2.2 Material Symmetries[1]

In general, the constitutive relations depend on the choice of reference configuration. However, certain rotations of the reference configuration may leave the constitutive relations unchanged. In physical terms, two samples taken from the same homogeneous sheet of material but which were oriented differently from each other may be indistinguishable in any material test.

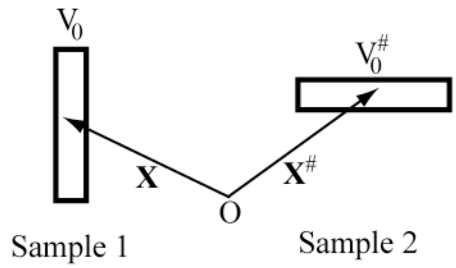

Fig. 2.2.1 Two samples of the same material

Sample 1 occupies a region V_0 with reference position $\mathbf{X}$. Sample 2 occupies a region $V_0^{\#}$ with reference position $\mathbf{X}^{\#}$:

$$\mathbf{X}^{\#} = \mathbf{Q} \cdot \mathbf{X} + \mathbf{c}. \tag{1}$$

Test 1. Sample 1 is subjected to a deformation

$$\begin{aligned} \mathbf{x} &= f(\mathbf{X}), \\ d\mathbf{x} &= \frac{\partial f}{\partial \mathbf{X}} \cdot d\mathbf{X} = \mathbf{F} \cdot d\mathbf{X}. \end{aligned} \tag{2}$$

Test 2. (a) Move sample 2 to coincide with V_0:

$$\begin{aligned} \mathbf{X} &= \mathbf{Q}^{\mathrm{T}} \cdot (\mathbf{X}^{\#} - \mathbf{c}), \\ d\mathbf{X} &= \mathbf{Q}^{\mathrm{T}} \cdot d\mathbf{X}^{\#}, \\ \frac{\partial \mathbf{X}}{\partial \mathbf{X}^{\#}} &= \mathbf{Q}^{\mathrm{T}}. \end{aligned} \tag{3}$$

(b) Apply the same deformation as in Test 1:

$$\begin{aligned} \mathbf{x} &= f(\mathbf{X}) = f(\mathbf{X}(\mathbf{X}^{\#})), \\ d\mathbf{x} &= \frac{\partial f}{\partial \mathbf{X}} \cdot \frac{\partial \mathbf{X}}{\partial \mathbf{X}^{\#}} \cdot d\mathbf{X}^{\#} = \mathbf{F}^{\#} \cdot d\mathbf{X}^{\#}, \end{aligned} \tag{4}$$

where

[1] This report provides an alternative to Section 2.2 of my book and makes a clearer statement of the physics of the matter.

$$\mathbf{F}^{\#} = \mathbf{F} \cdot \mathbf{Q}^{\mathrm{T}},$$
$$\mathbf{F} = \mathbf{F}^{\#} \cdot \mathbf{Q}. \tag{5}$$

While $\theta^{\#} = \theta$ and $\mathbf{g}^{\#} = \mathbf{g}$. Therefore,

$$\mathbf{C}^{\#} = (\mathbf{F}^{\#})^{\mathrm{T}} \cdot \mathbf{F}^{\#} = \mathbf{Q} \cdot \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} \cdot \mathbf{Q}^{\mathrm{T}} = \mathbf{Q} \cdot \mathbf{C} \cdot \mathbf{Q}^{\mathrm{T}} \tag{6}$$

and

$$\mathbf{E}^{\#} = \mathbf{Q} \cdot \mathbf{E} \cdot \mathbf{Q}^{\mathrm{T}}. \tag{7}$$

We now use the constitutive relations to calculate the internal energy, entropy, heat flux, and stress tensor in the deformed configuration for the two tests. The set of rotations that leaves these quantities unchanged is called the symmetry group of the material. If the symmetry group consists of all rotations, the material is called *isotropic*.

For an elastic material, the free energy in the tests of the two samples is

$$\psi = \hat{\psi}(\theta, \mathbf{E}) = \hat{\psi}(\theta, \mathbf{E}^{\#}). \tag{8}$$

Therefore

$$\hat{\psi}(\theta, \mathbf{E}) = \hat{\psi}(\theta, \mathbf{Q} \cdot \mathbf{E} \cdot \mathbf{Q}^{\mathrm{T}}) \tag{9}$$

identically in $\mathbf{E}$ for all rotations $\mathbf{Q}$ in the symmetry group.

Since the stress tensor is the same for the two samples we find

$$\mathbf{F}^{\#} \cdot \hat{\mathbf{T}}(\theta, \mathbf{E}^{\#}) \cdot \mathbf{F}^{\#\mathrm{T}} = \mathbf{F} \cdot \hat{\mathbf{T}}(\theta, \mathbf{E}) \cdot \mathbf{F}^{\mathrm{T}},$$
$$\mathbf{F} \cdot \mathbf{Q}^{\mathrm{T}} \cdot \hat{\mathbf{T}}(\theta, \mathbf{Q} \cdot \mathbf{E} \cdot \mathbf{Q}^{\mathrm{T}}) \cdot \mathbf{Q} \cdot \mathbf{F}^{\mathrm{T}} = \mathbf{F} \cdot \hat{\mathbf{T}}(\theta, \mathbf{E}) \cdot \mathbf{F}^{\mathrm{T}}, \tag{10}$$

identically in $\mathbf{F}$ for all rotations $\mathbf{Q}$ in the symmetry group. Therefore

$$\mathbf{Q} \cdot \hat{\mathbf{T}}(\theta, \mathbf{E}) \cdot \mathbf{Q}^{\mathrm{T}} = \hat{\mathbf{T}}(\theta, \mathbf{Q} \cdot \mathbf{E} \cdot \mathbf{Q}^{\mathrm{T}}) \tag{11}$$

identically in $\mathbf{E}$ for all rotations $\mathbf{Q}$ in the symmetry group.