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# The Shear Center and Kirchhoff's Theory of Rods

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As can be easily be seen by testing a rod, a load applied at the centroid of the free end of a cantilever with an unsymmetrical cross section causes both bending and twisting. Bending without twist only occurs when the load is applied at a particular point of the cross section which is called the *shear center.* This phenomenon is not predicted by Kirchhoff's theory of rods. According to Kirchhoff's theory, a load applied at the centroid causes bending without twist. This would appear to be a defect in the theory. However, we shall see here that, for slender rods with compact cross section, the terms accounting for the shear center are small compared to those retained in Kirchhoff's theory .

## Part I. Kirchhoff's Theory of rods

### *1. Basic Equations*

Kirchhoff's theory [1859, 1876] of bending and twisting of rods is the first approximation to the nonlinear equations of elasticity when the strains are small but the displacements may be large. A full presentation is given by Clebsch [1883] and Love [1892]. Studies[1] of the asymptotic solution of the equations of large displacement of elastic bodies have confirmed that the Kirchhoff theory is the correct first approximation for slender rods. The theory has been widely studied and used down to the present time. A modern treatment of Kirchhoff's theory is presented by Dill [1992]. We will first review the principle results from that theory.

Suppose that the undeformed body is a straight rod of uniform cross section. The line of centroids of the cross sections is called the axis of the rod. Let $(x_k) = (x, y, z)$ be a rectangular Cartesian coordinate system with origin at the centroid of one end of the rod and the $z$-axis directed along the axis of the rod. Let $\mathbf{a}_i$ be the associated base vectors and chose $\mathbf{a}_\alpha$ to be

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[1] Rigolot [1972], Parker [1979, 1984], Cimetière *et al* [1988].

directed along the principal axes of the cross sections.[1] The position vector of a point on the axis of the rod is $\boldsymbol{R}_0 = s\,\mathbf{a}_3$ where $s = z$ is the arc length. The position vector of a material point of the undeformed body can be expressed in the form

$$\mathbf{R}(x_1, x_2, s) = \mathbf{R}_o(s) + x_\alpha\ \mathbf{a}_\alpha(s)\ . \tag{1.1}$$

The arc length $s$ is measured from one end of the axis whose length is $L$: $0 \le s \le L$. Denote by $h$ a maximal dimension of the cross section so that $|x_\alpha| \le h$. We will refer to $h$ as the radius of the rod although the rod is not circular in general. The body is called a rod if the *slenderness ratio* $\alpha = h/L$ is small compared to 1.

When the rod is bent, the cross section forms a material surface which we call an s-section and the axis of the rod becomes a space curve with position vector

$$\boldsymbol{r}_0 = \boldsymbol{r}_0(s,t)\ . \tag{1.2}$$

Since $s$ is the arc length of the undeformed axis,

$$\frac{\partial \mathbf{r}_o}{\partial s} = (1+\varepsilon)\mathbf{t}\ , \tag{1.3}$$

where $\mathbf{t}$ is the unit vector tangent to the deformed axis and $\varepsilon$ is the extension of the axis.

The deformation of the rod can be described by the formula

$$\mathbf{r} = \mathbf{r}_0 + x_\alpha \mathbf{e}_\alpha + \hat{u}_k \mathbf{e}_k\ , \tag{1.4}$$

where $\mathbf{e}_k$ is a right-handed orthonormal system with $\mathbf{e}_3 = \mathbf{t}$. Since $\mathbf{e}_k(s, t)$ is a right-handed orthonormal system, there is a vector $\boldsymbol{\kappa}$ such that

$$\frac{\partial \mathbf{e}_k}{\partial s} = \boldsymbol{\kappa} \times \mathbf{e}_k\ , \tag{1.5}$$

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[1] Greek letter indices have the range (1,2), while other indicies have the range (1,2,3).

The components of $\boldsymbol{\kappa}$ on the basis $\mathbf{e}_k$ are $\kappa_1$ and $\kappa_2$, which are called the components of curvature of the rod, and $\kappa_3$ which is called the twist of the rod.

The case $\hat{u}_k(x, y, s, t) \equiv 0$ describes a special motion in which, the s-sections are plane and undeformed, and normal to the deformed axis. The s-sections are then cross sections of the deformed body and the vectors $\mathbf{e}_\alpha$ are the deformed directions of the fibers $\mathbf{a}_\alpha$. We therefore call $\hat{u}_k$ the extra displacements.

The extra displacements $\hat{u}_k(x, y, s, t)$ are such that $\hat{u}_k(0, 0, s, t) = 0$. At the time $t_o$ when the body is in the reference configuration, $\hat{\mathbf{u}} = 0$ and we choose $\boldsymbol{e}_k = \boldsymbol{a}_k$. In a rigid motion, we equate $\boldsymbol{e}_k$ with the convected fibers $\mathbf{a}_k$ so that $\hat{\mathbf{u}}$ is zero in such a motion.

In Kirchhoff's theory of rods, the extra displacements $\hat{u}_k$ are assumed to be small, so that the motion of the rod differs by small terms from one in which the cross sections remain plane, and undeformed, and normal to the deformed axis. The vectors $\mathbf{e}_i$ are then still the deformed directions of the fibers $\mathbf{a}_i$ to first order approximation. By small, we mean that

$$\frac{\hat{u}_i}{h} = O(\alpha) , \qquad \frac{\partial \hat{u}_i}{\partial x_\alpha} = O(\alpha) , \qquad \frac{\partial \hat{u}_i}{\partial s} = O(\alpha^2) . \tag{1.6}$$

The order of magnitude symbol is used here in the sense that the quantity is of that numerical magnitude or smaller.

To first order, neglecting terms of $O(\alpha^2)$, the extra displacements turn out to be the lateral contraction of the rod due to the extension of the fibers in bending and the warping of the cross section in torsion. For an isotropic elastic material,

$$\hat{u}_1 = -\nu\varepsilon x - \nu\kappa_1 xy + \nu\kappa_2 \frac{1}{2}\left(x^2 - y^2\right) , \tag{1.7}$$

$$\hat{u}_2 = -\nu\varepsilon y + \nu\kappa_2 xy + \nu\kappa_1 \frac{1}{2}\left(x^2 - y^2\right) , \tag{1.8}$$

$$\hat{u}_3 = \kappa_3 \Phi(x, y) , \tag{1.9}$$

The constant ν is the lateral contraction (Poisson) ratio in small extension. The function Φ is the warping function for torsion which depends only on the shape of the cross section. The assumption (1.6) is satisfied if the warping function is such that

$$\Phi(0,0) = 0 \ , \quad \frac{\Phi(x,y)}{h^2} = O(1) \, , \quad \frac{1}{h}\frac{\partial \Phi}{\partial x} = O(1) \, , \quad \frac{1}{h}\frac{\partial \Phi}{\partial y} = O(1) \, . \qquad (1.10)$$

and the deformation is such that

$$\varepsilon = O(\alpha), \quad \kappa_i h = O(\alpha), \qquad (1.11)$$

$$L\frac{\partial \varepsilon}{\partial s} = O(\alpha) \ , \quad L^2\frac{\partial \kappa_i}{\partial s} = O(\alpha) \ . \qquad (1.12)$$

That is, Kirchhoff's theory applies only to slender rods ($\alpha << 1$), with cross sections such that (1.10) applies, which undergo small bending and extension such that there is no locally large gradient of extension or curvature of the axis.

The components $\varepsilon_{ij}$ of the strain tensor[1] **E** with respect to the basis $\mathbf{a}_k \otimes \mathbf{a}_m$ are of $O(\alpha)$. Neglecting terms of $O(\alpha^2)$,

$$\varepsilon_{11} = \varepsilon_{22} = -\nu(\varepsilon + \kappa_1 y - \kappa_2 x) \ , \qquad (1.13)$$

$$\varepsilon_{12} = 0 \ , \qquad (1.14)$$

$$\varepsilon_{13} = \tfrac{1}{2}\kappa_3\left(\frac{\partial \Phi}{\partial x} - y\right) \ , \qquad (1.15)$$

$$\varepsilon_{23} = \tfrac{1}{2}\kappa_3\left(\frac{\partial \Phi}{\partial y} + x\right) \ , \qquad (1.16)$$

$$\varepsilon_{33} = \varepsilon + \kappa_1 y - \kappa_2 x \ . \qquad (1.17)$$

The components of the stress tensor **S** on the basis $\mathbf{a}_k \otimes \mathbf{a}_m$ , and **D** on the basis $\mathbf{a}_k \otimes \mathbf{e}_m$ , neglecting terms of $O(\alpha^2)$, are

$$\sigma_{11} = \sigma_{12} = \sigma_{22} = 0, \qquad (1.18)$$

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[1] A summary of the notation for nonlinear elasticity is presented in Appendix A.

$$\sigma_{31} = G\,\kappa_3\left(\frac{\partial \Phi}{\partial x} - y\right) \ , \tag{1.19}$$

$$\sigma_{32} = G\,\kappa_3\left(\frac{\partial \Phi}{\partial y} + x\right) \ , \tag{1.20}$$

$$\sigma_{33} = E(\varepsilon - \kappa_2\, x + \kappa_1 y) \ . \tag{1.21}$$

Note that the components $\sigma_{\alpha\beta}$ are not zero in Kirchhoff's theory, just small: $\sigma_{\alpha\beta}/E$ is of $O(\alpha^2)$.

The solution (1.7)-(1.21) is completely determined by the extension $\varepsilon$ and the curvatures $\kappa_i$. These are found by solving the differential equations of the balance of momentum together with the global constitutive relations.

When the general expression of mechanical equilibrium is applied to a slice of the rod between two s-sections, the following equilibrium equation for the resultant force on an s-section is obtained:

$$\frac{\partial \mathbf{F}}{\partial s} + \mathbf{f} = 0 \ , \tag{1.22}$$

where $\boldsymbol{f}$ is the resultant, per unit of arc length of the undeformed axis, of the body forces and the tractions on the side of the rod, and $\boldsymbol{F}$ is the resultant force on an s-section.

The moment equilibrium for a slice of the rod yields the following result:

$$\frac{\partial \mathbf{M}}{\partial s} + \frac{\partial \mathbf{r}_o}{\partial s} \times \mathbf{F} + \mathbf{m} = 0 \ . \tag{1.23}$$

where $\boldsymbol{M}$ is the resultant moment about a point on the deformed axis of the stress vector on the s-section, and $\boldsymbol{m}$ is the resultant moment of the surface tractions and the body forces, per unit length of the undeformed axis.

Let $\boldsymbol{F} = F_k\,\mathbf{e}_k$ . Neglecting terms of $O(\alpha^2)$, the constitutive relation for axial extension is

$$F_3 = AE\,\varepsilon \tag{1.24}$$

There are no constitutive relations for $F_1$ or $F_2$. They are determined by the balance of momentum as in the elementary linear theory of bending of rods.

Neglecting terms of $O(\alpha^2)$, the constitutive relations for the moment $\boldsymbol{M} = M_k\,\mathbf{e}_k$, are as follows:

$$M_1 = E I_1\,\kappa_1\,, \tag{1.25}$$

$$M_2 = E I_2\,\kappa_2\,, \tag{1.26}$$

$$M_3 = G J\,\kappa_3\,, \tag{1.27}$$

where the moments of inertia and the torsion constant are given by

$$I_1 = \int\!\!\int y^2\,dxdy. \tag{1.28}$$

$$I_2 = \int\!\!\int x^2\,dxdy. \tag{1.29}$$

$$J = \int\!\!\int_A (x^2 + y^2 + x\frac{\partial\Phi}{\partial y} - y\frac{\partial\Phi}{\partial x})\,dxdy\,. \tag{1.30}$$

Equations (1.25)-(1.30) are the same relations as occur in the linear theory of small displacement of rods. However, due to the large displacements the geometric relations are nonlinear, as well as the equation (1.23) of moment equilibrium.

The equilibrium equations (1.22)-(1.23) in component form are as follows.

$$\frac{\partial F_1}{\partial s} + \kappa_2 F_3 - \kappa_3 F_2 = 0\,, \tag{1.31}$$

$$\frac{\partial F_2}{\partial s} + \kappa_3 F_1 - \kappa_1 F_3 = 0\,, \tag{1.32}$$

$$\frac{\partial F_3}{\partial s} + \kappa_1 F_2 - \kappa_2 F_1 = 0\,, \tag{1.33}$$

$$\frac{\partial M_1}{\partial s} + \kappa_2 M_3 - \kappa_3 M_2 - (1+\varepsilon)F_2 = 0 , \tag{1.34}$$

$$\frac{\partial M_2}{\partial s} + \kappa_3 M_1 - \kappa_1 M_3 + (1+\varepsilon)F_1 = 0 , \tag{1.35}$$

$$\frac{\partial M_3}{\partial s} + \kappa_1 M_2 - \kappa_2 M_1 = 0 . \tag{1.36}$$

The approach of Kirchhoff leads to a satisfactory expression for the shear stress in torsion (1.19)-(1.20), regardless of the cross section, and for the normal stress due to bending (1.21). These components of stress are of $O(\alpha)$. The shear stress in bending is not determined in Kirchhoff's theory because, as we shall see in Part II, those components are of $O(\alpha^2)$.

## 2. Small Displacements

We will now consider the Kirchhoff theory when the displacements are small. The deformed axis can be described by the displacements ($u_o$, $v_o$, $w_o$):

$$\mathbf{r}_o = u_o \, \mathbf{a}_1 + v_o \, \mathbf{a}_2 + (s + w_o) \, \mathbf{a}_3 . \tag{2.1}$$

Neglecting all nonlinear terms in the displacements, it follows from the definition of $\mathbf{e}_i$ that[1]

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{a}_1 + \phi \mathbf{a}_2 - u_o' \mathbf{a}_3 , \\ \mathbf{e}_2 &= -\phi \mathbf{a}_1 + \mathbf{a}_2 - v_o' \mathbf{a}_3 , \\ \mathbf{e}_3 &= u_o' \mathbf{a}_1 + v_o' \mathbf{a}_2 + \mathbf{a}_3 . \end{aligned} \tag{2.2}$$

The angle $\phi$ is the angle of rotation of the cross section about the axis of the rod due to the twisting of the rod. From (1.5) and (2.2), neglecting all nonlinear terms, the components of curvature and the extension are

$$\kappa_1 = -v_o'' , \quad \kappa_2 = u_o'' , \quad \kappa_3 = \phi' , \quad \varepsilon = w_o' . \tag{2.3}$$

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[1] A prime denotes the derivative with respect to $s$.

The displacement vector of a particle of the rod is **u** = **r** - **R**. Using (2.1)-(2.3), equations (1.1), (1.4), and (1.7)-(1.9) give the following components of the displacement vector on the basis $\mathbf{a}_i$:

$$u = u_o - \phi y - \varepsilon \nu x - \kappa_1 \nu x y + \kappa_2 \frac{\nu}{2}\left(x^2 - y^2\right) , \tag{2.4}$$

$$v = v_o + \phi x - \varepsilon \nu y + \kappa_2 \nu x y + \kappa_1 \frac{\nu}{2}\left(x^2 - y^2\right) , \tag{2.5}$$

$$w = w_o - x u_o' - y v_o' + \kappa_3 \Phi(x, y) . \tag{2.6}$$

The components of the Cauchy stress tensor **T** are given by the expressions (1.18)-(1.21) in this linearized theory.

The equations of equilibrium of forces become

$$\begin{aligned} &\frac{dF_1}{ds} + f_1 = 0 , \\ &\frac{dF_2}{ds} + f_2 = 0 , \\ &\frac{dF_3}{ds} + f_3 = 0 . \end{aligned} \tag{2.7}$$

The equations of moment equilibrium become

$$\begin{aligned} &\frac{dM_1}{ds} - F_2 + m_1 = 0 , \\ &\frac{dM_2}{ds} + F_1 + m_2 = 0 , \\ &\frac{dM_3}{ds} + m_3 = 0 . \end{aligned} \tag{2.8}$$

The constitutive relations remain

$$\begin{aligned} &F_3 = AE\varepsilon , \\ &M_1 = E I_1 \kappa_1 , \\ &M_2 = E I_2 \kappa_2 , \\ &M_3 = G J \kappa_3 , \end{aligned} \tag{2.9}$$

but the components of curvature are given by the linearized expressions(2.3).

These equations agree with the elementary beam theory. However, the moment $M_3$ is the moment about the centroid of the cross section, so there will be no twist if the torque about the centroid is zero. In particular, for a rod loaded only by shear forces applied to one end, there is no twist for loads applied at the centroid (according to the Kirchhoff theory). This is in contrast to the known result that end loads acting at the centroid will produce twist unless the shear center coincides with the centroid. That is, the corrections for twist of the rod in the case when the shear center does not coincide with the centroid are not included in Kirchhoff's theory. We shall see why this is the case by comparing Kirchhoff's theory with the exact solution for the small displacement of rods with end loads which is presented in Part II.

The comparison will be made for a cantilever rod with given resultant shear force $V$ directed along the x-axis and a resultant twisting moment $T$ on the end $z = L$. Global equilibrium (2.7)-(2.8) requires that $F_1 = 0$, $F_2 = V$, $F_3 = 0$, $M_1 = 0$, $M_2 = V(L - z)$, and $M_3$ is constant. The constitutive equations (2.9) show that $\varepsilon = 0$, $\kappa_1 = 0$ and $\kappa_3$ is constant. Thus $v_o$ is a rigid motion at most. We take the rotation to be zero at the end $z = 0$ so that the geometric relations (2.3) provide $\phi = \kappa_3 z$.

The displacements (2-4)-(2.6) reduce to

$$u = u_o - \phi y + \kappa_2 \frac{\nu}{2}\left(x^2 - y^2\right) , \qquad (2.10)$$

$$v = +\phi x + \kappa_2 \nu x y , \qquad (2.11)$$

$$w = -x u_o' + \kappa_3 \Phi(x,y) . \qquad (2.12)$$

The stress components (1.18)-(1.21) simplify to the following

$$\sigma_{11} = \sigma_{12} = \sigma_{22} = 0, \qquad (2.13)$$

$$\sigma_{31} = G\, \kappa_3 \left(\frac{\partial \Phi}{\partial x} - y\right) , \qquad (2.14)$$

$$\sigma_{32} = G\, \kappa_3 \left(\frac{\partial \Phi}{\partial y} + x\right) , \qquad (2.15)$$

$$\sigma_{33} = - E \kappa_2 x . \tag{2.16}$$

Since the equations of small displacement are linear, we can consider the two load cases separately.

### *a. Twisting of a rod by a torque T.*

In this first case, the force acting on the end is zero, and equations (2.7) give $F_i = 0$. Since $M_1$ and $M_2$ are zero at the end $z = L$, equations (2.8) give $M_1 = 0$ and $M_2 = 0$. Equations (2.9) give $\kappa_1 = 0$ and $\kappa_2 = 0$, and by (2.2), $\upsilon_o$ and $u_o$ are rigid motions. The complete solution is

$$\begin{aligned} u &= -\phi y \, , \\ \upsilon &= +\phi x \, , \\ w &= \kappa_3 \Phi \, , \\ \phi &= \kappa_3 z \, , \\ T &= GJ \kappa_3 \, , \end{aligned} \tag{2.17}$$

apart from the rigid body motion.

### *b. Bending by a shear load V along the x-axis.*

In this second case, the twisting moment $M_3$ is zero at the end $z = L$, and the only end load is a resultant force $V$ along the x-axis. The equilibrium equations (2.7) and (2.8) give $F_1 = V$, $F_2 = 0$, $F_3 = 0$, $M_1 = 0$, $M_2 = V(L - z)$, $M_3 = 0$. The constitutive relations (2.9) show that $\kappa_3 = 0$ and $\kappa_1 = 0$. Consequently, by (2.3), $\phi$ is constant and corresponds to a rigid rotation, and $\upsilon_o$ is at most a rigid displacement. Solving the remaining relation for $u_o$ we find the following complete solution, apart from rigid displacements:

$$
\begin{aligned}
&u_o = \frac{V}{EI}\left(\frac{Lz^2}{2} - \frac{z^3}{6}\right), \\
&v_o = 0\,, \\
&w_o = 0\,, \\
&u = u_o + \nu\kappa_2 \frac{1}{2}\left(x^2 - y^2\right)\,, \qquad (2.18) \\
&v = \nu\kappa_2 xy\,, \\
&w = -u_o' x\,, \\
&M = EI\kappa_2\,.
\end{aligned}
$$

The components of stress are given by (1.18)-(1.21): $\sigma_{\alpha\beta} = 0$,

$$\sigma_{31} = 0, \tag{2.19}$$

$$\sigma_{32} = 0, \tag{2.20}$$

$$\sigma_{33} = -E\,\kappa_2\,x\ . \tag{2.21}$$

Recall that terms of $O(\alpha^2)$ are neglected.

## Part II. The linear theory of elasticity

### 3. *Bending and Twisting of Rods by Terminal Loads*

We review here a special problem in the linear theory of elasticity (the theory of small displacement gradients) such that a (non-circular) cylinder is loaded by tractions distributed over the end $z = L$, which are statically equivalent to a shear force $V$ applied at the centroid and a twisting moment $T$. The material is linearly elastic with tensile modulus $E$, shear modulus $G$, and Poisson ratio $\nu$.

Let the $z$-axis be directed along the line of centroids of the cross sections of the cylinder, and the $x$-axis and $y$-axis be directed along the principal axes of the cross sections. The origin is chosen at one end of the cylinder which has length $L$: $0 \le z \le L$. Each cross section covers a region $\mathcal{R}$ in the $x$-$y$ plane with bounding curve $\mathcal{C}$. The $z$-axis is called the axis of the rod. Again, let $h$ be the maximum value taken by $|x|$ or $|y|$, and define the slenderness ratio by $\alpha = h/L$. We are interested in the case when $\alpha << 1$.

We restrict attention to cross sections which are simply connected regions, and the case when $V$ is directed along the $x$-axis. Consequently, the only moment on each cross section is $M_2 \equiv M$. Global equilibrium shows that $M_1 = 0$, $M_2 = M = V(L - z)$, and $M_3 = T$.

The solution to the basic equations of the linear theory of elasticity which satisfies these conditions is presented by Sokolnikoff [1956]. It can be expressed in terms of the warping function in torsion $\Phi(x,y)$ and the bending function[1] $\Psi(x,y)$ for bending about the $y$-axis. These are functions which depend only on the geometry of the cross section. They have been determined for a few cross sections. For an ellipse, $\Phi = Cxy$. For a circle, of raius $a$, $\Psi = (\frac{3}{4} + \frac{\nu}{2})a^2 x + (\frac{1}{12} + \frac{\nu}{6})(x^3 - 3xy^2)$. We will restrict our attention to cross sections such that $\Phi$ satisfies (1.10) and the bending function is such that

$$\Psi(0,0) = 0 \quad , \quad \frac{\Psi(x,y)}{h^3} = O(1) \, , \quad \frac{1}{h^2}\frac{\partial \Psi}{\partial x} = O(1) \, , \quad \frac{1}{h^2}\frac{\partial \Psi}{\partial y} = O(1) \, . \qquad (3.1)$$

The moment of inertia of the cross section about the $y$-axis is denoted by $I$ and the torsion constant by $J$. They are defined by (1.29) and (1.30). The nonzero components of stress are given by

$$\sigma_{33} = -E\kappa_2 x \, , \qquad (3.2)$$

$$\sigma_{13} = G\kappa_3\left(\frac{\partial \Phi}{\partial x} - y\right) + \frac{GV}{EI}\left(\frac{\partial \Psi}{\partial x} + \nu y^2 - (1+\nu)x^2\right) \qquad (3.3)$$

$$\sigma_{23} = G\kappa_3\left(\frac{\partial \Phi}{\partial y} + x\right) + \frac{GV}{EI}\frac{\partial \Psi}{\partial y} \qquad (3.4)$$

These equations may be compared with (2.13 to (2.16). The new terms with $\Psi$ make contributions of $O(\alpha^2)$ . Terms of that order are neglected in

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[1] Solkolnikoff's function $\varphi_1$.

Kirchhoff's theory. The higher order terms in (3.3) and (3.4) are needed in order to find the shear center.

The components of displacement of the z-axis are

$$u_o = \frac{V}{EI}\left(\frac{Lz^2}{2} - \frac{z^3}{6}\right), \quad v_o = 0, \quad w_o = 0. \tag{3.5}$$

The rotation of a fiber of the axis is $\theta$, the curvature is $\kappa_2$, and the rotation of normal fibers about the z-axis is $\phi$:

$$\theta = u_o', \quad \kappa_2 = u_o'', \quad \phi = \kappa_3 z. \tag{3.6}$$

To within the rigid motions, the components of displacement are

$$u = u_o - \phi y + \kappa_2 \frac{\nu}{2}\left(x^2 - y^2\right). \tag{3.7}$$

$$v = +\phi x + \kappa_2 \nu x y. \tag{3.8}$$

$$w = -\theta x + \kappa_3 \Phi + \frac{V}{EI}\left(\Psi - \frac{2+\nu}{6}x^3 + \frac{\nu}{2}xy^2\right). \tag{3.9}$$

These expressions agree with the equations (2.10)-(2.12) of the Kirchhoff theory except for the term with $\Psi$. That term makes a contribution of $O(\alpha^3)$. Terms of that order are neglected in Kirchhoff's theory.

The resultant moment is

$$M_2 = EI\kappa_2. \tag{3.10}$$

which agrees with Kirchhoff's theory. The resultant torque about the centroid is

$$T = GJ\kappa_3 - Vy_o \tag{3.11}$$

where

$$y_o = \frac{1}{2(1+\nu)I}\iint_{\mathcal{R}}\left[x\frac{\partial\Psi}{\partial y} - y\frac{\partial\Psi}{\partial x} + (1+\nu)yx^2 - y^3\right]dA \tag{3.12}$$

Equation (3.11) does not agree with the constitutive relation $(2.9)_4$ of Kirchhoff's theory. We see from (3.11) that the twist $\kappa_3$ may be nonzero even when $T$ is zero. That is, a shear load $V$ acting through the centroid of the cross section will produce twist as well as bending unless $y_o = 0$. This feature is not contained in the theory of Kirchhoff. We are interested here in seeing why this is so.

When $\kappa_3$ is zero, we must have $T = - V y_o$. which is equivalent to a shear force acting at a distance $y_o$ from the centroid. A similar result holds for a load directed along the $y$-axis which determines $x_o$. The point $(x_o, y_o)$ is called the shear center. The twist is zero for an end load acting at the shear center However, a shear force applied at the centroid will produce a nonzero twist unless the shear center coincides with the centroid.

We again consider separately the case when the only load is a twisting moment on one end and the case when the only load is a shear force acting at the centroid. The complete solution for bending and twisting is the sum of the two.

### *a. Twisting of a rod by a torque T.*

When $V$ is zero, the complete solution agrees exactly with the theory of Kirchhoff, equations (2.17).

### *b. Bending by a shear load V along the x-axis.*

We are now interested in the deformations when the torque $T$ is zero and the shear load acts through the centroid. In that case, (3,11) gives

$$\kappa_3 = \frac{V}{GJ} y_o \,. \tag{3.13}$$

In order to display the corresponding stress and displacement in a form which makes the order of magnitude of each term easily seen, we will use nondimensional variables:

$$\bar{x} = \frac{x}{h}, \quad \bar{y} = \frac{y}{h}, \quad \bar{z} = \frac{z}{L}, \tag{3.14}$$

$$\bar{u} = \frac{u}{L}, \quad \bar{v} = \frac{v}{L}, \quad \bar{w} = \frac{w}{L}, \tag{3.15}$$

$$\bar{\kappa}_2 = \kappa_2 L, \quad \bar{\Phi} = \frac{\Phi}{h^2}, \quad \bar{\Psi} = \frac{\Psi}{h^3}, \tag{3.16}$$

$$\bar{V} = \frac{VL^2}{EI}, \quad \bar{M} = \frac{ML}{EI}, \quad \bar{T} = \frac{TL}{GJ}. \tag{3.17}$$

Note the $\bar{y}_o = O(1)$ and $\frac{EI}{GJ} = O(1)$ for Kirchhoff rods. The nondimensional coordinates $\bar{x}, \bar{y}, \bar{z}$, are of O(1). The nondimensional torsion and bending functions $\bar{\Phi}$ and $\bar{\Psi}$ are also of O(1). Kirchhoff's theory is for the case $\bar{V} = O(1)$ and $\bar{M} = O(1)$.

The nonzero stress components are then as follows.

$$\frac{\sigma_{33}}{E} = -\alpha\bar{\kappa}_2\bar{x}\,. \tag{3.18}$$

$$\frac{\sigma_{13}}{G} = \alpha^2\bar{V}\frac{EI}{GJ}\bar{y}_o\left(\frac{\partial\bar{\Phi}}{\partial\bar{x}} - \bar{y}\right) + \alpha^2\bar{V}\left(\frac{\partial\bar{\Psi}}{\partial\bar{x}} + \nu\bar{y}^2 - (1+\nu)\bar{x}^2\right) \tag{3.19}$$

$$\frac{\sigma_{23}}{G} = \alpha^2\bar{V}\frac{EI}{GJ}\bar{y}_o\left(\frac{\partial\bar{\Phi}}{\partial\bar{y}} + \bar{x}\right) + \alpha^2\bar{V}\left(\frac{\partial\bar{\Psi}}{\partial\bar{y}}\right) \tag{3.20}$$

These expressions may be compared with (2.19)-(2.21). We see that the extra shear stress when the shear center does not coincide with the centroid are of O($\alpha^2$), the same as the shear stress due to bending. Terms of O($\alpha^2$) are neglected in Kirchhoff's theory, and the terms introduced by the shear center phenomena are of that order.

The components of displacement are

$$\bar{u} = \bar{u}_o - \alpha^2\bar{V}\frac{EI}{GJ}\bar{y}_o\bar{y}\bar{z} + \alpha^2\bar{\kappa}_2\frac{\nu}{2}\left(\bar{x}^2 - \bar{y}^2\right) \tag{3.21}$$

$$\bar{v} = +\alpha^2\bar{V}\frac{EI}{GJ}\bar{y}_o\bar{x}\bar{z} + \alpha^2\bar{\kappa}_2\nu\bar{x}\bar{y}\,. \tag{3.22}$$

$$\overline{w} = -\alpha\theta\overline{x} + \alpha^3\overline{V}\frac{EI}{GJ}\overline{y}_o\overline{\Phi} + \alpha^3\overline{V}\left(\overline{\Psi} - \frac{2+\nu}{6}\overline{x}^3 + \frac{\nu}{2}\overline{xy}^2\right). \qquad (3.23)$$

These expressions may be compared with (2.18). The extra terms in $y_0$ are of the same order as the extra displacements and the correction due to the bending function. Kirchhoff's theory therefore agrees up through terms of $O(\alpha)$ for $u$ and $v$, and through $O(\alpha^2)$ for $w$.

The normal stress on the cross-section is proportional to the bending moment and is $O(\alpha)$. The shear force $\overline{V}$ causing the bending produces shear stress of $O(\alpha^2)$ which are not included in Kirchhoff's theory.

## 4. *Thin-walled cross sections*

There are cross sections for which the extra terms found in the complete solution by the equations of linear elasticity, section 3, are not negligible. In particular, the extra terms may be important for thin-walled cross sections.

Kirchhoff's theory only applies to rods with cross sections for which the warping function satisfies (1.10). The restrictions (1.10) implies that the torsion constant $J$, defined by (1.30), and the moment of inertia $I$, defined by (1.29), are of the same order of magnitude:

$$\frac{I}{J} = O(1). \qquad (4.1)$$

The extra twist due to a displaced shear center, given by (3.13), is

$$\kappa_3 L = \alpha\overline{y}_o\overline{V}\frac{EI}{GJ}. \qquad (4.2)$$

In the case that (4.1) is true,

$$\kappa_3 L = O(\alpha). \qquad (4.3)$$

However, equations (4.1), and therefore(4.3), may not hold for thin-walled cross sections because the torsion constant $J$ may be much smaller than the bending constant $I$. Consequently, Kirchhoff's theory does not apply to such rods.

To see this explicitly, let us consider the bending and twisting of a rod with the channel shaped cross section shown in fig. 1.

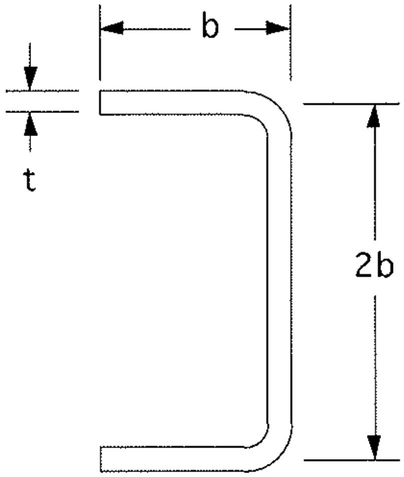

Fig. 1 Channel

The thickness ratio $t/h$ is a small quantity. To first order terms in the thickness ratio,

$$I = \frac{8}{3}th^3 \,, \quad J = \frac{4}{3}t^3h \quad . \tag{4.4}$$

That is,

$$\frac{I}{J} = 2\frac{h^2}{t^2} \,. \tag{4.5}$$

For small t, this ratio can be quite large, whatever the value of the slenderness ratio $\alpha = h/L$.

The shear center in this case is at $\overline{y}_o = 7/8$. The extra twist due to the displaced shear center is given by (4.2) and (4.5):

$$\kappa_3 L = \alpha \frac{h^2}{t^2}\frac{E}{G}\overline{V}\overline{y}_o \,. \tag{4.6}$$

Therefore, for small t, this correction due to the displaced shear center (not present in Kirchhoff's theory) is not negligible.

## 5. Conclusion

For slender rods of compact cross section, where the assumption (1.10) applies, the extra deformations which occur when the shear center does not coincide with the centroid of the cross section are negligible. Kirchhoff's theory may be used for such rods.

However, there are rods for which the shear center effect is not negligible. Kirchhoff's theory, which considers only the slenderness ratio, is not the correct first order approximation for such rods.

## Appendix A. Continuum Mechanics.[1]

Let $\mathcal{V}$ denote the region occupied by the body in the reference configuration and $x^k$ denote curvilinear coordinates in $\mathcal{V}$. The position vector with respect to a fixed point of a material particle in the reference configuration is denoted by $\boldsymbol{R}$. Let

$$\mathbf{G}_k = \frac{\partial \mathbf{R}}{\partial x^k} \tag{A.1}$$

denote the family of gradient vectors in the reference configuration. They are the covariant base vectors for the curvilinear coordinate system in $\mathcal{V}$.

Denote the position vector, from the same origin, of the material particle in the deformed configuration by $\boldsymbol{r} = \boldsymbol{r}(\boldsymbol{x},t)$. All fields will depend on the material coordinates $\boldsymbol{x}$ and on time. The system of gradient vectors associated with the deformed body are

$$\mathbf{g}_k = \frac{\partial \mathbf{r}}{\partial x^k} \quad . \tag{A.2}$$

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[1] For a complete treatment see Truesdell and Noll [1965].

The deformation gradient $\boldsymbol{A}(\boldsymbol{x},t)$ is the tensor

$$\mathbf{A} = \mathbf{g}_k \otimes \mathbf{G}^k \quad . \tag{A.3}$$

The symbol $\boldsymbol{F}$ is commonly used instead of $\boldsymbol{A}$, but the symbol $\boldsymbol{F}$ is reserved for resultant force in this paper. A strain tensor can be defined by

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{A}^{\mathrm{T}} \cdot \boldsymbol{A} - \mathbf{1}) \quad . \tag{A.4}$$

Let $\boldsymbol{T}$ denote the stress tensor. The stress vector per unit area of the deformed body is

$$\boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{n} \cdot \boldsymbol{T} \, . \tag{A.5}$$

where $\boldsymbol{n}$ is the unit normal to the deformed body. Let

$$\boldsymbol{D} = \frac{\rho_o}{\rho} \boldsymbol{A}^{-1} \cdot \boldsymbol{T} \quad . \tag{A.6}$$

The stress vector per unit area of the reference configuration is

$$\boldsymbol{p}_{(\boldsymbol{N})} = \boldsymbol{N} \cdot \boldsymbol{D} \, , \tag{A.7}$$

where $\boldsymbol{N}$ is the unit normal in the reference configuration. It is convenient to introduce a third tensor:

$$\mathbf{S} = \boldsymbol{D}\,(\boldsymbol{A}^{-1})^{\mathrm{T}} . \tag{A.8}$$

Let

$$\mathbf{S} = \sigma^{km}\, \boldsymbol{G}_k \otimes \boldsymbol{G}_m \quad , \tag{A.9}$$

then

$$\boldsymbol{D} = \sigma^{km}\, \boldsymbol{G}_k \otimes \boldsymbol{g}_m \quad . \tag{A.10}$$

The constitutive equation of a homogeneous elastic material can be expressed in the form

$$\mathbf{S} = \mathcal{F}(\mathbf{E}) = \partial_{\boldsymbol{E}} \mathcal{W}(\mathbf{E}). \tag{A.11}$$

where $\mathcal{W}$ and $\mathcal{F}$ are functions which characterize the material. An example for an isotropic material is

$$\boldsymbol{S} = 2\mu\, \boldsymbol{E} + \lambda(\mathrm{tr}\, \boldsymbol{E})\, \mathbf{1} \quad . \qquad \text{(A.12)}$$

The relation (A.19) may be assumed as the constitutive equation for this investigation of rods. However, it doesn't matter which elastic law is used since we will only be interested in approximations which involve the linearized form for small $\boldsymbol{E}$. The scalars $\lambda$ and $\mu$ are the elastic constants of Lamé:

$$\mu = G = \frac{E}{2(1+\nu)}\,, \quad \lambda = \frac{2\mu\nu}{1-2\nu}\,. \qquad \text{(A.13)}$$

where $G$ is the shear modulus, $E$ is the tensile modulus, and $\nu$ is the transverse contraction (Poisson) ratio.

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