Reading Assignments
Chapter 5: Section 5.4.
Chapter 8: Sections 8.1, 8.2.1, 8.3.1, 8.3.2, 8.3.4, 8.3.5, 8.5.1.
Chapter 9: Sections 9.1, 9.2.

Text Homework Problems
Chapter 5: 5.19, 5.21 – 5.27.
Chapter 8: 8.1 – 8.3, 8.5 – 8.8, 8.12, 8.13, 8.15, 8.17 – 8.21, 8.23 – 8.25, 8.27 – 8.30, 8.31.a, 8.32 – 8.35, 8.37, 8.38, 8.45, 8.46, 8.49, 8.51.

19.1 Let $X_1, \ldots, X_n$ be i.i.d. $Poisson(\lambda)$ for $\lambda > 0$, and let $\hat{\gamma}_1$ and $\hat{\gamma}_2$ represent, respectively, the UMVUE and the MLE for $\gamma = e^\lambda$.
   i) Find the mean and variance of $\hat{\gamma}_1$ and of $\hat{\gamma}_2$.
   ii) Show directly that $\text{var}(\hat{\gamma}_1) > 1/I(\gamma; X)$.
   iii) Show $\text{var}(\hat{\gamma}_1) \rightarrow 1$ as $n \rightarrow \infty$.
   iv) Compare the MSEs of $\hat{\gamma}_1$ and $\hat{\gamma}_2$.

19.2 Prove that for a $k = 1$ dimensional exponential family in natural form, $I(\eta; X) = -d''_o(\eta)$.

20.1 Suppose $X_1, \ldots, X_n$ are i.i.d. Normal($\mu, \sigma^2$) with $\Theta = \{ (\mu, \sigma) \mid \mu \in \mathbb{R}, \sigma > 0 \}$.
   i) Find the confidence level of the interval $\bar{X} - 1 < \mu < \bar{X} + 1$.
   ii) Find a 95% confidence upper bound for $\sigma$.

20.2 For $X \sim Geometric(\theta)$, find a $(1 - \alpha) \times 100\%$ upper confidence bound for $\theta$.

21.1 Suppose $X_1, \ldots, X_n$ are i.i.d. Normal($\mu, \sigma^2$) with $\mu \in \mathbb{R}$ and $\sigma > 0$ both being unknown parameter. Consider testing $H_0 : \mu = 0$ vs. $H_1 : \mu > 0$.
   i) Find the significance level of the test with rejection region $\bar{X} > 5$.
   ii) Derive the $\alpha = 0.05$ level likelihood ratio test.

21.2 For a general parametric model, show that any likelihood ratio statistic must be a function of any sufficient statistic.

21.3 Let $t_\nu(\delta)$ represent the non-central t-distribution on $\nu$ degrees of freedom and non-centrality parameter $\delta$.
   i) Show that for a fixed $x$, the cumulative distribution function $F(x; \delta) = P[ t_\nu(\delta) \leq x ]$ is strictly decreasing in $\delta$.
   ii) Show that for a fixed $x$, $P[ | t_\nu(\delta) | > x ]$ is strictly increasing in $|\delta|$.

21.4 Suppose $X_1, \ldots, X_n$ are i.i.d. Normal($\mu, \sigma^2$) with $\mu \in \mathbb{R}$ and $\sigma > 0$ both being unknown parameters. Let $T_n(X) = \sqrt{n} \bar{X}/s$.
   i) Show $T_n(X) \sim t_{n-1}(\delta_n)$, where $\delta_n = \sqrt{n} \mu/\sigma$.
   ii) Consider testing $H_0 : \mu = 0$ vs. $H_1 : \mu \neq 0$. Show that the test which rejects when $| T_n(X) | > t_{n-1,\alpha/2}$ is a consistent $\alpha$-level test.
   iii) For $m < n$, show that the test which rejects when $| T_m(X) | > t_{m-1,\alpha/2}$, is inadmissible.
21.5 Suppose $X_1, \ldots, X_n$ are i.i.d. $Normal(\mu, \sigma^2)$ with $\mu \in \mathbb{R}$ and $\sigma > 0$ both unknown.

i) Construct the $\alpha$ level likelihood ratio test for $H_o : \sigma^2 = \sigma_o^2$ versus $H_1 : \sigma^2 < \sigma_o^2$

ii) Recall that the likelihood ratio test for $H_o : \sigma^2 = \sigma_o^2$ versus $H_1 : \sigma^2 \neq \sigma_o^2$ has the form

$$
\psi(X) = \begin{cases} 
1, & s^2/\sigma_o^2 < k_1 \text{ or } s^2/\sigma_o^2 > k_2 \\
0, & \text{otherwise}
\end{cases}
$$

where $s^2$ is the unbiased sample variance. Find the exact values of $k_1$ and $k_2$ when $\alpha = 0.05$ and $n = 5$.

22.1 Construct the most powerful $\alpha$ level test for the following cases, and find $\beta$, the probability of a Type II error, in each case.

i) $X_1, \ldots, X_n$ i.i.d. $Normal(\mu, \sigma^2)$, with $\sigma^2$ known. Test $H_o : \mu = \mu_o$ versus $H_1 : \mu = \mu_1$, where $\mu_1 < \mu_o$.

ii) $X_1, \ldots, X_n$ i.i.d. $Normal(\mu, \sigma^2)$, with $\mu$ known. Test $H_o : \sigma^2 = \sigma_o^2$ versus $H_1 : \sigma^2 = \sigma_1^2$, where $\sigma_1^2 > \sigma_o^2$.

iii) $H_o : X \sim Normal(0, 1)$ versus $H_o : X \sim Cauchy(0, 1)$.

22.2 Suppose $X_1, \ldots, X_n$ are i.i.d. $Uniform(0, \theta)$ for $\theta > 0$.

Consider testing $H_o : \theta = \theta_o$ vs. $H_1 : \theta = \theta_1$.

i) For the case $\theta_1 > \theta_o$,

a) Construct a most powerful $\alpha = 0.05$ level test.

b) Is the following test inadmissible? Clearly justify your answer.

$$
\psi(X) = \begin{cases} 
1, & X_{(n)} > \theta_o \text{ or } X_{(n)} < \theta_o/2 \\
0, & \text{otherwise}
\end{cases}
$$

ii) For the case $\theta_1 < \theta_o$, construct a most powerful $\alpha = 0.05$ level test.

iii) Does a uniformly most powerful $\alpha = 0.05$ level test exist for testing $H_o : \theta = \theta_o$ vs. $H_1 : \theta < \theta_o$? If not, why not? If yes, construct such a test.

23.1 Suppose $X_1, \ldots, X_n$ are i.i.d. $Normal(\mu, \sigma^2)$. Justify your answers below.

i) For unknown $\sigma$, does a UMP $\alpha$ level test of $H_o : \mu = 0$ versus $H_1 : \mu = 1$ exist?

ii) For known $\sigma$, say w.lo.g. $\sigma = 1$, does a UMP $\alpha$ level test of $H_o : \mu = 0$ versus $H_1 : \mu \neq 0$ exist?

23.2 Suppose $X_1, \ldots, X_3$ are i.i.d. $Poisson(\lambda)$, and consider $H_o : \lambda = 1$ versus $H_1 : \lambda = 0.5$. Is the following test inadmissible? Clearly justify your answer.

$$
\psi(X) = \begin{cases} 
1, & \sum_{i=1}^9 X_i \leq 3 \text{ or } X_1 = 0 \\
0, & \text{otherwise}
\end{cases}
$$

24.1 Read the proof of the Karlin-Rubin theorem. Also, state the theorem for the four cases:

i) $H_o : \theta \leq \theta_o$ vs. $H_1 : \theta > \theta_o$ when $T(x)$ has an increasing MLR.

ii) $H_o : \theta \leq \theta_o$ vs. $H_1 : \theta > \theta_o$ when $T(x)$ has a decreasing MLR.

iii) $H_o : \theta \geq \theta_o$ vs. $H_1 : \theta < \theta_o$ when $T(x)$ has an increasing MLR.

iv) $H_o : \theta \geq \theta_o$ vs. $H_1 : \theta < \theta_o$ when $T(x)$ has a decreasing MLR.
25.1 Suppose $X_1, \ldots, X_n$ are i.i.d. $\text{Poisson}(\lambda)$ for $\lambda > 0$. Suppose $n = 10$ and we observe $(X_1, \ldots, X_{10}) = (1, 0, 0, 0, 2, 0, 0, 1, 1, 0)$.
   i) Construct a 95% upper confidence limit for $\lambda$.
   ii) Find the P-value for testing $H_0 : \lambda = 1$ versus $H_1 : \lambda < 1$.

25.2 Suppose $x \sim \text{Binomial}(n = 10, \theta)$.
   i) Find a 95% lower bound if we observe $X = 6$.
   ii) Find an $\alpha = 0.05$ level test for testing $H_0 : \theta = 0.1$ for each of the following cases:
       (a) $H_1 : \theta > 0.1$, (b) $H_1 : \theta < 0.1$, and (c) $H_1 : \theta \neq 0.1$.

25.3 Suppose $X_1, \ldots, X_n$ be i.i.d. with c.d.f. $F$. The empirical distribution function is defined by $F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq x)$. Let $x_1 < x_2$.
   i) Find the joint distribution of $F_n(x_1)$ and $F_n(x_2)$.
   ii) Find $\text{cov}(F_n(x_1), F_n(x_2))$.

26.1 Show that: $\beta(\alpha_1, \ldots, \alpha_k) = \frac{\prod_{j=1}^{k} \Gamma(\alpha_j)}{\Gamma(\alpha)}$, where $\alpha = \alpha_1 + \cdots + \alpha_k$.

26.2 Let $X_1, \ldots, X_n$ be i.i.d. $\text{Exponential}(\beta)$. Find the joint distribution of the gaps:

$$Y_1 = X_{(1)}, \ Y_2 = (X_{(2)} - X_{(1)}), \ \ldots, \ Y_n = (X_{(n)} - X_{(n-1)}).$$