10.1. The generalized eigenvalues problem. For a $p \times p$ positive definite symmetric matrix $\Sigma$ and a $p \times p$ symmetric matrix $M$, we say that $\lambda$ and $x$ are an eigenvalue/eigenvector pair of $M$ relative to $\Sigma$ if $Mx = \lambda \Sigma x$ or equivalently if $M_o x = \lambda x$ for $x \neq 0$, where $M_o = \Sigma^{-1} M$.

By relating the eigenvalues/eigenvectors of $M_o$ to those of $\Sigma^{-1/2} M \Sigma^{-1/2}$, show that for $M_o$:

i) All the eigenvalues of are real, and henceforth denoted by $\lambda_1 \geq \cdots \geq \lambda_p$.

ii) The algebraic and geometric multiplicities of the eigenvalues are equal.

iii) The eigenspace $S(\lambda)$ can be spanned by a set of linearly independent real eigenvectors.

iv) For $\lambda_j \neq \lambda_k$, the eigenspaces $S(\lambda_j)$ and $S(\lambda_k)$ are orthogonal to each other in the metric of $\Sigma$. That is, if $M_o x = \lambda_j x$ and $M_o y = \lambda_k y$, then $\langle x, y \rangle_\Sigma = 0$ where the inner product $\langle x, y \rangle_\Sigma = x^T \Sigma y$.

v) The generalized spectral value decomposition. There exists a set of orthonormal vectors relative to $\langle \cdot, \cdot \rangle_\Sigma$, say $h_1, \ldots, h_p$, such that $M_o = H \Delta H^{-1}$, where $H = [h_1 \cdots h_p]$ and $\Delta = \text{diagonal}\{\lambda_1, \ldots, \lambda_p\}$.

10.2. Uniqueness of the generalized spectral value decomposition.

i) Suppose the $p \times p$ matrix $C$ can be expressed as $C = H \Delta H^{-1}$, with $\Delta = \text{diagonal}\{\eta_1, \ldots, \eta_p\}$ for $\eta_1 \geq \cdots \geq \eta_p$. Show that $\eta_1, \ldots, \eta_p$ are the eigenvalues of $C$ and the columns of $H = [h_1 \cdots h_p]$ are the corresponding eigenvectors, i.e. $Ch_j = \eta_j h_j$. Furthermore, show that $C$ is symmetric relative to $\langle \cdot, \cdot \rangle_\Sigma$ where $\Sigma = (HH^T)^{-1}$, i.e. $\langle Cx, y \rangle_\Sigma = \langle x, Cy \rangle_\Sigma$, or equivalently $C = \Sigma^{-1} M$ for some $p \times p$ symmetric matrix $M$.

ii) We say that the $p \times p$ real matrix $C$ is a normal matrix if all the eigenvalues of $C$ are real, and their algebraic and geometric multiplicities are equal. Show that $C$ is a normal matrix if and only if it can be expressed in the form $C = H \Delta H^{-1}$, with $\Delta$ being a diagonal matrix.

10.3. The versicolor variety of the Fishers iris data corresponds to the R data set Iris[51 : 100]. This data set consists of 50 observation in $\mathbb{R}^4$ Find the canonical correlations between

$$X = (Sepal.Length, Sepal.Width)^T$$ and $$Y = (Petal.Length, Petal.Width)^T,$$

as well as the corresponding canonical variables. The canonical variables should have variances equal to one.