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Introduction

The 1852 paper[1] by Kirchhoff on the equations of equilibrium of an elastic body when the displacements of its points are not infinitesimal occupies an important place in the history of continuum mechanics. It has been cited[2] in connection with the introduction of auxiliary stress tensors related to the reference configuration, the tensors which Truesdell[3] calls the first and second Piola-Kirchhoff tensors. However, Todhunter and Pearson [1893] criticized Kirchhoff's paper, especially the first part, and pointed out this paper was not included in Kirchhoff's collected works. They suggested that Kirchhoff was possibly dissatisfied with its method and results. It is therefore of interest to review Kirchhoff's paper to see exactly how his results fit into modern continuum mechanics.

The title of Kirchhoff's paper is misleading since he immediately assumes small strains. His method of proof is to set forth the fundamental relations for the special case of small strains directly from the definitions without ever displaying the exact relations which the small strain relations are supposed to approximate. It is perhaps not very satisfying to see the approximation but not the thing which is being approximated. But, if one accepts Kirchhoff's conceptual proofs as a valid method then his derivations are complete and his results are correct for small strains but large rotations, the case which he addressed.

Since Kirchhoff's paper deals from the beginning only with small strains, it is not immediately clear what results of his are applicable in the general theory. Furthermore, the concept of tensors as linear mappings of vectors into vectors had not been developed in 1852 and so Kirchhoff could not have explicitly introduced any tensor. It was known from a theorem of Cauchy that the three stress vectors on the coordinate surfaces completely determine the stress vector on any other surface. Thus the nine components of these three vectors characterize the state of stress at a point. Kirchhoff used the stress vectors on the convected coordinate surfaces to characterize the state of stress. This leads directly to the first (non-symmetric) Piola-Kirchhoff tensor. He introduces the components of the second (symmetric) Piola-Kirchhoff tensor only as a convenient combination of algebraic terms which occur in the basic equations. He introduced the components of the non-linear strain tensor as the coefficients in an invariant quadratic form.

Kirchoff's paper is in two rather distinct parts. He first developes a complete theory from the fundamental principles of mechanics, and then he formulates non-linear elasticity as a variational principle. He shows that the two formulations agree for small strains. Kirchhoff introduced two examples of non-linear consitutive relations for an isotropic elastic material. He assumed first that the principal stresses can be expressed as a linear isotropic function of the principal extensions. For the energy principle, he assumed that the stored energy is an isotropic quadratic function of the principle extensions. He shows that these two assumptions agree for small strains. We will first restate his results in modern notation.[4]

[1] Über die Gleichungen des Gleichgewichts eines elastischen Korpers bei nicht unendlich kleinen Verschiebungen seiner Theile. A loose translation is attached as appendix B.
[2] e.g., Truesdell and Toupin[1960], page 553, and Novozhilov[1948].
[3] Truesdell and Noll [1965], pg. 124.
[4] Truesdell [1952, §49] has presented the general theory and the history of the theories of infinitesimal strain but finite rotation.

Equations of small strain

The fundamental relations of continuum mechanics are summarized in appendix A. The deformation gradient **A** is composed of a rotation tensor **R** and the right stretch tensor **U**. The symmetric tensor **U** completely determines the extension of every fiber and the shear of every pair of fibers. If **U** equals the unit tensor at some point, the extensions and shears are zero at that point. Let **H** denote the difference between **U** and the unit tensor:

$$\mathbf{H} := \mathbf{U} - \mathbf{1}, \tag{1}$$

We will call **H** the extension tensor. The magnitude of **H** is

$$\varepsilon = \|\mathbf{H}\| = \sqrt{tr(\mathbf{H}^2)}. \tag{2}$$

Kirchhoff considers motions in which **R** is finite, but **H** is everywhere small (infinitesimal),

$$\varepsilon \ll 1. \tag{3}$$

Consequently, the shears and extensions are everywhere small. For instance, from A(17), the extension $\delta=\lambda-1$ of a fiber is

$$\delta = \mathbf{N}\cdot\mathbf{H}\cdot\mathbf{N} = O(\varepsilon). \tag{4}$$

Material bodies which have one or two dimensions that are very small compared to the others (rods, plates, shells) may experience such motions: there may be large displacements although the extensions and shears are everywhere small. In this situation, simplifications can be made to the exact equations of the continuum theory of materials.

From A(6), we find for the deformation gradient,

$$\mathbf{A} = \mathbf{R} + \mathbf{R}\cdot\mathbf{H} = \mathbf{R} + O(\varepsilon) . \tag{5}$$

From A(7), we find for the right deformation tensor,

$$\mathbf{C} = \mathbf{1} + 2\mathbf{H} + O(\varepsilon^2) = \mathbf{1} + O(\varepsilon). \tag{6}$$

Therefore, A(8) gives for the strain tensor,

$$\mathbf{E} = \mathbf{H} + O(\varepsilon^2) = O(\varepsilon). \tag{7}$$

By A(9), the deformed direction **n** of a fiber **N** of the reference configuration is

$$\mathbf{n} = \mathbf{R}\cdot\mathbf{N} + O(\varepsilon), \tag{8}$$

so that pairs of fibers which are orthogonal in the reference configuration remain approximately orthogonal. Kirchhoff uses this result several times without an explicit statement of the relation. The volume change is small; by A(4) and A(7),

$$J = 1 + O(\varepsilon). \tag{9}$$

The change of area elements is also small; by A(15),

$$a_i = 1 + O(\varepsilon), \tag{10}$$

an essential result assumed by Kirchhoff. It therefore follows from A(25) that the nominal stress vector $\mathbf{p}_i$ is approximately equal to the true stress vector $\mathbf{t}_i$ on the convected coordinate surfaces:

$$\mathbf{p}_i = \mathbf{t}_i + \mathbf{O}(\varepsilon). \tag{11}$$

If we neglect the terms of $\mathbf{O}(\varepsilon)$, the equations A(24) of balance of linear momentum are

$$\frac{\partial \mathbf{t}_i}{\partial x_i} + \rho_o \mathbf{b} = \rho_o \ddot{\mathbf{r}} \ . \tag{12}$$

This establishes equation B(1) of Kirchhoff.

Kirchhoff introduced the components X_x, Y_x, ..., Z_z of the stress vector $\mathbf{t}_i$ on the convected coordinate surfaces as the fundamental components of stress. However, we see from (11) that, for the case of small strains, his components are approximately equal to those of the nominal stress vector on those surfaces, and therefore to the components of the (first Piola-Kirchhoff) stress tensor **D** on the global rectangular Cartesian basis. With this definition of the stress components, his equation of balance of momentum is approximate, but it is exact for the stress per unit of reference area, as is clear from his method of derivation.

For the special constitutive relation A(33), $\mathbf{S}/2\mu = O(\varepsilon)$, and we assume this to be true in general. Then, from A(31), the tensor **S** is related to the principal stresses by

$$\frac{\mathbf{S}}{2\mu} = \sum_k \frac{\sigma_k}{2\mu} \mathbf{N}_k \otimes \mathbf{N}_k + O(\varepsilon^2). \tag{13}$$

Comparing this expression with equation B(7) of Kirchhoff, we see that, if we neglect the terms of $O(\varepsilon^2)$, Kirchhoff has introduced the components of the second Piola-Kirchhoff tensor **S** by this means, and his equation B(8) becomes

$$\mathbf{D} = \mathbf{S} \cdot \mathbf{A}^T \tag{14}$$

in agreement with A(30).

Kirchhoff introduces the strain tensor **E** by his equation B(11). He assumes, in the relation peceding B(9), that the material is isotropic and that the principal stresses are linear functions of the principal extensions[5]. In this case, it follows from (7), (4), and (13) that

[5] Such a relation is not possible for an elastic material with a stored energy function (hyperelastic), as the compatibility relation for existence of a strain energy shows ([1952, eq. 41.11], unless Poisson's ratio is 1/3.

$$\frac{\mathbf{S}}{2\mu} = \mathbf{E} + \frac{\lambda}{2\mu}(tr\mathbf{E})\mathbf{1} + O(\varepsilon^2). \tag{15}$$

This is Kirchhoff's result for the constitutive relation which is the second equation after B(12). That is, Kirchhoff shows that A(33) holds for small strain of an isotropic material, such that the principal stress is a linear function of the principal extensions, if the extensions are small.

The restriction to small strains, and therefore negligible change in areas, eliminates the differences in the components of the various stress tensors on their natural basis. From A(20) and A(30) , we find

$$\begin{aligned} \frac{1}{2\mu}\mathbf{D} &= \frac{1}{2\mu}\mathbf{S}\cdot\mathbf{R}^{T} + O(\varepsilon^2), \\ \frac{1}{2\mu}\mathbf{T} &= \frac{1}{2\mu}\mathbf{R}\cdot\mathbf{S}\cdot\mathbf{R}^{T} + O(\varepsilon^2). \end{aligned} \tag{16}$$

That is, the components of the various stress tensors agree up to terms of first order except for a rotation of the basis vectors. This makes it difficult to discern which tensor is indicated in papers like Kirchhoff's which use a component notation.

Energy method

In the second part of his paper , from eq. B(13) onward, Kirchhoff presents an alternative approach to the derivation of the equations of motion. He begins with the variational statement

$$\delta P = \delta\Omega, \tag{17}$$

where δP is the virtual work of the exterior forces and inertial forces,

$$\delta P = \iint_{\partial V} \mathbf{p}_{(N)}\cdot\delta\mathbf{r}\, dA + \iiint_V \rho_0(\mathbf{b} - \ddot{\mathbf{r}})\cdot\delta\mathbf{r}\, dV, \tag{18}$$

while $\delta\Omega$ is the first variation of the integral over the reference configuration of a strain energy function $\mathcal{W}$.

$$\delta\Omega = \delta\iiint \mathcal{W}\, dV. \tag{19}$$

The function $\mathcal{W}$ is assumed to be an isotropic quadratic function of the principal extensions[6]. Since the extensions are determined by the deformation gradient **A**, the function $\mathcal{W}$ may be regarded as a function of the components of **A**. Therefore, using A(5),

$$\delta\mathcal{W} = tr(\partial_{\mathbf{A}}\mathcal{W}\cdot\nabla\delta\mathbf{r}). \tag{20}$$

Inserting this result into equation (19) and integration by parts yields,

[6] Kirchhoff denotes the strain energy density $\mathcal{W}$ by F. While this is a restrictive special assumption in the general theory, it is a general constitutive relation for isotropic materials for the case of small strain which Kirchhoff considers.

$$\delta\Omega = \iint_{\partial\mathcal{V}} \mathbf{N}\cdot(\partial_{\mathbf{A}}\mathcal{W})^{\mathrm{T}}\cdot\delta\mathbf{r}\,dA - \iiint_{\mathcal{V}} \nabla\cdot(\partial\mathcal{W})^{\mathrm{T}}\cdot\delta\mathbf{r}\,dV. \tag{21}$$

The variational principle (17) now leads to equations of balance of momentum and boundary conditions which match equations (12) if we define the asymmetric Piola-Kirchhoff tensor **D** by

$$\mathbf{D} = (\partial_{\mathbf{A}}\mathcal{W})^{\mathrm{T}}. \tag{22}$$

This relation is in agreement with equation A(35).

Kirchhoff concluded that $\mathcal{W}$ is a function of **A** from the assumption that $\mathcal{W}$ is an isotropic quadratic function of the principal extensions. However, his result is independent of the specific form of the depemdence on the gradient of deformation **A.** In order to show that the complete set of equations derived by the two approaches agree, it is necessary to show that the constitutive relations generated by equation (22) agree with the relations (14) and (15). This can be seen for his special consitutive assumptions as follows.

Kirchhoff's assumption on $\mathcal{W}$ is equivalent to[7]

$$\frac{1}{\mu}\mathcal{W} = \ tr\,\mathbf{H}^2 + \frac{1}{2}\frac{\lambda}{\mu}(tr\,\mathbf{H})^2. \tag{23}$$

Using the relation (7) between the strain tensor **E** and the extension tensor **H**, we find that

$$\frac{1}{\mu}\mathcal{W} = \ \mathrm{tr}\,\mathbf{E}^2 + \frac{1}{2}\frac{\lambda}{\mu}(\mathrm{tr}\,\mathbf{E})^2 + \mathrm{O}(\varepsilon^3). \tag{24}$$

Neglecting the higher order terms, and applying equation (22) leads directly to equation (14) if equation (15) is regarded as a definition of **S**.

Conclusions

Kirchhoff's principal results are the equations of balance of linear momentum B(1),

$$\nabla\cdot\mathbf{D} + \rho_o\mathbf{b} = \rho_o\ddot{\mathbf{r}}, \tag{25}$$

the boundary conditions on traction B(2),

$$\mathbf{p} = \mathbf{N}\cdot\mathbf{D}, \tag{26}$$

the definition of the strain tensor B(11),

$$\mathbf{E} = \frac{1}{2}(\mathbf{A}^{\mathrm{T}}\cdot\mathbf{A} - \mathbf{1}), \tag{27}$$

and the constitutive relations on his last page,

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[7] If (23) is taken as an exact description of the material, the stress-strain relation becomes $\mathbf{S} = (2\mu\mathbf{H}+\lambda\, tr\mathbf{H})\mathbf{U}^{-1}$, [1977, eq. 1.18]

$$\mathbf{D} = (\partial_{\mathbf{A}} \mathcal{W})^{\mathrm{T}}. \tag{28}$$

In the context of the energy formulation, these relations are exact except for the assumption for the special form of the strain energy. However, Kirchhoff derived the equations for the mechanical formulation only for small strains. If his stress components had been defined per unit reference area, the fundamental equations would be exact except for the constitutive relation . The relation B(8) which relates the first and second Piola-Kirchhoff tensors then agrees with the definition of Truesdell.

Apparently Kirchhoff could not conceive of a theory of real materials which allowed large strains. Consequently, he introduced that approximation into the theory at an early stage. It is clear that he possessed the mathematical tools and the concept of stress and strain which would have enabled him to set forth the exact relations if he could have freed himself from the restrictive assumption of small strains.

Kirchhoff's paper is not of much use in formulating the appropriate equations governing large rotations but small strains. He has not carried the simplifications far enough, nor introduced the rotations explicitly. His most significant contribution to mechanics in this paper appears to be the step taken toward the formulation of the general equations, albeit clouded by the small strain approximation and the special constitutive relation.

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Appendix A. Continuum Mechanics

The following relations can be found in Green and Zerna [1954], Truesdell and Toupin [1960] and/or Truesdell and Noll[1965] with small differences in notation.

Kinematics. We use a global rectangular Cartesian coordinate system with basis vectors $\mathbf{e}_k$. The position vector of a material particle in the reference configuration is denoted by

$$\mathbf{R} = x_k \, \mathbf{e}_k \, . \tag{1}$$

Denote the position vector of the material particle in the deformed configuration by

$$\mathbf{r} = \xi_k \, \mathbf{e}_k \, . \tag{2}$$

We regard the coordinates in the reference configuration as the fundamental independent variables: $\mathbf{r} = \mathbf{r}(x,t)$. Associated with the deformed body is the system of gradient vectors

$$\mathbf{g}_k = \frac{\partial \mathbf{r}}{\partial x^k} \quad . \tag{3}$$

The system x^k may be regarded as an imbedded coordinate system for the deformed body which is convected by the body. The vectors $\mathbf{g}_k$ are then covariant base vectors for the imbedded system. The standard notation for the metric tensor g_{km}, its determinant g, and the contravariant vectors $\mathbf{g}^k$ is used. The quantity

$$J = \sqrt{g} \quad . \tag{4}$$

occurs frequently.

The deformation gradient $\mathbf{A}(\mathbf{x},t)$ is the tensor[8]

$$\mathbf{A} = \mathbf{g}_k \otimes \mathbf{e}_k = \frac{\partial \xi_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j \quad . \tag{5}$$

The deformation gradient is composed of the rotation tensor $\mathbf{R}$, which is an orthogonal tensor, and the stretch tensor $\mathbf{U}$ which is positive definite symmetric:

$$\mathbf{A} = \mathbf{R} \cdot \mathbf{U} . \tag{6}$$

The deformation tensor $\mathbf{C}$ is defined by

$$\mathbf{C} = \mathbf{A}^{T} \cdot \mathbf{A} = \mathbf{U}^2 = g_{ij} \, \mathbf{e}_i \otimes \mathbf{e}_j \quad . \tag{7}$$

A strain tensor is defined by

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[8] $\mathbf{A}$ is the tensor which Truesdell and Noll denote by $\mathbf{F}$.

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}) \ . \tag{8}$$

A fiber of material is rotated and extended by the deformation. The ratio of length after deformation to the length before deformation is the stretch λ, and $\lambda - 1 = \delta$ is the extension of the fiber. A fiber with initial direction **N** is deformed into a fiber in the direction **n** such that

$$\mathbf{n} = \frac{1}{\lambda}\mathbf{A} \cdot \mathbf{N} \ . \tag{9}$$

Therefore,

$$\lambda^2 = \mathbf{N} \cdot \mathbf{C} \cdot \mathbf{N} \ . \tag{10}$$

In particular, for the three fibers directed along the coordinate lines,

$$(1 + \delta_i)^2 = g_{ii} \ \text{(no sum)}. \tag{11}$$

The deformed directions of these fibers are

$$\mathbf{n}_i = \frac{\mathbf{g}_i}{\sqrt{g_{ii}}} \ \text{(no sum)} \ . \tag{12}$$

An element of area $\mathbf{N}\, da_0$ becomes the area element $\mathbf{n}\, da$ and

$$\mathbf{n}\, da = J\, \mathbf{N} \cdot \mathbf{A}^{-1}\, da_o \ . \tag{13}$$

The magnitudes of the areas are related by

$$da^2 = J^2\, \mathbf{N} \cdot \mathbf{C}^{-1} \cdot \mathbf{N}\, da_o^{\ 2} \tag{14}$$

In particular, for the area elements of the convected coordinate planes, the ratio of the area after deformation to the initial area is

$$a_i = J\sqrt{g^{ii}} \ \ (\textit{no sum}) \ . \tag{15}$$

The ratio of the volume element after deformation to the volume element before deformation is J. If ρ is the density of the deformed body and ρ_0 is the density of the reference configuration, the conservation of mass is expressed by

$$\rho_o = \rho\, J \ . \tag{16}$$

Suppose λ is a principal extension and **N** is the initial direction of corresponding fiber:

$$\mathbf{U} \cdot \mathbf{N} = \lambda\, \mathbf{N} \ . \tag{17}$$

The direction of the deformed fiber is

$$\mathbf{n} = \mathbf{R} \cdot \mathbf{N} \ \cdot \tag{18}$$

The three fibers experiencing principal extensions are orthogonal in the undeformed configuration and the deformed configuration. The vector **N** is also a principal axis of **E** and **C**.

Stress and Momentum. Let **T** denote the stress tensor. The stress vector on a deformed surface, per unit area of the deformed body, is

$$\mathbf{t}_{(\mathbf{n})} = \mathbf{n} \cdot \mathbf{T} . \tag{19}$$

where **n** is the unit normal to the surface in the deformed body. This vector may be called the *true* stress vector.

Let[9]

$$\mathbf{D} = J\, \mathbf{A}^{-1} \cdot \mathbf{T} . \tag{20}$$

The stress vector on the deformed surface, per unit area of the reference configuration, is

$$\mathbf{p}_{(\mathrm{N})} = \mathbf{N} \cdot \mathbf{D}, \tag{21}$$

where **N** is the unit normal to the surface in the reference configuration. This vector may be called the *nominal* stress vector.

The nominal stress vector on a convected coordinate surface is

$$\mathbf{p}_i = \mathbf{e}_i \cdot \mathbf{D} = \sigma_{im} \mathbf{e}_m \quad . \tag{22}$$

The balance of momentum can then be expressed by integration over the reference configuration:

$$\iint_{\partial\mathcal{V}} \mathbf{p}_{(\mathrm{N})}\, dA + \iiint_{\mathcal{V}} \rho_o \mathbf{b}\, dV = \iiint_{\mathcal{V}} \rho_o\, \ddot{\mathbf{r}}\, dV \cdot \tag{23}$$

For smooth fields, this leads to the following differential equation:

$$\frac{\partial \mathbf{p}_i}{\partial x_i} + \rho_o \mathbf{b} = \rho_o \ddot{\mathbf{r}} \ . \tag{24}$$

The balance of angular momentum for the body implies that **T** and **S** are symmetric and that will be used as a condition on the constitutive relations.

If $\mathbf{t}_i$ denotes the true stress vector on the convected coordinate surfaces,

$$\mathbf{p}_i = a_i\, \mathbf{t}_i \ \ (no\ sum) . \tag{25}$$

For a general surface,

$$\mathbf{t}_{(\mathrm{n})} = \sqrt{g_{11}}\, n_1\, \mathbf{t}_1 + \sqrt{g_{22}}\, n_2\, \mathbf{t}_2 + \sqrt{g_{33}}\, n_3\, \mathbf{t}_3 \ , \tag{26}$$

[9]**D** is the transpose of what Truesdell and Noll call the first Piola-Kirchhoff tensor.

$$\mathbf{p}_{(N)} = N_1\,\mathbf{p}_1 + N_2\,\mathbf{p}_2 + N_3\,\mathbf{p}_3, \tag{27}$$

where $\mathbf{N} = N_i\,\mathbf{e}_i$ and $\mathbf{n} = n_i\,\mathbf{g}_i$.

If **n** is a principal axis of the stress tensor **T** and σ is the corresponding principal stress, then

$$\mathbf{t}_{(\mathbf{n})} = \sigma\,\mathbf{n}\ . \tag{28}$$

One can always establish a set of three orthogonal principal axes. The three fibers which have the directions of the principal axes of stress in the deformed body are not initially orthogonal unless they are also principal axes of strain. For an isotropic material, the principal axes of stress and strain coincide. Let $\mathbf{n}_k$ denote the three principal axes of stress and σ_k the corresponding principal stresses, then

$$\mathbf{T} = \sum_k \sigma_k\,\mathbf{n}_k \otimes \mathbf{n}_k \tag{29}$$

It is convenient to introduce a third stress tensor[10]:

$$\mathbf{S} = \mathbf{D}\cdot\left(\mathbf{A}^{-1}\right)^{\mathrm{T}} = J\mathbf{A}^{-1}\cdot\mathbf{T}\cdot\left(\mathbf{A}^{-1}\right)^{\mathrm{T}}, \tag{30}$$

which plays a fundamental role in thermodynamics and constitutive equations. The formula for **S** in terms of the principal stresses can be found as follows. The initial directions of the fibers having the principal directions $\mathbf{n}_k$ in the deformed configuration are $\mathbf{N}_k$ given by eq. (9), and we find from eqs. (30) that

$$\mathbf{S} = \sum_k \frac{J\sigma_k}{\left(\lambda_k\right)^2}\,\mathbf{N}_k \otimes \mathbf{N}_k \tag{31}$$

The $\mathbf{N}_k$ are not orthogonal in general, unless the principal axes of strain and stress coincide, as they will for an isotropic material.

Material Properties. The constitutive equation of an elastic material can be expressed in the form

$$\mathbf{S} = \mathcal{F}(\mathbf{E}) = \partial_{\mathbf{E}}\mathcal{W}(\mathbf{E}). \tag{32}$$

where $\mathcal{W}$ and $\mathcal{F}$ are functions which characterize the material. An example for an isotropic material is one for which[11]

$$\mathbf{S} = 2\mu\mathbf{E} + \lambda(\mathrm{tr}\mathbf{E})\mathbf{1}. \tag{33}$$

For small strain, this relation reduces to that for isotropic materials in the linear theory and the scalars λ and μ are the elastic constants of Lamé :

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[10] S is the tensor which Truesdell and Noll call the second Piola-Kirchhoff tensor.

[11] The material constant λ is of course not the same as the stretch λ introduced above.

$$\mu = G = \frac{E}{2(1+\nu)} \ , \ \lambda = \frac{2\mu\nu}{1\text{-}2\nu} \ , \tag{34}$$

where G is the shear modulus, E is the tensile modulus, and ν is the transverse contraction (Poisson) ratio.

It follows from the definition of **D** and **E** that

$$\mathbf{D} = \left(\partial_{\mathbf{A}} \mathcal{W}(\mathbf{E})\right)^{\mathrm{T}} \tag{35}$$

which is an alternate form of the general constitutive relation of elasticity.

Appendix B.

On the Equations of Equilibrium of an elastic body when the displacements of its points are not infinitesimal[12]

by Prof. Dr. Kirchhoff
of Breslau

St. Venant has, in his *Mémoire sur l'équilibre des corps solides ....Compt. Rend. XXIV, pg. 260*, indicated[13] a way by which one can derive the equations that express the condition of equilibrium for an elastic body in the case that the displacements of its parts which are subjected to exterior forces are not infinitely small, a case which can occur for a body such that one dimension is infinitely small without exceeding the limit of perfect elasticity. I have derived these equations in two different ways, of which the first essentially agrees with that of St. Venant, and the second is a further development of my established formulas (Crelle's Journ. XL).

I do not take the displacements of points as independent variables, as is usually done, but the coordinates of the points after deformation; one gains nothing through the introduction of the displacements if they are not infinitely small, on the contrary one loses brevity and clarity. I denote the coordinates of points after deformation by ξ, η, ζ, and the coordinates of points before deformation by x, y, z. In the natural state of the body, I consider three planes parallel to the coordinate planes. The parts of these planes which are near the considered point go by the deformation into planes which are inclined at a finite angle to the coordinate planes, but are to each other shifted infinitely little[14] from 90°. I resolve the pressures[15] on these planes after deformation along the coordinate axes and denote these components: X_x, Y_x. Z_x, X_y, Y_y. Z_y, X_z, Y_z. Z_z, in such a way that, e. g., Y_x is the y component of the stress acting on the plane which was perpendicular to the x axis before deformation. These nine pressures[16] are in general inclined to the plane across which they act, and no three of them are equal to another three[17], as is the case for infinitely small displacements. If one forms the condition that each part of the body is in equilibrium, one finds for the infinitely small parallelepiped which before deformation has its edges parallel to the coordinate axes, and the lengths dx, dy, dz, the equations

[12] This is a loose translation of Kirchhoff's paper and should not be relied upon for the exact nuance of his remarks.

[13] The only equations in this publication are the non-linear strain-displacement relations.

[14] Kirchhoff immediately limits considerations to finite rotations but small strains, using the approximation (8) that $\mathbf{n} = \mathbf{R}\cdot\mathbf{N}$.

[15] "Drucke." Kirchhoff takes pressure as positive which will require negative signs in the constitutive relation below. This will be translated as "stress" although retaining his sign convention.

[16] Kirchhoff clearly states the meaning of X_x, Y_x, ... as the components of the true stress vectors on the convected coordinate surfaces. If we use that definition some of the following relations are exact and others are approximate. However, since he considers only small strains, these quantities are approximately equal to the components of the material stress vector $\mathbf{p}_i$ and therefore they are the components of the non-symmetric Piola-Kirchhoff tensor $\mathbf{D}$ on the rectangular Cartesian coordinate system. With this second interpretation equation (1) coincides with A(23).

[17] That is, $\mathbf{D}$ is not symmetric.

$$\rho X = \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z},$$
$$\rho Y = \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z}, \qquad (1)$$
$$\rho Z = \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z},$$

where ρ denotes the density of the body and X, Y, Z denote the components of the acceleration force[18] which acts at the point (ξ, η, ζ). One comes to these equations if one notes that angles and the edges of the parallelepiped have only infinitely small changes and one makes the same considerations by which one proves these equations for infinitely small displacements. If one further sets forth the conditions for equilibrium of a part of the body which, before deformation, is an infinitely small pyramid whose three lateral surfaces were parallel to the coordinate planes and whose base surface is perpendicular to a line s which forms angles (s, x), (s, y), (s, z) with the axes, and denote by X_s, Y_s, Z_s the components of stress along the coordinate axes, which act on the base surface after the deformation, one finds, if one further considers that the pyramid has only infinitely small changes in its shape, that

$$X_s = X_x \cos(s,x) + X_y \cos(s,y) + X_z \cos(s,z),$$
$$Y_s = Y_x \cos(s,x) + Y_y \cos(s,y) + Y_z \cos(s,z), \qquad (2)$$
$$Z_s = Z_x \cos(s,x) + Z_y \cos(s,y) + Z_z \cos(s,z).$$

It therefore follows that, if n is the unit normal to an element of surface in its unstressed state, and (X), (Y), (Z) are the components of pressure on this element after the deformation, one has the following relation[19]

$$(X) = X_x \cos(n,x) + X_y \cos(n,y) + X_z \cos(n,z),$$
$$(Y) = Y_x \cos(n,x) + Y_y \cos(n,y) + Y_z \cos(n,z), \qquad (3)$$
$$(Z) = Z_x \cos(n,x) + Z_y \cos(n,y) + Z_z \cos(n,z).$$

We must now seek expressions for the stress X_x, X_y, etc., in terms of $\partial\xi/\partial x$, $\partial\xi/\partial y$, etc., and substitute these into the differential equations (1) and the boundary conditions (3). We arrive easily at this expressions by considering the principal stress and principal strain. The state of an infinitely small part of the body after deformation can be distinguished from the state before deformation by the manner that the part has submitted to a displacement, a rotation, and a finite dilatation in three mutually perpendicular directions which must coincide with the those of the principal stresses, that is, those which act perpendicular; for planes which are perpendicular to one another have necessarily to undergo perpendicular stresses, assuming that the structure of the body is the same in all directions[20]. If σ is an infinitely small line, which lies in one of these three directions, s is the line in the unstressed state of the body, which goes by the deformation into σ, P one of the principal

---

[18] Body forces are not included.

[19] In the notation of appendix A, $\mathbf{p}_{(\mathbf{N})} = \mathbf{N}\cdot\mathbf{D}$.

[20] Kirchhoff considers isotropic materials.

stresses, namely the stress, which acts on the plane perpendicular to σ, then we have that P acts in the direction of σ, the components X_s, Y_s, Z_s of P, have the following expression:

$$
\begin{aligned}
X_s &= P\cos(\sigma,x)\,, \\
Y_s &= P\cos(\sigma,y)\,, \\
Z_s &= P\cos(\sigma,z)\,.
\end{aligned}
\tag{4}
$$

The quantities cos(σ, x), cos(σ, y), and cos(σ, z), can be expressed in terms of cos(s, x), cos(s, y), and cos(s, z). Let (x, y, z) and ($x+dx$, $y+dy$, $z+dz$) be two points on the line s, and also let (ξ, η, ζ) and (ξ+dξ, η+dη, ζ+dζ) be two points on the line σ, then we have[21]

$$
\begin{aligned}
d\xi &= \frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial y}dy + \frac{\partial \xi}{\partial z}dz\,, \\
d\eta &= \frac{\partial \eta}{\partial x}dx + \frac{\partial \eta}{\partial y}dy + \frac{\partial \eta}{\partial z}dz\,, \\
d\zeta &= \frac{\partial \zeta}{\partial x}dx + \frac{\partial \zeta}{\partial y}dy + \frac{\partial \zeta}{\partial z}dz\,.
\end{aligned}
$$

We set $dx^2 + dy^2 + dz^2 = e^2$ and $d\xi^2 + d\eta^2 + d\zeta^2 = \varepsilon^2$, so that we have

$$
\begin{aligned}
&dx = e\cos(s,x),\ dy = e\cos(s,y),\ dz = e\cos(s,z), \\
&d\xi = \varepsilon\cos(\sigma,x),\ d\eta = \varepsilon\cos(\sigma,x),\ d\zeta = \varepsilon\cos(\sigma,x).
\end{aligned}
$$

It therefore follows, if we take into account that ε and e are only infinitesimally different, that

$$
\begin{aligned}
\cos(\sigma,x) &= \frac{\partial \xi}{\partial x}\cos(s,x) + \frac{\partial \xi}{\partial y}\cos(s,y) + \frac{\partial \xi}{\partial z}\cos(s,z)\,, \\
\cos(\sigma,y) &= \frac{\partial \eta}{\partial x}\cos(s,x) + \frac{\partial \eta}{\partial y}\cos(s,y) + \frac{\partial \eta}{\partial z}\cos(s,z)\,, \\
\cos(\sigma,z) &= \frac{\partial \zeta}{\partial x}\cos(s,x) + \frac{\partial \zeta}{\partial y}\cos(s,y) + \frac{\partial \zeta}{\partial z}\cos(s,z)\,.
\end{aligned}
\tag{5}
$$

Since these equations are true if s is an arbitrary line, one easily recognizes the geometric meaning of the nine partial derivatives $\frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}$, etc. If one takes s parallel to the x axis, one sees[22] that $\frac{\partial \xi}{\partial x}, \frac{\partial \eta}{\partial x}, \frac{\partial \zeta}{\partial x}$ are the cosines of the angles which an infinitesimal line

---

[21] In the notation of appendix A, **dr** = **A**·**dR**.

[22] Assuming small strains. In the notation of appendix A, eq. (5) is **n** = (1/λ) **A**·**N**.

segment forms with the axes after deformation which was parallel to the x axis in the undeformed state, and similarly for $\frac{\partial \xi}{\partial y}, \frac{\partial \eta}{\partial y}, \frac{\partial \zeta}{\partial y}$ and $\frac{\partial \xi}{\partial z}, \frac{\partial \eta}{\partial z}, \frac{\partial \zeta}{\partial z}$.

If one substitutes the values of $\cos(\sigma,x)$, $\cos(\sigma,y)$, $\cos(\sigma,z)$, from (5), into eq. (4) and combines this with eq. (2), one obtains, with $\cos(s,x)=a$, $\cos(s,y)=b$, $\cos(s,z)=c$,

$$\begin{aligned}
X_x a + X_y b + X_z c &= P\left(\frac{\partial \xi}{\partial x} a + \frac{\partial \xi}{\partial y} b + \frac{\partial \xi}{\partial z} c\right), \\
Y_x a + Y_y b + Y_z c &= P\left(\frac{\partial \eta}{\partial x} a + \frac{\partial \eta}{\partial y} b + \frac{\partial \eta}{\partial z} c\right), \qquad (6) \\
Z_x a + Z_y b + Z_z c &= P\left(\frac{\partial \zeta}{\partial x} a + \frac{\partial \zeta}{\partial y} b + \frac{\partial \zeta}{\partial z} c\right).
\end{aligned}$$

I will denote the three principal stresses by P_1, P_2, P_3, and give the quantities a, b, c the indices 1, 2, 3, according to which one or the other they are related. These equations are then true if one gives uniformly the index 1, or 2, or 3, to the quantities P, a, b, c. The three directions determined by the quantities a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, are perpendicular to each other[23] and therefore known relations exist between these quantities. With the help of these relations one can by the system (6) express X_x, X_y, etc., in terms of $P_1, P_2, P_3, a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3$. One now multiplies each of the equations

$$\begin{aligned}
X_x a_1 + X_y b_1 + X_z c_1 &= P_1\left(\frac{\partial \xi}{\partial x} a_1 + \frac{\partial \xi}{\partial y} b_1 + \frac{\partial \xi}{\partial z} c_1\right), \\
X_x a_2 + X_y b_2 + X_z c_2 &= P_2\left(\frac{\partial \xi}{\partial x} a_2 + \frac{\partial \xi}{\partial y} b_2 + \frac{\partial \xi}{\partial z} c_2\right), \\
X_x a_3 + X_y b_3 + X_z c_3 &= P_3\left(\frac{\partial \xi}{\partial x} a_3 + \frac{\partial \xi}{\partial y} b_3 + \frac{\partial \xi}{\partial z} c_3\right).
\end{aligned}$$

by a_1, a_2, a_3, then by b_1, b_2, b_3, then by c_1, c_2, c_3, and add them together, to obtain

$$\begin{aligned}
X_x = {} & \frac{\partial \xi}{\partial x}\left(P_1 a_1^2 + P_2 a_2^2 + P_3 a_3^2\right) \\
& + \frac{\partial \xi}{\partial y}\left(P_1 a_1 b_1 + P_2 a_2 b_2 + P_3 a_3 b_3\right) \\
& + \frac{\partial \xi}{\partial z}\left(P_1 a_1 c_1 + P_2 a_2 c_2 + P_3 a_3 c_3\right),
\end{aligned}$$

[23] Kirchhoff considers only isotropic materials so that the principal axes of stress and strain coincide and those axes are orthogonal in the reference configuration and the deformed configuration.

$$X_y = \frac{\partial\xi}{\partial x}(P_1a_1b_1 + P_2a_2b_2 + P_3a_3b_3)$$
$$+ \frac{\partial\xi}{\partial y}(P_1b_1^2 + P_2b_2^2 + P_3b_3^2)$$
$$+ \frac{\partial\xi}{\partial z}(P_1b_1c_1 + P_2b_2c_2 + P_3b_3c_3),$$

$$X_z = \frac{\partial\xi}{\partial x}(P_1a_1c_1 + P_2a_2c_2 + P_3a_3c_3)$$
$$+ \frac{\partial\xi}{\partial y}(P_1b_1c_1 + P_2b_2c_2 + P_3b_3c_3)$$
$$+ \frac{\partial\xi}{\partial z}(P_1c_1^2 + P_2c_2^2 + P_3c_3^2).$$

If one sets[24]

$$\begin{aligned}
P_1a_1^2 + P_2a_2^2 + P_3a_3^2 &= (aa)\\
P_1b_1^2 + P_2b_2^2 + P_3b_3^2 &= (bb)\\
P_1c_1^2 + P_2c_2^2 + P_3c_3^2 &= (cc)\\
P_1b_1c_1 + P_2b_2c_2 + P_3b_3c_3 &= (bc)\\
P_1a_1c_1 + P_2a_2c_2 + P_3a_3c_3 &= (ac)\\
P_1a_1b_1 + P_2a_2b_2 + P_3a_3b_3 &= (ab)
\end{aligned} \qquad (7)$$

these equations, and the ones corresponding to the second and third equations of the system (6), become the following[25] :

$$\begin{aligned}
X_x &= \frac{\partial\xi}{\partial x}(aa) + \frac{\partial\xi}{\partial y}(ab) + \frac{\partial\xi}{\partial z}(ac),\\
X_y &= \frac{\partial\xi}{\partial x}(ab) + \frac{\partial\xi}{\partial y}(bb) + \frac{\partial\xi}{\partial z}(bc),\\
X_z &= \frac{\partial\xi}{\partial x}(ac) + \frac{\partial\xi}{\partial y}(bc) + \frac{\partial\xi}{\partial z}(cc),\\
Y_x &= \frac{\partial\eta}{\partial x}(aa) + \frac{\partial\eta}{\partial y}(ab) + \frac{\partial\eta}{\partial z}(ac),\\
Y_y &= \frac{\partial\eta}{\partial x}(ab) + \frac{\partial\eta}{\partial y}(bb) + \frac{\partial\eta}{\partial z}(bc),\\
Y_z &= \frac{\partial\eta}{\partial x}(ac) + \frac{\partial\eta}{\partial y}(bc) + \frac{\partial\eta}{\partial z}(cc),\\
Z_x &= \frac{\partial\zeta}{\partial x}(aa) + \frac{\partial\zeta}{\partial y}(ab) + \frac{\partial\zeta}{\partial z}(ac),
\end{aligned} \qquad (8)$$

[24] For large strains, (aa),...,(ab) are the components of the symmetric Piola-Kirchhoff tensor **S**.

[25] In the notation of appendix A, $\mathbf{D} = \mathbf{S}\cdot\mathbf{A}^T$.

$$Z_y = \frac{\partial \zeta}{\partial x}(ab) + \frac{\partial \zeta}{\partial y}(bb) + \frac{\partial \zeta}{\partial z}(bc),$$
$$Z_z = \frac{\partial \zeta}{\partial x}(ac) + \frac{\partial \zeta}{\partial y}(bc) + \frac{\partial \zeta}{\partial z}(cc).$$

We denote the values of the principal extensions by λ_1, λ_2, λ_3, so that P_1, P_2, P_3 are functions of λ_1, λ_2, λ_3. We assume the relation to be linear so that we can set[26]

$$P_1 = -2K(\lambda_1 + \theta(\lambda_1 + \lambda_2 + \lambda_3))$$
$$P_2 = -2K(\lambda_2 + \theta(\lambda_1 + \lambda_2 + \lambda_3))$$
$$P_3 = -2K(\lambda_3 + \theta(\lambda_1 + \lambda_2 + \lambda_3))$$

Therefore eq. (7) takes the following form:

$$\begin{aligned}
(aa) &= -2K(\lambda_1 a_1^2 + \lambda_2 a_2^2 + \lambda_3 a_3^2 + \theta(\lambda_1 + \lambda_2 + \lambda_3)) \\
(bb) &= -2K(\lambda_1 b_1^2 + \lambda_2 b_2^2 + \lambda_3 b_3^2 + \theta(\lambda_1 + \lambda_2 + \lambda_3)) \\
(cc) &= -2K(\lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2 + \theta(\lambda_1 + \lambda_2 + \lambda_3)) \\
(bc) &= -2K(\lambda_1 b_1 c_1 + \lambda_2 b_2 c_2 + \lambda_3 b_3 c_3) \\
(ac) &= -2K(\lambda_1 c_1 a_1 + \lambda_2 c_2 a_2 + \lambda_3 c_3 a_3) \\
(ab) &= -2K(\lambda_1 a_1 b_1 + \lambda_2 a_2 b_2 + \lambda_3 a_3 b_3)
\end{aligned} \tag{9}$$

We denote by e the length of the line element between the points (x, y, z) and $(x+dx, y+dy, z+dz)$ before deformation, and ε denotes this quantity after deformation. Therefore the extension[27] λ, which the line e undergoes, is equal to $\frac{\varepsilon}{e} - 1$. Assuming that λ is infinitely small, we can write $\lambda = \frac{1}{2}\left(\frac{\varepsilon^2}{e^2} - 1\right)$. In this equation, one has to put

$$\varepsilon^2 = d\xi^2 + d\eta^2 + d\zeta^2,$$
$$d\xi = \frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial y}dy + \frac{\partial \xi}{\partial z}dz,$$
$$d\eta = \frac{\partial \eta}{\partial x}dx + \frac{\partial \eta}{\partial y}dy + \frac{\partial \eta}{\partial z}dz,$$
$$d\zeta = \frac{\partial \zeta}{\partial x}dx + \frac{\partial \zeta}{\partial y}dy + \frac{\partial \zeta}{\partial z}dz.$$

[26]Such a relation is not possible for an elastic material with a stored energy function (hyperelastic) unless Poisson's ratio is 1/3..

[27] The extension is denoted by δ in appendix A where λ denotes the stretch.

Moreover, one sets $\frac{dx}{e} = a, \frac{dy}{e} = b, \frac{dz}{e} = c$, so that one obtains

$$\lambda = \frac{1}{2}\left(\left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial x}\right)^2 - 1\right)a^2$$
$$+\frac{1}{2}\left(\left(\frac{\partial\xi}{\partial y}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2 - 1\right)b^2$$
$$+\frac{1}{2}\left(\left(\frac{\partial\xi}{\partial z}\right)^2 + \left(\frac{\partial\eta}{\partial z}\right)^2 + \left(\frac{\partial\zeta}{\partial z}\right)^2 - 1\right)c^2$$
$$+\left(\frac{\partial\xi}{\partial y}\frac{\partial\xi}{\partial z} + \frac{\partial\eta}{\partial y}\frac{\partial\eta}{\partial z} + \frac{\partial\zeta}{\partial y}\frac{\partial\zeta}{\partial z}\right)bc$$
$$+\left(\frac{\partial\xi}{\partial z}\frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial z}\frac{\partial\eta}{\partial x} + \frac{\partial\zeta}{\partial z}\frac{\partial\zeta}{\partial x}\right)ca$$
$$+\left(\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y} + \frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial x}\frac{\partial\zeta}{\partial y}\right)ab.$$

We can write this equation in the following form[28]

$$\lambda = La^2 + Mb^2 + Nc^2 + 2lbc + 2mca + 2nab, \tag{10}$$

where we have put[29]

[28] In the absence of tensor analysis, Kirchhoff introduces a tensor through the coefficients in a quadratic form.

[29] $L, M, N, l, m,$ and n are the components of the strain tensor $\mathbf{E}=(\mathbf{A}^{T}\cdot\mathbf{A}-\mathbf{1})/2$.

$$
\begin{aligned}
L &= \frac{1}{2}\left(\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial x}\right)^2-1\right)\\
M &= \frac{1}{2}\left(\left(\frac{\partial\xi}{\partial y}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2-1\right)\\
N &= \frac{1}{2}\left(\left(\frac{\partial\xi}{\partial z}\right)^2+\left(\frac{\partial\eta}{\partial z}\right)^2+\left(\frac{\partial\zeta}{\partial z}\right)^2-1\right)\\
l &= \frac{1}{2}\left(\frac{\partial\xi}{\partial y}\frac{\partial\xi}{\partial z}+\frac{\partial\eta}{\partial y}\frac{\partial\eta}{\partial z}+\frac{\partial\zeta}{\partial y}\frac{\partial\zeta}{\partial z}\right)\\
m &= \frac{1}{2}\left(\frac{\partial\xi}{\partial z}\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial z}\frac{\partial\eta}{\partial x}+\frac{\partial\zeta}{\partial z}\frac{\partial\zeta}{\partial x}\right)\\
n &= \frac{1}{2}\left(\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\zeta}{\partial x}\frac{\partial\zeta}{\partial y}\right).
\end{aligned}
\tag{11}
$$

In order to find the principal extensions λ_1, λ_2, λ_3, for which the associated values of a, b, c are a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, we must find the maximum and minimum values of λ. To find them, we have to solve the following system of equations:

$$
\begin{aligned}
0 &= (L-\lambda)a+nb+mc\\
0 &= na+(M-\lambda)b+lc\\
0 &= ma+lb+(N-\lambda)c\\
a^2&+b^2+c^2=1
\end{aligned}
\tag{12}
$$

These equations are satisfied by the quantities λ, a, b, c with the index 1, 2, or 3. If we apply to them the operations similar to those which we applied to the system (6), we find that

$$
\begin{aligned}
L &= \lambda_1 a_1^2+\lambda_2 a_2^2+\lambda_3 a_3^2\\
M &= \lambda_1 b_1^2+\lambda_2 b_2^2+\lambda_3 b_3^2\\
N &= \lambda_1 c_1^2+\lambda_2 c_2^2+\lambda_3 c_3^2\\
l &= \lambda_1 b_1 c_1+\lambda_2 b_2 c_2+\lambda_3 b_3 c_3\\
m &= \lambda_1 c_1 a_1+\lambda_2 c_2 a_2+\lambda_3 c_3 a_3\\
n &= \lambda_1 a_1 b_1+\lambda_2 a_2 b_2+\lambda_3 a_3 b_3\\
L+M+N &= \lambda_1+\lambda_2+\lambda_3\ .
\end{aligned}
$$

Eq. (9) therefore becomes[30]

[30] In the notation of appendix A, $S=2\mu E+\lambda\ \mathrm{tr}E$.

$$(aa) = -2K(L + \theta(L + M + N))$$
$$(bb) = -2K(M + \theta(L + M + N))$$
$$(cc) = -2K(N + \theta(L + M + N))$$
$$(bc) = -2Kl$$
$$(ac) = -2Km$$
$$(ab) = -2Kn \ .$$

and therefore eqs. (8) become

$$X_x = -2K\left(\frac{\partial\xi}{\partial x}(L + \theta(L + M + N)) + \frac{\partial\xi}{\partial y}n + \frac{\partial\xi}{\partial z}m\right),$$
$$X_y = -2K\left(\frac{\partial\xi}{\partial x}n + \frac{\partial\xi}{\partial y}(M + \theta(L + M + N)) + \frac{\partial\xi}{\partial z}l\right),$$
$$X_z = -2K\left(\frac{\partial\xi}{\partial x}m + \frac{\partial\xi}{\partial y}l + \frac{\partial\xi}{\partial z}(N + \theta(L + M + N))\right),$$
$$Y_x = -2K\left(\frac{\partial\eta}{\partial x}(L + \theta(L + M + N)) + \frac{\partial\eta}{\partial y}n + \frac{\partial\eta}{\partial z}m\right),$$
$$Y_y = -2K\left(\frac{\partial\eta}{\partial x}n + \frac{\partial\eta}{\partial y}(M + \theta(L + M + N)) + \frac{\partial\eta}{\partial z}l\right),$$
$$Y_z = -2K\left(\frac{\partial\eta}{\partial x}m + \frac{\partial\eta}{\partial y}l + \frac{\partial\eta}{\partial z}(N + \theta(L + M + N))\right),$$
$$Z_x = -2K\left(\frac{\partial\zeta}{\partial x}(L + \theta(L + M + N)) + \frac{\partial\zeta}{\partial y}n + \frac{\partial\zeta}{\partial z}m\right),$$
$$Z_y = -2K\left(\frac{\partial\zeta}{\partial x}n + \frac{\partial\zeta}{\partial y}(M + \theta(L + M + N)) + \frac{\partial\zeta}{\partial z}l\right),$$
$$Z_z = -2K\left(\frac{\partial\zeta}{\partial x}m + \frac{\partial\zeta}{\partial y}l + \frac{\partial\zeta}{\partial z}(N + \theta(L + M + N))\right).$$

If one puts these expressions into eq. (1) and eq. (3) and sets L, M, N, l, m, and n to their values from eq. (11), then one has the sought differential equations and boundary conditions.

In my paper "Uber das Gleichgewicht und die Bewegung einer elastischen Scheibe" I have established the equilibrium conditions for an elastic body whose parts undergo finite displacements in another form, which one may use to find the equations developed here by a relatively simple calculation. I will also set forth here those calculations.

The equilibrium condition is

$$\delta P - K\,\delta\Omega = 0\,,$$
$$\Omega = \int dV\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \theta(\lambda_1 + \lambda_2 + \lambda_3)^2\right) \tag{13}$$

Here δP denotes the "momentum" of the exterior forces which may act on the body. Therefore

$$\delta P = \iiint dx\,dy\,dz\,\rho(X\,\delta\xi + Y\,\delta\eta + Z\,\delta\zeta)$$
$$+ \int df((X)\delta\xi + (Y)\delta\eta + (Z)\delta\zeta)\,,$$

where *df* denotes an element of surface of the body, and d*V* is an element of volume of the body, so that $dV = dx\,dy\,dz$.

The factor of *dV* in the second integral is a function of the nine derivatives $\frac{\partial\xi}{\partial x}, \frac{\partial\xi}{\partial y}$, etc., which we must formulate, but which I will for the moment write[31]

$$F\left(\frac{\partial\xi}{\partial x}, \frac{\partial\xi}{\partial y}, \frac{\partial\xi}{\partial z}, \frac{\partial\eta}{\partial x}, \frac{\partial\eta}{\partial y}, \frac{\partial\eta}{\partial z}, \frac{\partial\zeta}{\partial x}, \frac{\partial\zeta}{\partial y}, \frac{\partial\zeta}{\partial z}\right)$$

which for brevity will be denoted by F. Since $\frac{\partial(\xi+\delta\xi)}{\partial x} = \frac{\partial\xi}{\partial x} + \frac{\partial\delta\xi}{\partial x}$, $\frac{\partial(\xi+\delta\xi)}{\partial y} = \frac{\partial\xi}{\partial y} + \frac{\partial\delta\xi}{\partial y}$, etc., therefore

$$\delta\Omega = \delta\iiint dx\,dy\,dz\,F$$
$$= \iiint dx\,dy\,dz\left\{\frac{\partial F}{\partial\frac{\partial\xi}{\partial x}}\frac{\partial\delta\xi}{\partial x} + \frac{\partial F}{\partial\frac{\partial\eta}{\partial x}}\frac{\partial\delta\eta}{\partial x} + \frac{\partial F}{\partial\frac{\partial\zeta}{\partial x}}\frac{\partial\delta\zeta}{\partial x}\right\}$$
$$+ \iiint dx\,dy\,dz\left\{\frac{\partial F}{\partial\frac{\partial\xi}{\partial y}}\frac{\partial\delta\xi}{\partial y} + \frac{\partial F}{\partial\frac{\partial\eta}{\partial y}}\frac{\partial\delta\eta}{\partial y} + \frac{\partial F}{\partial\frac{\partial\zeta}{\partial y}}\frac{\partial\delta\zeta}{\partial y}\right\}$$
$$+ \iiint dx\,dy\,dz\left\{\frac{\partial F}{\partial\frac{\partial\xi}{\partial z}}\frac{\partial\delta\xi}{\partial z} + \frac{\partial F}{\partial\frac{\partial\eta}{\partial z}}\frac{\partial\delta\eta}{\partial z} + \frac{\partial F}{\partial\frac{\partial\zeta}{\partial z}}\frac{\partial\delta\zeta}{\partial z}\right\}$$

The first of the three integrals on the right hand side may be split into three terms, and apply to each of them a theorem expressed by the equation

[31] That is, F is a function of **A**. F is denoted by $\mathcal{W}$ in appendix A.

$$\iiint dx\,dy\,dz\,H\frac{\partial G}{\partial x} = -\iiint dx\,dy\,dz\,G\frac{\partial H}{\partial x} - \int df\,H\,G\cos(n,x)$$

in which we first put G equal to $\delta\xi$, then $\delta\eta$, then $\delta\zeta$. Next, we do corresponding operations on the other two integrals to obtain

$$\begin{aligned}
K\,\delta\Omega = -K\iiint dx\,dy\,dz\{ & \left(\frac{\partial}{\partial x}\frac{\partial F}{\partial\frac{\partial\xi}{\partial x}} + \frac{\partial}{\partial y}\frac{\partial F}{\partial\frac{\partial\xi}{\partial y}} + \frac{\partial}{\partial z}\frac{\partial F}{\partial\frac{\partial\xi}{\partial z}}\right)\delta\xi \\
& + \left(\frac{\partial}{\partial x}\frac{\partial F}{\partial\frac{\partial\eta}{\partial x}} + \frac{\partial}{\partial y}\frac{\partial F}{\partial\frac{\partial\eta}{\partial y}} + \frac{\partial}{\partial z}\frac{\partial F}{\partial\frac{\partial\eta}{\partial z}}\right)\delta\eta \\
& + \left(\frac{\partial}{\partial x}\frac{\partial F}{\partial\frac{\partial\zeta}{\partial x}} + \frac{\partial}{\partial y}\frac{\partial F}{\partial\frac{\partial\zeta}{\partial y}} + \frac{\partial}{\partial z}\frac{\partial F}{\partial\frac{\partial\zeta}{\partial z}}\right)\delta\zeta\ \} \\
- K\int df\{ & \left(\frac{\partial F}{\partial\frac{\partial\xi}{\partial x}}\cos(n,x) + \frac{\partial F}{\partial\frac{\partial\xi}{\partial y}}\cos(n,y) + \frac{\partial F}{\partial\frac{\partial\xi}{\partial z}}\cos(n,z)\right)\delta\xi \\
& + \left(\frac{\partial F}{\partial\frac{\partial\eta}{\partial x}}\cos(n,x) + \frac{\partial F}{\partial\frac{\partial\eta}{\partial y}}\cos(n,y) + \frac{\partial F}{\partial\frac{\partial\eta}{\partial z}}\cos(n,z)\right)\delta\eta \\
& + \left(\frac{\partial F}{\partial\frac{\partial\zeta}{\partial x}}\cos(n,x) + \frac{\partial F}{\partial\frac{\partial\zeta}{\partial y}}\cos(n,y) + \frac{\partial F}{\partial\frac{\partial\zeta}{\partial z}}\cos(n,z)\right)\delta\zeta
\end{aligned}$$

From the equation $\delta P - K\,\delta\Omega = 0$, one has only to set the coefficients of $\delta\xi$, $\delta\eta$, $\delta\zeta$ in the expressions for δP and $K\,\delta\Omega$ equal to teach other, and one finds the equations which are identical to eq. (1) and (3) if one sets[32]

[32] In the notation of appendix A, $\mathbf{D} = \left(\partial_{\mathrm{A}}\mathcal{W}(\mathbf{E})\right)^{\mathrm{T}}$

$$X_x = -K\frac{\partial F}{\partial \frac{\partial \xi}{\partial x}} \quad X_y = -K\frac{\partial F}{\partial \frac{\partial \xi}{\partial y}} \quad X_z = -K\frac{\partial F}{\partial \frac{\partial \xi}{\partial z}}$$

$$Y_x = -K\frac{\partial F}{\partial \frac{\partial \eta}{\partial x}} \quad Y_y = -K\frac{\partial F}{\partial \frac{\partial \eta}{\partial y}} \quad Y_z = -K\frac{\partial F}{\partial \frac{\partial \eta}{\partial z}}$$

$$Z_x = -K\frac{\partial F}{\partial \frac{\partial \zeta}{\partial x}} \quad Z_y = -K\frac{\partial F}{\partial \frac{\partial \zeta}{\partial y}} \quad Z_z = -K\frac{\partial F}{\partial \frac{\partial \zeta}{\partial z}}$$

It remains to show how these values of X_x, X_y, etc., agree with those established above. In order to formulate the function F, we use the established expression (10) for λ, the values $\lambda_1, \lambda_2, \lambda_3$ are found if we eliminate a, b, c from the system (12) and find the roots of the cubic equation that we then obtain for λ. This cubic equation is

$$0 = (L-\lambda)(M-\lambda)(N-\lambda) - (L-\lambda)l^2$$
$$- (M-\lambda)m^2 - (N-\lambda)n^2 + 2lmn.$$

There is also

$$\lambda_1 + \lambda_2 + \lambda_3 = L + M + N$$
$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = LM + MN + NL - l^2 - m^2 - n^2$$

and therefore

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = L^2 + M^2 + N^2 + 2l^2 + 2m^2 + 2n^2$$

Consequently,

$$F = L^2 + M^2 + N^2 + 2l^2 + 2m^2 + 2n^2 + \theta(L + M + N)^2$$

One forms the derivatives in question from this expression in view of the values of L, M, N, l, m, n, from (11). Thus, one shows that the given values of X_x, X_y, etc., are identical with those established above.

The agreement of my equation (13) with my equations (1) and (3), which are identical with those of St. Venant, is accordingly established. I believe however the former will generally be more convenient for the applications. I have used them to obtain the equilibrium equations for the finitely curved plate and for a rod in a similar manner.