_k) = \prod_{i=1}^{l}(1 - 2Q_{ik} + I_{ik})$$

and,

$$\begin{aligned}\Lambda_l(\mathbf{x}_k) = &\left[\prod_{i=1}^{l} I_{ik} + \sum_{j=1}^{l}\frac{\prod_{i=1}^{l} I_{ik}}{I_{jk}} + \sum_{j=1}^{l}\sum_{\substack{u=1\\u\neq j}}^{l}\frac{\prod_{j=1}^{l} I_{ik}}{I_{jk}I_{uk}} + \cdots + \sum_{i=1}^{l} I_{ik}\right]\\ &-2\left[\sum_{j=1}^{l} Q_{jk}\frac{\prod_{i=1}^{l} I_{ik}}{I_{jk}} + \sum_{j=1}^{l}\sum_{\substack{j=1\\u\neq j}}^{l} Q_{jk}\frac{\prod_{i=1}^{l} I_{ik}}{I_{jk}I_{uk}} + \cdots + \sum_{j=1}^{l} Q_{jk}\sum_{i=1}^{l} I_{ik}\right]\\ &+(-2)^2\left[\sum_{j=1}^{l}\sum_{\substack{u=1\\u\neq j}}^{l} Q_{jk}Q_{uk}\frac{\prod_{i=1}^{l} I_{ik}}{I_{jk}I_{uk}} + \cdots + \sum_{j=1}^{l}\sum_{\substack{u=1\\u\neq j}}^{l} Q_{jk}Q_{uk}\sum_{\substack{i\neq 1\\i\neq u}} I_{ik}\right]\\ &+\cdots+(-2)^{l-1}\left[\sum_{j=1}^{l} I_{jk}\frac{\prod_{i=1}^{l} Q_{ik}}{Q_{jk}}\right].\end{aligned}$$

5. Solution Methodology

The testing is conducted in distinct runs. As discussed in Sec. 3.1, Run 0, is devoted to determine initial component reliability estimates while subsequent phases are conducted in accordance with the solutions obtained to minimize the system reliability estimate variance. If sufficient data were available from previous testing or field experience, then C_0 can be set to zero and the procedure can be initiated in Run 1. The methodology allocates testing units in an efficient schedule and follows the subsequent steps.

Step 1.

- Allocate budget C to each of the runs such that $C_0 + C_1 + C_2 + \cdots + C_g = C$.
- Allocate budget C_0 to determine n_{ij0} (see Sec. 3.1) conforming with:

$$\sum_{i=1}^{m}\sum_{j=1}^{s_i} c_{ij}x_{ij0} \le C_0$$

and

$$x_{ij0} > 0 \; \forall \; i, j \, .$$

- For each component, test n_{ij0} for time t and record failures and survivals.
- Set $k = 1$.

Step 2.

- After testing is completed compute:

$$\hat{r}_{ijk}(t) = 1 - \sum_{l=0}^{k-1} f_{ijl} \Bigg/ \sum_{l=0}^{k-1} x_{ijl}$$

$$\hat{q}_{ijk}(t) = 1 - \hat{r}_{ijk} \, .$$

Step 3.

- If $k = g$ stop, the test allocation methodology has been completed.
- Otherwise, solve the following problem (using a nonlinear programming solver, e.g., LINGO) or equivalent:

$$\min_{\mathbf{x}_k} \Lambda_m(\mathbf{x}_k)$$

subject to:

$$\sum_{i=1}^{m} \sum_{j=1}^{s_i} c_{ij} x_{ijk} \leq C_k \quad x_{ijk} \in \{1, 2, 3, \ldots\}$$

- Obtain $\mathbf{x}_k^*$ and go to Step 4.

Step 4.

- Conduct component testing according to the optimal solution $\mathbf{x}_k^*$.
- Observe and record the number of failures (f_{ijk}) and survivals ($x_{ijk} - f_{ijk}$).
- Set $k = k + 1$.
- Go to Step 2.

6. Numerical Example

Two example series-parallel systems were subject to the proposed methodology. Example 1, consists of four subsystems connected in series; it is depicted in Fig. 1. Each subsystem has two to four components in parallel. A total budget of 1200 cost units was available and a four-run test schedule was planned. An initial budget of $C_0 = 600$ was allocated to obtain initial reliability estimates. The initial test allocation was determined subjectively (see Sec. 3.1) based on prior information regarding the component reliability. The remainder of the budget was allocated in equal amounts for runs 1, 2 and 3.

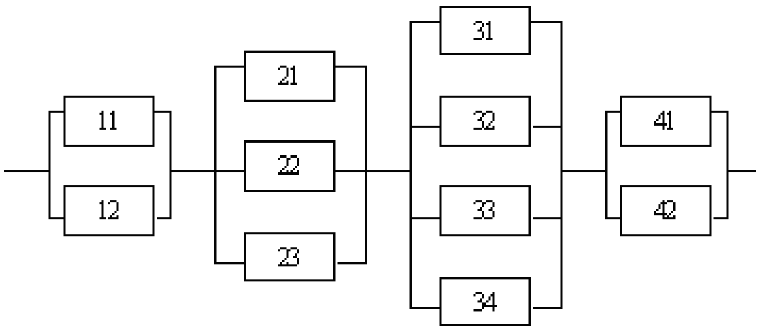

Fig. 1. Example 1 system.

Table 1. Example 1 Unit test cost & cumulative units tested.

subsystem	component	unit test cost	Cumulative Number of Components Tested at the End of Each Run			
i	j	c_{ijk}	Run 0	Run 1	Run 2	Run 3
1	1	2	50	50	81	81
	2	1	30	159	159	159
2	1	2	25	25	49	49
	2	2	40	40	40	40
	3	0.5	25	53	61	61
3	1	2	20	20	20	20
	2	1.5	40	40	40	40
	3	0.5	35	37	37	37
	4	2	40	40	40	40
4	1	2	45	45	80	180
	2	1	40	96	112	112
Cumulative Test Cost			600	800	1000	1200

Table 1 presents the component testing costs for each of the components. Moreover, it presents the cumulative number of components and the cumulative test cost after the end of each run. Potentially, each of the components can be tested at each run. The total number of components tested using the proposed methodology can be obtained by referring to the last column of Table 1. Table 2 presents example test data observed at each run and the updated reliability estimates after each test has been completed. To simulate the testing for each run, Monte Carlo simulation was used. In practice, actual testing would be conducted.

Example 2 consists of four subsystems connected in series. Each subsystem has two components connected in parallel. The system is shown in Fig. 2. A budget of 1250 was allocated to a four run scheduling plan. An initial budget of 500 was used to obtain preliminary component reliability estimates and each of the remaining runs was assigned 250 as the testing budget. Table 3 presents the component

Table 2. Example 1 Components testing schedule & estimates by run.

Component Data			Run 0				Run 1			
sub system i	component j	unit test cost c_{ij}	no. tested n_{ij}	no. survivals $n_{ij} - f_{ij}$	$\hat{r}_{ij}(t)$	$\hat{\mathrm{Var}}(\hat{r}_{ij}(t))$	no. tested n_{ij}	no. survivals $n_{ij} - f_{ij}$	$\hat{r}_{ij}(t)$	$\hat{\mathrm{Var}}(\hat{r}_{ij}(t))$
1	1	2	50	46	0.920	0.00147	0	0	0.920	0.00147
	2	1	30	28	0.933	0.00208	129	119	0.925	0.00044
2	1	2	25	20	0.800	0.00640	0	0	0.800	0.00640
	2	2	40	34	0.850	0.00319	0	0	0.850	0.00319
	3	0.5	25	23	0.920	0.00294	28	22	0.849	0.00242
3	1	2	20	18	0.900	0.00450	0	0	0.900	0.00450
	2	1.5	40	36	0.900	0.00225	0	0	0.900	0.00225
	3	0.5	35	30	0.860	0.00344	2	2	0.865	0.00316
	4	2	40	37	0.925	0.00173	0	0	0.925	0.00173
4	1	2	45	42	0.930	0.00145	0	0	0.930	0.00145
	2	1	40	37	0.925	0.00168	56	51	0.917	0.0080

Component Data			Run 2				Run 3			
sub system i	component j	unit test cost c_{ij}	no. tested n_{ij}	no. survivals $n_{ij} - f_{ij}$	$\hat{r}_{ij}(t)$	$\hat{\mathrm{Var}}(\hat{r}_{ij}(t))$	no. tested n_{ij}	no. survivals $n_{ij} - f_{ij}$	$\hat{r}_{ij}(t)$	$\hat{\mathrm{Var}}(\hat{r}_{ij}(t))$
1	1	2	31	29	0.926	0.00085	0	0	0.926	0.00085
	2	1	0	0	0.925	0.00044	0	0	0.925	0.00044
2	1	2	24	18	0.776	0.00355	0	0	0.776	0.00355
	2	2	0	0	0.850	0.00319	0	0	0.850	0.00319
	3	0.5	8	6	0.836	0.00225	0	0	0.836	0.00225
3	1	2	0	0	0.900	0.00450	0	0	0.900	0.00450
	2	1.5	0	0	0.900	0.00225	0	0	0.900	0.00225
	3	0.5	0	0	0.865	0.00316	0	0	0.865	0.00316
	4	2	0	0	0.925	0.00173	0	0	0.925	0.00173
4	1	2	35	31	0.913	0.00100	100	90	0.906	0.00048
	2	1	16	15	0.920	0.00066	0	0	0.920	0.00066

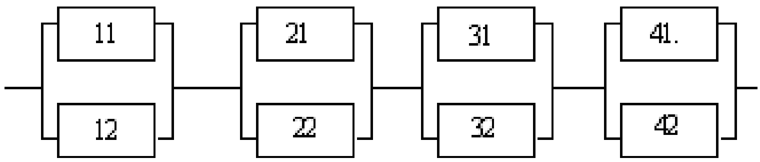

Fig. 2. Example 2 system.

Table 3. Example 2 Unit test cost & cumulative units tested.

subsystem	component	unit test cost	Cumulative Number of Components Tested at the End of Each Run			
i	j	c_{ij}	Run 0	Run 1	Run 2	Run 3
1	1	2	50	50	50	50
	2	1	30	30	41	41
2	1	2	25	94	143	245
	2	2	40	93	151	151
3	1	2	20	23	34	57
	2	1.5	40	40	40	40
4	1	2	45	45	46	46
	2	1	50	50	51	51
Cumulative Test Cost			500	750	1000	1250

testing costs for each of the components and it describes the cumulative number of components and the cumulative test cost after the end of each run. Table 4 presents the optimal scheduling plan at each run and the updated reliability estimates after the test has been completed.

It is important to note that, after the methodology is completed, the variance of the components reliability estimate is considerably reduced, and thus, component reliability estimates are enhanced. From Table 4, consider as examples the variance for the reliability estimates of components x_{12}, x_{21}, x_{31} and x_{41}, (i.e., component 2 in subsystem 1, component 1 in subsystem 2, component 1 in subsystem 3 and component 1 in subsystem 4) as each run testing progresses. For these components the preliminary reliability estimates were 0.933, 0.8, 0.9 and 0.93 respectively. After the end of run 3, the reliability estimates for these components were updated to 0.925, 0.776, 0.947 and 0.906. Notice that after testing has been completed, the variability of reliability estimates has been considerably reduced. Thus, component and system reliability estimates become statistically robust when the proposed methodology is applied.

The methodology was implemented using commercially available nonlinear solver (LINGO) software on a Pentium laptop. The problems are solved in an efficient way; solutions are obtained in a matter of seconds.

Table 4. Example 2 Components testing schedule & estimates by run.

Component Data			Run 0				Run 1			
sub system i	component j	unit test cost c_{ij}	no. tested n_{ij}	no. survivals $n_{ij} - f_{ij}$	$\hat{r}_{ij}(t)$	$\hat{\text{Var}}(\hat{r}_{ij}(t))$	no. tested n_{ij}	no. survivals $n_{ij} - f_{ij}$	$\hat{r}_{ij}(t)$	$\hat{\text{Var}}(\hat{r}_{ij}(t))$
1	1	2	50	47	0.940	0.00113	0	0	0.940	0.00113
	2	1	30	28	0.933	0.00207	0	0	0.933	0.00207
2	1	2	25	20	0.800	0.00640	69	54	0.787	0.00178
	2	2	40	33	0.825	0.00361	53	42	0.806	0.00168
3	1	2	20	18	0.900	0.00450	3	3	0.913	0.00345
	2	1.5	40	38	0.950	0.00119	0	0	0.950	0.00119
4	1	2	45	41	0.911	0.00180	0	0	0.911	0.00180
	2	1	50	48	0.960	0.00077	0	0	0.960	0.00077

Component Data			Run 2				Run 3			
sub system i	component j	unit test cost c_{ij}	no. tested n_{ij}	no. survivals $n_{ij} - f_{ij}$	$\hat{r}_{ij}(t)$	$\hat{\text{Var}}(\hat{r}_{ij}(t))$	no. tested n_{ij}	no. survivals $n_{ij} - f_{ij}$	$\hat{r}_{ij}(t)$	$\hat{\text{Var}}(\hat{r}_{ij}(t))$
1	1	2	0	0	0.940	0.00113	0	0	0.940	0.00113
	2	1	11	11	0.951	0.00113	0	0	0.951	0.00113
2	1	2	49	43	0.818	0.00104	102	85	0.824	0.00059
	2	2	58	44	0.788	0.00111	0	0	0.788	0.00111
3	1	2	11	11	0.941	0.00163	23	22	0.947	0.00087
	2	1.5	0	0	0.950	0.00119	0	0	0.950	0.00119
4	1	2	1	1	0.913	0.00173	0	0	0.913	0.00173
	2	1	1	1	0.961	0.00074	0	0	0.961	0.00074

7. Conclusion

Due to the limited testing resources, allocation of testing units becomes increasingly important for competitive industries. A methodology has been described and demonstrated to allocate test units to minimize an approximation of the variance of system reliability estimate. The methodology allocates testing units in an efficient schedule. Moreover, by testing in sequential runs, the variability of the estimates is reduced, enhancing the reliability estimates of the components. Given that the proposed methodology uses nonlinear integer programming tools, the solution cannot be claimed to be the global optimum solution, although it provides a good approximation to such optimum.

This approach was demonstrated on series-parallel systems. It is also applicable to other complex systems as long as they can be decomposed into a series-parallel or parallel-series systems.

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About the Authors

Jose E. Ramirez-Marquez is an Assistant Professor in the Department of Systems Engineering and Engineering Management at Stevens Institute of Technology. He was a doctoral student at Rutgers University, where he was responsible for conducting research related to the development of analytical methodologies for improving system reliability performance and mathematical approaches for computing system reliability when considering multiple performance levels for both systems and components. He received his B.S. degree in Actuarial Science from the UNAM in Mexico City in 1998 and M.S. degrees in Industrial Engineering and Statistics from Rutgers University. His areas of interest include system reliability and quality assurance, applied probability and statistical models and applied operations research.

David W. Coit is an Associate Professor in the Department of Industrial & Systems Engineering at Rutgers University. He received a B.S. degree in mechanical engineering from Cornell University, an MBA from Rensselaer Polytechnic Institute and M.S. and Ph.D. in industrial engineering from the University of Pittsburgh. He also has over ten years of experience working for IIT Research Institute (IITRI), Rome NY where he was a reliability analyst and project manager, and in his final position, the Manager of Engineering at IITRI's Assurance Technology Center. In 1999, he was awarded a CAREER grant from NSF to study reliability optimization. His current research involves reliability prediction and optimization, risk analysis, and multi-criteria optimization considering uncertainty. He is a member of IIE, INFORMS and IEEE.

Tongdan Jin is currently working as a reliability engineer at Digital Process Lab at Teradyne to improve field semiconductor test equipment reliability via corrective actions and accelerated life testing. He obtained his Ph.D. from the Industrial and Systems Engineering Department at Rutgers University in 2001. He received his M.S. (1994) in electromechanical engineering from Beijing Institute of Technology, specializing in Computer-Aided Design. He received a B.S. (1991) in electrical engineering from Northwest Institute of Light Industry, PR China. His research interests are system and component reliability analysis and prediction, reliability optimization.