Topic 7 - Expected Value

Statistics for Managers

June 3, 1999

Expected Value or Mathematical Expectation

Suppose Y is a random variable with probability distribution function f(y). The mathematical expectation, or expected value, E(Y) is defined as:

$$E(Y) = \sum_{i=1}^{n} y_i f(y_i) = \mu$$

The sum is taken over all values of the random variable.

Example 1. Number of heads in one coin toss

Toss a balanced coin once. Let Y denote the number of heads. Find the expected value of Y.

Expected Value Calculation

y	f(y)	yf(y)
0	½	0
1	½	½
	E(Y) =	**½**

Example 2. Number of heads in two coin tosses

Toss a balanced coin twice. Let Y denote the number of heads. Find the expected value of Y.

Expected Value Calculation

y	f(y)	yf(y)
0	¼	0
1	½	½
2	¼	½
	E(Y) =	**1**

Variance

The variance of the probability mass function *f*(*y*) is

$$\sigma^2 = \sum (y-\mu)^2 f(y)$$

The standard deviation is

$$\sigma = \sqrt{\sum (y-\mu)^2 f(y)}$$

An equivalent formula for the variance

$$\sigma^2 = \sum y^2 f(y) - \mu^2$$

Example 3.: Fair Coin Toss Thrice

y	f(y)	yf(y)	y-μ	$(y-\mu)^2$	$(y-\mu)^2 f(y)$
0	0.125	0	-1.5	2.25	0.28125
1	0.375	0.375	-0.5	0.25	0.09375
2	0.375	0.750	+0.5	0.25	0.09375
3	0.125	0.375	+1.5	2.25	0.28125

$\mu = 1.5$ $\sigma^2 = 0.75$

$\sigma = 0.8660254$

Expected Monetary Value

Suppose the random variable Y represents the monetary profit or loss from an economic decision. Then E(Y) is called the expected monetary value.

Example 4. Number of tosses until the number six comes up

Repeatedly toss a die until the number six comes up. Let Y denote the number of tosses; the player wins Y dollars. Find the expected monetary value of the game.

Probability mass function

$$f(y) = \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{y-1}$$
$$y = 1,2,\ldots$$

Approximate Expected Value Calculation

y	f(y)	yf(y)	y	f(y)	yf(y)	y	f(y)	yf(y)
1	0.1667	0.1667	21	0.0043	0.0913	41	0.0001	0.0046
2	0.1389	0.2778	22	0.0036	0.0797	42	0.0001	0.0040
3	0.1157	0.3472	23	0.0030	0.0694	43	0.0001	0.0034
4	0.0965	0.3858	24	0.0025	0.0604	44	0.0001	0.0029
5	0.0804	0.4019	25	0.0021	0.0524	45	0.0001	0.0025
6	0.0670	0.4019	26	0.0017	0.0454	46	0.0000	0.0021
7	0.0558	0.3907	27	0.0015	0.0393	47	0.0000	0.0018
8	0.0465	0.3721	28	0.0012	0.0340	48	0.0000	0.0015
9	0.0388	0.3489	29	0.0010	0.0293	49	0.0000	0.0013
10	0.0323	0.3230	30	0.0008	0.0253	50	0.0000	0.0011
11	0.0269	0.2961	31	0.0007	0.0218	51	0.0000	0.0009
12	0.0224	0.2692	32	0.0006	0.0187	52	0.0000	0.0008
13	0.0187	0.2430	33	0.0005	0.0161	53	0.0000	0.0007
14	0.0156	0.2181	34	0.0004	0.0138	54	0.0000	0.0006
15	0.0130	0.1947	35	0.0003	0.0119	55	0.0000	0.0005
16	0.0108	0.1731	36	0.0003	0.0102	56	0.0000	0.0004
17	0.0090	0.1532	37	0.0002	0.0087	57	0.0000	0.0003
18	0.0075	0.1352	38	0.0002	0.0074	58	0.0000	0.0003
19	0.0063	0.1189	39	0.0002	0.0064	59	0.0000	0.0003
20	0.0052	0.1043	40	0.0001	0.0054	60	0.0000	0.0002
							E(Y) ≅	**5.9988**

Exact Expected Value Calculation

Preliminary Formulas

$$\sum_{y=0}^{\infty}(1-p)^{y}=\frac{1}{p}$$

$$\sum_{y=1}^{\infty}y(1-p)^{y-1}=\frac{1}{p^{2}}$$

Expected Value Formula

$$E(Y)=\sum_{y=1}^{\infty}yp(1-p)^{y-1}$$

$$=\frac{p}{p^{2}}=\frac{1}{p}$$

For p = 1/6, we have Expected Monetary Value = E(Y) = 1/(1/6) = 6 dollars.

Relation to Present Value

With interest rate r per period, the Present Value of an infinite number of periodic cash payments of one dollar is 1/r dollars.

$$\sum_{y=0}^{\infty}(1-r)^{y}=\frac{1}{r}$$

With the above formula, we see the present value of an infinite number of periodic cash payments of one dollar when the interest rate is 1/6 per period is 6 dollars.

Therefore, this is monetarily equivalent to playing a game that can terminate with probability of p = r, and that pays one dollar for each round played.

Example 5. Decision to build a new facility

A small lumber company is considering building a new sawmill. It is faced with uncertain future economic conditions and it has three alternatives: build a large facility, build a small facility or do nothing. However, the company knows how much money can be made depending on what the economic conditions may be. The various payoffs are displayed in the following table. What should the company do?

Payoff Table with Calculations

		The Future		
		Favorable Market	Unfavorable Market	
		Probability		
		0.5	0.5	
		Payoffs		EMV
Alternatives	Large Facility	200,000	-180,000	10,000
	Small Facility	100,000	-20,000	40,000
	Do Nothing	0	0	0

The company has no real information about the future economy so it assigns a subjective probability 0.5 to each of two possibilities. Using the probability distribution function for the payoffs for each of the alternative decisions, it determined the expected monetary value (EMV) of each set of payoffs. It turns out that building a small facility yields the largest EMV of 40,000 dollars.

Fair or Equitable Game

A gambling game is said to be a fair or equitable game if the expected amount won or lost is equal to zero.

Example 6. Number of tosses until all tails come up

Take ten fair coins; repeatedly toss them until all tails come up. Let Y denote the number of tosses and get back Y dollars. What wager amount would make this a fair game?

First must find the probability that a game will terminate. In this case, this is the probability of getting all tails. By the special multiplication law of probability, the probability, p, that the game will terminate in a given round is

P(Tail, Tail, Tail, Tail, Tail, Tail, Tail, Tail, Tail, Tail) =

P(Tail) P(Tail) P(Tail) P(Tail) P(Tail) P(Tail) P(Tail) P(Tail) P(Tail) P(Tail) =

(1/2)(1/2)(1/2)(1/2)(1/2) (1/2)(1/2)(1/2)(1/2)(1/2)=1/1024

Thus p = 1/1024, and E(Y) = 1/p = 1024. Therefore, the fair wager amount should be 1,024 dollars.

Example 7. Sum of numbers on two dice

Bet 7 dollars and toss a pair of dice. Get back the amount of dollars equal to the sum of numbers on the two dice. Does the wager amount of 7 dollars make this a fair game?

Yes. Let Y denote the amount won or lost after playing the game once. The expected monetary value is E(Y) = 0.

Expected Value Calculation

y	f(y)	yf(y)
-5	1/36	(-5 X 1) / 36 = -5/36
-4	2/36	(-4 X 2) / 36 = -8/36
-3	3/36	(-3 X 3) / 36 = -9/36
-2	4/36	(-2 X 4) / 36 = -8/36
-1	5/36	(-1 X 5) / 36 = -5/36
0	6/36	(0 X 6) / 36 = 0
1	5/36	(1 X 5) / 36 = 5/36
2	4/36	(2 X 4) / 36 = 8/36
3	3/36	(3 X 3) / 36 = 9/36
4	2/36	(4 X 2) / 36 = 8/36
5	1/36	(5 X 1) / 36 = 5/36
	E(Y)=	**0**

Odds and fair wagers

A gambling game pays W to one and the probability of winning is p. What should W be to make this a fair game?

As stated, a gambler bets one dollar in order to win W dollars with probability p. Let Y denote the amount won or lost. The probability distribution function and expected value of Y is:

Expected Value Calculation

y	f(y)	yf(y)
W	p	Wp
-1	1-p	-(1-p)
	E(Y)=	**Wp - (1-p)**

For a fair game, the expected monetary value must be zero:

Wp - (1-p) = 0

W = (1-p)/p

That is, W-to-one are the odds of loosing. In addition, one-to-W are the odds of winning.

Example 8. Horseracing

Suppose a horse pays 20 to 1, which means upon winning it will pay 20 dollars for each one-dollar bet in addition to the original bet. The pay out is determined by the number of people that bet on the same horse. For a fair bet, the odds of winning are one-to-20; i.e. the horse must win with probability 1/21.

For probabilities this small, research has shown that people severely underestimate the true probability of a horse winning due to choosing horses that are more attractive too often. Therefore, in fact, long-shot bets are in favor of the gambler, giving an expected monetary value significantly larger than the bet.

Suppose you went to the horse races and consistently bet the 20-to-one horses, one per race. The apparent probability of wining is 1/21, assume the true probability is 1/20. The expected monetary value is 1/20 = 0.05 dollars. This is better than any casino game.

Expected Value Calculation

y	f(y)	yf(y)
20	1/20	1
-1	19/20	-19/20
	E(Y)=	**1/20**

However, it takes too long win: The expected number of races to the first win is 20 races, about two days worth.

Example 9. American-style roulette

Blaise Pascal may have invented the original roulette game. In American-style roulette, there are 36 numbers, 1- 36, and the 0. Half the numbers are red and half are black. A straight up bet is on a single number and pays 35 to 1. Is this a fair game?

No. The expected monetary value of the game is -1/37 dollars for a one dollar wager.

Expected Value Calculation

y	f(y)	yf(y)
35	1/37	35/37
-1	36/37	-36/37
	E(Y) =	**-1/37**

Homework 1 Roulette

The red or black bet pays 1 to 1. What is its monetary value?

The line bet is on six numbers, pays 5 to 1. What is its monetary value?

Find out one other roulette bet. What is its monetary value?

Example 10. Three-Card Swindle

In many gambling games, trusting your intuition about probability can be disastrous. A simple betting game with three cards proves it.

Three cards are manufactured to special specifications. The first card has a spade on both sides. The second card has a diamond on both sides. The third card has a spade on one side and a diamond on the other.

The banker shakes the cards in a hat and lets you draw one card randomly, putting it on the table. He then bets even money that the underside suit is the same as the top.

To con you into thinking it is a fair game, the banker tells you that your card cannot be the spade-spade card. Therefore, it is either the spade-diamond card or the diamond-diamond card, so you and he have equal chances of winning.

What is the expected monetary value of the game?

Suppose the bet is one dollar and the diamond is the showing suit. Let Y denote the money you win including your bet. Label the sides of the diamond-diamond card A or B. In the following the first symbol is the side that is on top.

Expected Value Calculation

y	Elementary Events	f(y)	yf(y)
1	(♦,♠)	1/3	1/3
-1	(♦A, ♦B) (♦B, ♦A)	2/3	-2/3
		E(Y)=	**-1/3**

Example 11. Chuck-A-Luck Swindle

An old carnival and casino game dating back to the early 1800's, chuck-a-luck appears to be a fair game. To play the game, players bet one dollar and choose a number from one to six; three dice are tossed and the bankers pays a dollar for each die showing the players' number.

If there were only one die, a number has winning probability 1/6. However, with two dice, it seems the number has winning probability 2/6. Even more so, with three dice, it seems the number has winning probability of 3/6, so since the game pays one to one, this seems to be a fair game. However, the banker has the advantage when there are doubles or triples since he must pay back the original bet only once even though the winning number is showing two or three times. When all numbers are different, the banker breaks even.

Suppose there are six players, each choosing a different number. Y represents the banker's wins and losses. For a triplet, say 111, the banker gives back three to the winning player plus her original bet, for a net profit of two dollars. For a double, say 121, the banker gives back two dollars to the double plus one, and one dollar plus one to the single, for a net profit of one dollar. For all singles, say 412, the banker gives back one dollar plus one to each winning player for a net profit of zero. The expected monetary value for the banker is $102/216 \cong 0.47$ dollars for six players, or 0.078 dollars per player. The following table demonstrates the calculations.

Expected Value Calculation

Type of event	y	f(y)	yf(y)	Elementary Events	
Triplet	2	$\frac{6}{6\times6\times6} = \frac{6}{216}$	$\frac{2\times6}{216}$	111 222 333	444 555 666
Double	1	$\frac{6\times5\times3}{6\times6\times6} = \frac{90}{216}$	$\frac{1\times90}{216}$	112 121 211 113 131 311 114 141 411 115 151 511 116 161 611 221 212 122 223 232 322 224 242 422 225 252 522 226 262 622 331 313 133 332 323 233 334 343 433 335 353 533 336 363 633	441 414 144 442 424 244 443 434 344 445 454 544 446 464 644 551 515 155 552 525 255 553 535 355 553 535 355 554 545 455 661 616 166 662 626 266 663 636 366 664 646 466 665 656 566
All Singles	0	$\frac{6\times5\times4}{6\times6\times6} = \frac{120}{216}$	$\frac{0\times120}{216}$	123 124 125 126 132 134 135 136 142 143 145 146 152 153 154 156 162 163 164 165 213 214 215 216 231 233 234 235 241 243 245 246 251 253 254 256 261 263 264 265 312 314 315 316 321 324 325 326 341 342 345 346 351 352 354 356 361 362 364 365	412 413 415 416 421 423 425 426 431 432 435 436 451 452 453 456 461 462 463 465 512 513 514 516 521 523 524 526 531 532 534 536 541 542 543 546 561 562 563 564 612 613 614 615 621 623 624 625 631 632 634 635 641 642 643 645 651 652 653 654
		E(Y) =	$\frac{102}{216}$		