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On the stress-based and strain-based methods for predicting optimal orientation of orthotropic materials

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Abstract In this paper, the strain-based and stress-based methods for solving optimal orientation problems of orthotropic materials are studied. Closed form solutions for both methods are derived and classified. The optimal orientation of both shear "weak" and some shear "strong" orthotropic materials may coincide with the major principal stress direction in the stress-based method. Similar results from the strain-based method are also derived. From the derivations, however, it can also be shown that both methods may fail under a special condition called "repeated global minimum" condition.

Key words orthotropic materials, topology optimization

1 Introduction

In recent years, issues regarding the determination of the optimal orientation of orthotropic materials have drawn considerable attention. Two main thrusts behind this are (1) orthotropic materials, such as fiber reinforced composite materials, have found many applications in practical engineering activities because of their high stiffness/weight and strength/weight ratios; (2) the orthotropic material-based topology optimization model has been successfully applied to the determination of the optimal structural layout. Gibiansky and Cherkaev (1988) and Suzuki and Kikuchi (1991) pointed out that the optimal orientation of a relatively shear "weak" type of orthotropic material could be where it is co-aligned along its major principal stress direction. This method is based on the assumption of the stress field being invariant with respect to the orientation variable. On the other hand, Pedersen (1989, 1990, 1991) used a strain-based method to solve the optimal orientation problem of orthotropic material, in which the strain field is assumed to be fixed with respect to the variation of the orientation variable. Later, Diaz and Bendsøe (1992) presented another approach based on a given stress field (described by its principal stresses) to deal with the problem of shape optimization with a multiple loading condition. Cheng *et al.* (1994) introduced an improved stress-based approach which described the stress field by using its three components to solve the multi-modal optimization problem. Although the assumption of stress/strain being invariant can never be realized, these methods showed pretty good results. Some intuitive explanations were provided, but the formal mathematical study is still missing.

In this paper, the mathematical foundation of the optimal orientation of orthotropic materials is reviewed. Closed form solutions of the optimal orientation for both strain-based and stress-based methods are derived and classified. We will demonstrate that under some special conditions, the optimal orientation cannot be uniquely determined by either the stress-based method or the strain-based method. This special condition is called the "repeated global minimum" condition. Furthermore, we will show that the optimal orientation of both shear "weak" and some shear "strong" orthotropic materials may coincide with the major principal stress direction in the stress-based method. Similar results for the strain-based method are also derived. The remainder of this paper is organized as follows: the optimal orientation problem of orthotropic materials is stated in Sect. 2; the stress-based method is discussed in Sect. 3, where the existence of the "repeated global minimum phenomenon" is proved at first, and then the closed form of optimal orientation is derived and classified into three cases. Then, in Sect. 4,

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we provide the classification of optimal orientation for the strain-based method. Finally, some concluding remarks are presented.

2
Problem formulation

Consider the optimal orientation problem for a two-dimensional, linearly elastic, structure subjected to applied body force f, with surface traction t along the boundary Γ_t, and with displacement boundary condition Γ_d as shown in Fig. 1. Using the principle of virtual displacements, the weak form of the linearly elastostatic problem can be written as:

$$\int_{\Omega} C_{ijkl} \frac{\partial u_i}{\partial x_j} \frac{\partial v_k}{\partial x_l} \, \mathrm{d}\Omega = \int_{\Omega} f_i v_i \, \mathrm{d}\Omega + \int_{\Gamma_t} t_i v_i \, \mathrm{d}\Gamma \tag{1}$$

where C_{ijkl} is the elastic coefficient of orthotropic material, which depends on both material property and orientation variable θ; u_i represents the displacement that satisfies this equation of motion, and v_i represents the virtual displacement that belongs to the kinematically admissible displacement set.

To design the stiffest structure under static loading, the mean compliance or the elastic energy is often chosen as the design objective function. The mean compliance Π can be expressed as:

$$\Pi = \int_{\Omega} \sigma_{ij} \varepsilon_{ij} \, \mathrm{d}\Omega = \int_{\Omega} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \, \mathrm{d}\Omega = \int_{\Omega} C_{ijkl} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \, \mathrm{d}\Omega \tag{2}$$

where σ_{ij} represents a stress component and ε_{ij} is a strain component of the structure. The stiffest structure is defined as the structure that stores the minimum amount of total internal elastic energy and so has minimum mean compliance. Therefore, the optimality condition for optimal orientation θ is defined as $\partial\Pi/\partial\theta = 0$. To obtain its explicit form, we first take the derivative of the objective function Π with respect to the orientation variable θ in (2) as:

$$\frac{\partial \Pi}{\partial \theta} = \int_{\Omega} \left[\frac{\partial C_{ijkl}}{\partial \theta} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} + 2 C_{ijkl} \frac{\partial}{\partial \theta} \left(\frac{\partial u_i}{\partial x_j} \right) \frac{\partial u_k}{\partial x_l} \right] \mathrm{d}\Omega \tag{3}$$

Then, by finding the first derivative of the weak formulation in (1) with respect to the orientation variable θ, and setting the virtual displacement v_k equal to u_k, we can obtain:

$$\int_{\Omega} \frac{\partial C_{ijkl}}{\partial \theta} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \, \mathrm{d}\Omega = -C_{ijkl} \frac{\partial}{\partial \theta} \left(\frac{\partial u_i}{\partial x_j} \right) \frac{\partial u_k}{\partial x_l} \, \mathrm{d}\Omega \tag{4}$$

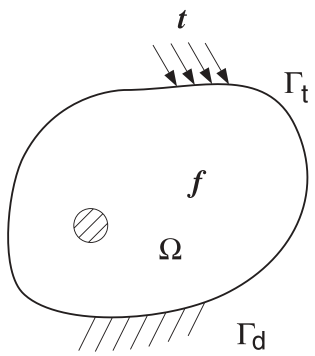

Fig. 1 Two dimensional elastic orthotropic structure under static loading

Combining the results of (3) and (4), the optimality condition can be expressed as:

$$\frac{\partial \Pi}{\partial \theta} = -\int_{\Omega} \frac{\partial C_{ijkl}}{\partial \theta} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \, \mathrm{d}\Omega = 0 \tag{5}$$

Eq. (5) is the basic equation used to obtain the optimal orientation of orthotropic materials under static loading; however, it is very difficult to solve in its integral form. Therefore, we reformat it using the finite element method. Suppose we take m elements as design cells; then the optimality condition can be stated as:

$$\frac{\partial \Pi}{\partial \theta_e} = -\int_{\Omega^e} \frac{\partial C_{ijkl}}{\partial \theta_e} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \, \mathrm{d}\Omega^e = -\int_{\Omega^e} \frac{\partial C_{ijkl}}{\partial \theta_e} \varepsilon_{ij} \varepsilon_{kl} \, \mathrm{d}\Omega^e = 0 \tag{6}$$

where Ω^e represents the spatial extent of the e^{th} design cell.

If the finite element mesh is dense enough, we can safely assume the strain and stress are constant inside each design cell, and they can be approximated by the values at the centroid. Then, the strain term in the above equation can be taken out of the integral as:

$$\frac{\partial \Pi}{\partial \theta_e} = -\boldsymbol{\varepsilon}_e^T \frac{\partial \mathbf{C}}{\partial \theta_e} \boldsymbol{\varepsilon}_e A^e = 0 \qquad e = 1, 2, \ldots, m \tag{7}$$

where $\boldsymbol{\varepsilon}_e$ represents the strain vector and $\mathbf{C}$ is the rotated orthotropic stiffness matrix. A^e is the area of the e^{th} design cell and is set as unity below for simplicity.

Instead of expressing the optimality condition in the strain form, (7) can also be represented in the stress form as:

$$\frac{\partial \Pi}{\partial \theta_e} = \boldsymbol{\sigma}_e^T \frac{\partial \mathbf{S}}{\partial \theta_e} \boldsymbol{\sigma}_e = 0 \qquad e = 1, 2, \ldots, m \tag{8}$$

Here $\mathbf{S}$ is the rotated orthotropic compliance matrix which can be expressed by the unrotated orthotropic compliance matrix $\mathbf{S}_0$ and standard rotation matrix $\mathbf{T}(\theta_e)$ as:

$$\mathbf{S} = \mathbf{T}^{-1}(\theta_e)\mathbf{S}_0\mathbf{T}^{-T}(\theta_e) \tag{9}$$

where $\mathbf{S}_0$ is:

$$\mathbf{S}_0 = \begin{bmatrix} s_{11}^0 & s_{12}^0 & 0 \\ s_{12}^0 & s_{22}^0 & 0 \\ 0 & 0 & s_{66}^0 \end{bmatrix} \tag{10}$$

Eqs. (7) and (8) are coupled equations because $\boldsymbol{\varepsilon}_e$ and $\boldsymbol{\sigma}_e$ are implicit functions of orientation variables in all design cells. To solve these equations, two commonly used methods, the stress-based method and the strain-based method, are discussed in the next sections.

3 Stress-based method

In the stress-based method, stress $\boldsymbol{\sigma}_e$ is assumed to be fixed with respect to the variation of orientation variables, so (8) can be rewritten as:

$$\frac{\partial \Pi}{\partial \theta_e} = \frac{\partial}{\partial \theta_e}(\boldsymbol{\sigma}_e^T \mathbf{S} \boldsymbol{\sigma}_e) = 0 \qquad e = 1, 2, \ldots, m \tag{11}$$

Denoting σ_1, σ_2, and σ_{12} as components of stress vector $\boldsymbol{\sigma}_e$, and using the expression for the rotated orthotropic compliance matrix $\mathbf{S}$, the above equations can be expressed explicitly by the orientation variable θ_e as:

$$A \sin 2\theta_e + B \cos 2\theta_e + D \sin 4\theta_e + E \cos 4\theta_e = 0 \tag{12}$$

The coefficients A, B, D, and E can be computed as:

$$\begin{aligned} A &= c(\sigma_2^2 - \sigma_1^2) \\ B &= 2c\sigma_{12}(\sigma_1 + \sigma_2) \\ D &= d\left(4\sigma_{12}^2 - (\sigma_1 - \sigma_2)^2\right) \\ E &= 4d\sigma_{12}(\sigma_1 - \sigma_2) \end{aligned} \tag{13}$$

where c and d are material property parameters:

$$c = s_{11}^0 - s_{22}^0 \tag{14}$$

$$d = \frac{s_{11}^0 - 2s_{12}^0 + s_{22}^0 - s_{66}^0}{2} \tag{15}$$

It should be noted that coefficients A, B, D, and E are not independent and their relationship can be established by eliminating all stress components in (13) as:

$$(A^2 - B^2)E = 2ABD \tag{16}$$

Once the coefficients A,B,D,E are determined, the roots of (12) can be obtained analytically from the roots of the following fourth order polynomial in $\tan\theta_e$:

$$\begin{aligned} &(E - B)\tan^4\theta_e + (2A - 4D)\tan^3\theta_e - 6E\tan^2\theta_e + \\ &(2A + 4D)\tan\theta_e + B + E = 0 \end{aligned} \tag{17}$$

Since several roots may exist for (17), the optimal orientation can be determined by comparing the values of the evaluation function,

$$f(\theta_e) = -2A\cos 2\theta_e + 2B\sin 2\theta_e - D\cos 4\theta_e + E\sin 4\theta_e \tag{18}$$

with respect to all the real solutions of (17) and $\theta_e = \pi/2(-\pi/2)$ respectively. It should be noted that $\theta_e = \pi/2$ and $\theta_e = -\pi/2$ correspond to the same material orientation, so we only need to calculate the value of the evaluation function at $\theta_e = \pi/2$ or $\theta_e = -\pi/2$. However, numerical methods for finding the roots of (17) may fail under the "repeated global minimum" condition. In the following sections, we will verify the existence of the "repeated global minimum" and derive the roots of (17). Then, the close form of optimal orientations are classified into three different categories.

3.1 Repeated global minimum

In this section, we will prove the existence of the "repeated global minimum". The "repeated global minimum" is described as:

Proposition 1. *If (17) has four real roots, evaluation function $f(\theta_e)$ must have the same value at two root points and they must be both global minimum when $d > 0$.*

We will prove the proposition in three steps. In the first step, we show that $f(\theta_e)$ can get the same extreme value at two different points; in the second step, we demonstrate that these two points are the roots of (17); finally, we prove that the value of $f(\theta_e)$ at these two points are the global minimum.

Step 1: If $f(\theta_e)$ has the same extreme value $\overline{f}$ at two different points θ_1 and θ_2, then θ_1 and θ_2 must be the two repeated roots of the following equation:

$$f(\theta_e) - \overline{f} = 0 \tag{19}$$

Using trigonometry relationships, the above equation can be expressed in terms of $\tan\theta_e$:

$$\begin{aligned} &\left(2A - D - \overline{f}\right)\tan^4\theta_e + (4B - 4E)\tan^3\theta_e + \\ &\left(6D - 2\overline{f}\right)\tan^2\theta_e + (4B + 4E)\tan\theta_e - \\ &2A - D - \overline{f} = 0 \end{aligned} \tag{20}$$

Eq. (19) can also be represented in the following form if both θ_1 and θ_2 are its repeated roots:

$$\left(2A-D-\overline{f}\right)(\tan\theta_e-x_1)^2(\tan\theta_e-x_2)^2=0 \tag{21}$$

here $x_1=\tan\theta_1$ and $x_2=\tan\theta_2$.

Eqns. (21) and (20) are equivalent only when the following conditions can be satisfied:

$$\begin{aligned}
&-2\left(2A-D-\overline{f}\right)(x_1+x_2)=4(B-E)\\
&\left(2A-D-\overline{f}\right)\left(x_1^2+x_2^2+4x_1x_2\right)=\left(6D-2\overline{f}\right)\\
&-2\left(2A-D-\overline{f}\right)x_1x_2(x_1+x_2)=4(B+E)\\
&\left(2A-D-\overline{f}\right)x_1^2x_2^2=-2A-D-\overline{f}
\end{aligned} \tag{22}$$

The solutions of (22) are:

$$\begin{aligned}
x_1&=\frac{-BE+\sqrt{B^2E^2-A^2B^2+A^2E^2}}{A(B-E)}\\
x_2&=\frac{-BE-\sqrt{B^2E^2-A^2B^2+A^2E^2}}{A(B-E)}\\
\overline{f}&=\frac{A(B+E)^2}{BE}-2A-D
\end{aligned} \tag{23}$$

$$(A^2-B^2)E=2ABD \tag{24}$$

It is obvious that (24) is the sufficient condition for (19) to have two repeated roots. Since (24) is always satisfied in the stress-based method, $f(\theta_e)$ can get the same extreme value at two different points. Moreover, (23) are the solutions of these two points and their evaluation function value.

Step 2: To check if θ_1 and θ_2 are the roots of (17), we simply factorize (17) by $(\tan\theta_e-x_1)(\tan\theta_e-x_2)$ as:

$$(E-B)(\tan\theta_e-x_1)(\tan\theta_e-x_2)\times\left(\tan^2\theta_e-\frac{2A}{B}\tan\theta_e-1\right)=0 \tag{25}$$

Apparently, (17) has two roots θ_1 and θ_2, and the other two roots can be calculated from the equation:

$$\tan^2\theta_e-\frac{2A}{B}\tan\theta_e-1=0 \tag{26}$$

Step 3: Now we need to prove $\overline{f}$ is the global minimum of the valuation function $f(\theta_e)$ in $(-\pi/2,\pi/2]$ when $d>0$. Since (21) is the equivalent form of (19), $f(\theta_e)-\overline{f}$ has the same sign as $2A-D-\overline{f}$, so we only need to check if $2A-D-\overline{f}\geq 0$. This can be easily shown:

$$2A-D-\overline{f}=4A-\frac{A(B+E)^2}{BE}=-\frac{A}{BE}(B-E)^2=\frac{1}{8d\sigma_{12}^2}(B-E)^2\geq 0 \tag{27}$$

From the above three steps, we have shown the existence of the "repeated global minimum". It should be noted that the condition that (17) has four real roots is necessary to make the "repeated global minimum" condition exist. The two roots calculated from (26) are always real because $(\frac{2A}{B})^2+4>0$; however, the other two roots are real only if $B^2E^2-A^2B^2+A^2E^2>0$, or in the stress components form, $4d^2[(\sigma_1-\sigma_2)^2+4\sigma_{12}^2]>c^2(\sigma_1+\sigma_2)^2$. Furthermore, the two orientations predicted by (26) are actually two principal stress directions. By rearranging (26) we get:

$$\frac{2\tan\theta_e}{1-\tan^2\theta_e}=-\frac{B}{A} \tag{28}$$

Noting the relationship $\tan 2\theta_e=2\tan\theta_e/(1-\tan^2\theta_e)$, and replacing A and B with their stress expressions, we have:

$$\tan 2\theta_e=\frac{2\sigma_{12}}{\sigma_1-\sigma_2} \tag{29}$$

(29) is the well-known expression which is used to determine the principal stress direction, so the orientations calculated from (26) are two principal stress directions.

Suppose θ_I,θ_{II} are two principal stress directions. We now need to determine which one is the desired orientation (the one that givess a smaller evaluation function value $f(\theta_e)$). From (28), we get:

$$\begin{aligned}
\cos 2\theta_{I,II}&=\pm\frac{A}{\sqrt{A^2+B^2}}\\
\sin 2\theta_{I,II}&=\mp\frac{B}{\sqrt{A^2+B^2}}\\
\cos 4\theta_{I,II}&=\frac{A^2-B^2}{A^2+B^2}\\
\sin 4\theta_{I,II}&=-\frac{2AB}{A^2+B^2}
\end{aligned} \tag{30}$$

Then the evaluation function values $f(\theta_I)$,$f(\theta_{II})$ can be expressed as:

$$\begin{aligned}
f(\theta_I)&=-2\sqrt{A^2+B^2}-\frac{2ABE+(A^2-B^2)D}{A^2+B^2}\\
f(\theta_{II})&=2\sqrt{A^2+B^2}-\frac{2ABE+(A^2-B^2)D}{A^2+B^2}
\end{aligned} \tag{31}$$

Obviously $f(\theta_I)$ is smaller than $f(\theta_{II})$, so θ_I is the principal stress direction, which possibly gives the optimal orientation.

3.2 Optimal orientation classification

With the derivations we had in the previous sections, the optimal orientation can be summarized via three cases.

Case 1. $d>0$ and $4d^2\left[(\sigma_1-\sigma_2)^2+4\sigma_{12}^2\right]>c^2(\sigma_1+\sigma_2)^2$

As discussed in Sect. 3.1, "repeated global minimum" happens in this case; the two optimal orientations θ_1 and θ_2 can be obtained from (23) or directly from the following stress component expressions:

$$\theta_1 = \arctan \left(\frac{4d\sigma_{12} + \sqrt{4d^2[(\sigma_1-\sigma_2)^2+4\sigma_{12}^2] - c^2(\sigma_1+\sigma_2)^2}}{2d(\sigma_1-\sigma_2) - c(\sigma_1+\sigma_2)} \right)$$

$$\theta_2 = \arctan \left(\frac{4d\sigma_{12} - \sqrt{4d^2[(\sigma_1-\sigma_2)^2+4\sigma_{12}^2] - c^2(\sigma_1+\sigma_2)^2}}{2d(\sigma_1-\sigma_2) - c(\sigma_1+\sigma_2)} \right)$$

Since "repeated global minimum" produces two different optimal orientations for the related design cell, it may generate many "optimal" layouts that obviously contradict physical reality. Therefore, numerical methods of solving (17) may fail. In practice, since the stress field cannot be fixed, iteration techniques must be applied in order to obtain the optimal orientation. The occurrence of "repeated global minimum" may cause oscillation of the object function, or even yield a wrong solution.

Case 2. $d > 0$ and $4d^2 \left[(\sigma_1-\sigma_2)^2 + 4\sigma_{12}^2\right] < c^2(\sigma_1+\sigma_2)^2$

In this case, (17) has only two real roots and they are the principal stress orientations θ_I and θ_{II}. We have shown that the major principal stress direction θ_I is a possible candidate for the optimal orientation. To determine if θ_I or $\pi/2$ is the optimal orientation, we need to compare the values of evaluation function $f(\theta_e)$ at these two orientations.

Since $f(\theta_e)$ has only two extreme points θ_I and θ_{II} in $[-\pi/2, \pi/2]$, and also $f(\theta_I) < f(\theta_{II})$, we can say θ_I is the minimum point and θ_{II} is the maximum point of $f(\theta_e)$. Considering the fact that $f(-\pi/2) = f(\pi/2)$, we can further claim that θ_I is the global minimum point and θ_{II} is the global maximum point. The proof is: if $f(-\pi/2) < f(\theta_I)$, then $f(\pi/2) < f(\theta_I)$. Since θ_I is the minimum point of $f(\theta_e)$, at least one local maximum point must exist between $(-\pi/2, \theta_I)$ if $\theta_I < \theta_{II}$ or between $(\theta_I, \pi/2)$ if $\theta_I > \theta_{II}$, and this apparently contradicts the fact that $f(\theta_e)$ has only two extreme points in $[-\pi/2, \pi/2]$. We can prove θ_{II} is the global maximum point in the same manner.

Case 3. $d < 0$

In Case 3, we will not distinguish between the cases $4d^2[(\sigma_1-\sigma_2)^2 + 4\sigma_{12}^2] > c^2(\sigma_1+\sigma_2)^2$ and $4d^2[(\sigma_1-\sigma_2)^2 + 4\sigma_{12}^2] < c^2(\sigma_1+\sigma_2)^2$, because in both cases θ_1 and θ_2 cannot be the optimal orientations corresponding to the global minimum. Actually, in the first situation, θ_1 and θ_2 are orientations corresponding to the global maximum; in the second situation, both tangents of θ_1 and θ_2 are not real, which make them meaningless. Therefore we only need to compare the evaluation function $f(\theta_e)$ at θ_I and $\pi/2$.

Rewriting (31) in the stress components form, we obtain the following expression for $f(\theta_I)$:

$$f(\theta_I) = -2|c(\sigma_1+\sigma_2)|\sqrt{(\sigma_1-\sigma_2)^2 + 4\sigma_{12}^2} + d\left[(\sigma_1-\sigma_2)^2 + 4\sigma_{12}^2\right] \quad (32)$$

$f(\pi/2)$ can also be expressed as:

$$f(\pi/2) = 2c(\sigma_2^2 - \sigma_1^2) - 4d\sigma_{12}^2 + d(\sigma_1-\sigma_2)^2 \quad (33)$$

Since $4d\sigma_{12}^2 < -4d\sigma_{12}^2$ when $d < 0$, and

$$-2|c(\sigma_1+\sigma_2)|\sqrt{(\sigma_1-\sigma_2)^2 + 4\sigma_{12}^2} \le -2|c(\sigma_2^2-\sigma_1^2)| \le 2c(\sigma_2^2-\sigma_1^2) \quad (34)$$

it can easily be proved that $f(\theta_I) < f(\pi/2)$, so θ_I is the optimal orientation when $d < 0$.

Suzuki and Kikuchi (1991) applied the principal stress direction as the optimal orientation for the optimal material distribution problem. We can find a mathematical basis for this approach in the above discussion. If the stress field is constant with respect to the orientational variable, for relatively shear "weak" types of orthotropic material ($d < 0$), the principal material axis should be oriented along the principal stress direction in order to obtain the stiffest structure, and this is also true for relatively shear "strong" types of orthotropic material ($d > 0$) satisfying stress condition $4d^2[(\sigma_1-\sigma_2)^2 + 4\sigma_{12}^2] < c^2(\sigma_1+\sigma_2)^2$. However in the case when "repeated global minimum" occurs, the principal stress direction is no longer the optimal orientation.

4 Strain-based method

In the strain-based method, the strain field, $\boldsymbol{\varepsilon}_e$, is assumed to be invariant with respect to the variation of orientation variables. Therefore, (7) can be rewritten as:

$$\frac{\partial \Pi}{\partial \theta_e} = -\frac{\partial}{\partial \theta_e}(\boldsymbol{\varepsilon}_e^T \mathbf{C} \boldsymbol{\varepsilon}_e) = 0 \qquad e = 1, 2, \dots, m \quad (35)$$

Let ε_1, ε_2, and γ_{12} be the three components of strain field $\boldsymbol{\varepsilon}_e$; the above equation can then be expressed in the same form as (12) of the stress-based method, with the coefficients A, B, D, and E:

$$A = -2a\left(\varepsilon_2^2 - \varepsilon_1^2\right)$$

$$B = -2a\gamma_{12}(\varepsilon_1 + \varepsilon_2)$$

$$D = -2b\left(\gamma_{12}^2 - (\varepsilon_1 - \varepsilon_2)^2\right)$$

$$E = -4b\gamma_{12}(\varepsilon_1 - \varepsilon_2) \quad (36)$$

where a, b are material property parameters:

$$a = \frac{c_{11}^0 - c_{22}^0}{2} \quad (37)$$

$$b = \frac{c_{11}^0 - 2c_{12}^0 + c_{22}^0 - 4c_{66}^0}{4} \quad (38)$$

By eliminating the strain components ε_1, ε_2,γ_{12}, the same relationship between coefficients A,B,D,E can be established in the same way as (16).

Close forms of optimal orientation can also be found using the strain-based method and they can be found in the same way as for the stress-based method. The optimal orientations are categorized into the following two cases:

Case 1. $b < 0$ and $4b^2 \left[(\varepsilon_1 - \varepsilon_2)^2 + \gamma_{12}^2\right] > a^2(\varepsilon_1 + \varepsilon_2)^2$

"Repeated global minimum" occurs in this case, and it can be calculated from:

$$\theta_1 = \arctan \left(\frac{2b\gamma_{12} + \sqrt{4b^2[(\varepsilon_1 - \varepsilon_2)^2 + \gamma_{12}^2] - a^2(\varepsilon_1 + \varepsilon_2)^2}}{2b(\varepsilon_1 - \varepsilon_2) - a(\varepsilon_1 + \varepsilon_2)} \right)$$

$$\theta_2 = \arctan \left(\frac{2b\gamma_{12} - \sqrt{4b^2[(\varepsilon_1 - \varepsilon_2)^2 + \gamma_{12}^2] - a^2(\varepsilon_1 + \varepsilon_2)^2}}{2b(\varepsilon_1 - \varepsilon_2) - a(\varepsilon_1 + \varepsilon_2)} \right)$$

Case 2. $b > 0$ or $4b^2 \left[(\varepsilon_1 - \varepsilon_2)^2 + \gamma_{12}^2\right] < a^2(\varepsilon_1 + \varepsilon_2)^2$

In this case, the optimal orientation coincides with the major principal strain direction θ_I.

5 Conclusion

The mathematical foundation of the strain-based and stress-based methods of determining the optimal orientation of orthotropic materials was discussed. The closed form of optimal orientation was first derived for the stress-based method, and it was proven that for relatively "weak" shear types of orthotropic material, and for some shear "strong" material, the optimal orientation coincides with the major principal stress direction. Optimal solutions were also derived for the strain-based method in the same manner. It was shown that the use of the strain-based or stress-based methods will fail when the "repeated global minimum" occurs. To avoid this problem, an energy-based method developed by Luo and Gea (1998) can be used to find the optimal orientation.

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