Optimal bead orientation of 3D shell/plate structures

Jian Hui Luo, Hae Chang Gea*

Rutgers, The State University of New Jersey, Piscataway, NJ 08855-0909, USA

Abstract

In this paper, the optimal bead orientation problem of 3D shell/plate structures is investigated for both static and dynamic cases. Using a bending equivalent orthotropic shell design cell model, the optimal bead orientation problem is converted to an optimal orientation problem of bending equivalent orthotropic materials, and it is solved by a new energy based method. In the energy based method, the dependency of strain and stress fields on material orientation is explored by introducing an energy factor in the inclusion model. From the derivations, the strain- and stress-based methods are recovered and their limitations are discussed. Three examples are presented to show the applications of the proposed method. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Shell stiffener; Orthotropic material; Topology optimization

1. Introduction

To satisfy the requirements of resource and energy savings, products with light-weight design are desirable in automobile and aerospace industries. One prominent example is the application of shell/plate structures. However, shell/plate structures exhibit poor stiffness as well as NVH (noise, vibration and harshness) performance due to their flexibility. One common approach to improving the stiffness and NVH performance of shell/plate structures is the addition of bead stiffener. The main advantage of bead stiffener over other types of stiffeners is its low manufacturing cost and light weight. Design of bead involves the determination of bead location, bead orientation and bead geometry such as height and width. Topology optimization based approach can be used to identify the bead location but cannot provide enough information about the bead orientation and bead geometry. Yang et al. [1] investigated the optimal bead orientation by adding beam elements to base shell element in pre-selected directions; a topology optimization-based approach is then used to determine the most desirable beam direction with which the orientation of bead will go.
Along. Although this approach is very easy to implement, it has several drawbacks: (1) bead orientation can only be chosen from a few pre-set discrete angles, which may not include the optimal one; (2) topology optimization approach may produce several equally desired beam directions in one element, in which case bead orientation cannot be uniquely determined; (3) the computational efforts can be tremendously increased since add-on beams in each element bring several times design variables in optimization process.

Compared to only preliminary research works of optimal bead orientation, the literature on optimal orientation of orthotropic materials is vast. Pedersen [2–4] first studied the optimal orientation problems by using the strain based method, in which the strain field is assumed to be fixed with respect to the variation of the orientation variable. By performing the sensitivity analysis of the elastic energy, a closed-form solution of the extreme energy was found. Suzuki and Kikuchi [5] pointed out that the optimal orientation of a relatively shear weak type of orthotropic material should be co-aligned in its major principal stress direction, this is the so-called stress based method. Later, Diaz and Bendsøe [6] presented an approach based on a given stress field (described by its principal stresses) to deal with multiple loading problems. Recently, Cheng et al. [10] introduced an improved stress-based approach which described the stress field by using its three components to solve multi-modal optimization problems.

In this paper, the optimal bead orientation problem of 3D shell/plate structures is investigated for both static and dynamic cases. Based upon the stress resultant-strain and stress couple-curve relations of bead-stamped shell element, a bending equivalent orthotropic shell design cell model is introduced to supersede the bead-stamped element for calculating the flexural deformations. This transformation converts the optimal bead orientation problem to the optimal orientation problem of bending equivalent orthotropic materials. To solve the optimal orientation problem of orthotropic materials, an energy based method [7] is developed. Instead of assuming the strain or stress field is fixed with respect to the orientation variables, the dependency of strain and stress fields on material orientation is explored by introducing an energy factor in the energy based method. From the derivations, the strain- and stress-based methods are recovered and their limitations are discussed.

The remainder of the paper is organized as follows: Section 2 discusses the bead-stamped shell/plate structures and introduces the bending equivalent orthotropic shell design cell model; Section 3 derives the optimality conditions of optimal orientation of orthotropic materials in both static and dynamic cases, the results are then extended to the bead orientation problems; the energy-based method is introduced in Section 4, where the strain- and stress-based methods are briefly reviewed and discussed. Finally, computational results from the proposed method are presented.

2. Bead-stamped shells and plates

Bead-stamped shell/plate structures are shells or plates to which folds (beads) have been added to supplement the rigidity of the base structures. One of the most important parameters of designing bead is the bead orientation. In this paper, we will investigate the optimal bead orientation in the shell/plate structures, and the geometry of bead is assumed to be defined beforehand.
In general, bead is restricted to some prescribed regions in the base shell/plate structure due to its geometric constraints. These regions are called as the bead design domain. By discretizing the bead design domain into a number of finite elements, and assuming only one bead can be added in each element, the bead orientation problem can be looked into at the element level. These elements are also referred to as bead-stamped shell design cells.

2.1. Stress resultant-strain and stress couple-curvature relations

In shell/plate structure problems, the constitutive equations are required to relate the stress resultants and stress couples, instead of merely stresses, to the corresponding strains and curvatures. The stress resultants and stress couples on two faces of a middle surface element are shown in Fig. 1. If we neglect the membrane-bending coupling effect, for a base shell/plate structure with Young’s modulus $E$, Possion ratio $\nu$, and uniform shell thickness $t$, the stress resultant-strain and stress couple-curvature relations can be represented as [8]:

$$
\begin{bmatrix}
N_m \\
M_b \\
Q_s
\end{bmatrix} =
\begin{bmatrix}
D_m & 0 & 0 \\
0 & D_b & 0 \\
0 & 0 & D_s
\end{bmatrix}
\begin{bmatrix}
\varepsilon_m \\
\kappa_b \\
\gamma_s
\end{bmatrix},
$$

where $N_m$ is the in-plane membrane force vector, $M_b$ is the flexural moment vector, and $Q_s$ is the transverse shear force vector; $\varepsilon_m$ stands for the in-plane strains, $\kappa_b$ represents the curvatures, and $\gamma_s$ stands for the transverse shear strains; $D_m$ represents the membrane stiffness, $D_b$ is the flexural rigidity, and $D_s$ is the transverse shear stiffness; i.e.

$$
\begin{align}
\{N_m\}^T &= \{N_x, N_y, N_{xy}\}, & \{M_b\}^T &= \{M_x, M_y, M_{xy}\}, & \{Q_s\}^T &= \{Q_x, Q_y\}, \\
\{\varepsilon_m\}^T &= \{\varepsilon_x, \varepsilon_y, \omega\}, & \{\kappa_b\}^T &= \{\kappa_x, \kappa_y, \tau\}, & \{\gamma_s\}^T &= \{\gamma_x, \gamma_y\}, \\
D_m &= \int_{-t/2}^{t/2} C^0 \, dz, & D_b &= \int_{-t/2}^{t/2} z^2 C^0 \, dz, & D_s &= \int_{-t/2}^{t/2} C^1 \, dz,
\end{align}
$$
where

\[
C^0 = \begin{bmatrix}
\frac{E}{1-v^2} & \frac{vE}{1-v^2} & 0 \\
\frac{vE}{1-v^2} & \frac{E}{1-v^2} & 0 \\
0 & 0 & \frac{E}{2(1+v)}
\end{bmatrix}, \quad C^1 = \begin{bmatrix}
\frac{E}{2(1+v)} & 0 & \frac{E}{2(1+v)} \\
0 & \frac{E}{2(1+v)} & 0
\end{bmatrix},
\]

and the explicit form of flexural rigidity \( D_b \) is

\[
D_b = \begin{bmatrix}
\frac{Et^3}{12(1-v^2)} & \frac{vEt^3}{12(1-v^2)} & 0 \\
\frac{vEt^3}{12(1-v^2)} & \frac{Et^3}{12(1-v^2)} & 0 \\
0 & 0 & \frac{Et^3}{24(1+v)}
\end{bmatrix}.
\]

Since the stress resultant-strain and stress couple-curvature relations can fully characterize the material properties of shell/plate structures in the element level, two shell/plate structures are equivalent if they have the same stress resultant-strain and stress couple-curvature relations. In the following sections, we will derive these relations for bead-stamped shell design cell using split rigidity concept, and from there, a bending equivalent orthotropic shell design cell model will be introduced.

2.2. Split rigidity concept

For a bead-stamped shell design cell as shown in Fig. 2, although it is theoretically possible to model it as ensembles of several smaller shell components, it is often more economical to derive the equivalent thicknesses of the shell design cell to form the stress resultant-strain and stress couple-curvature relations. A promising approach for this equivalence problem is the split rigidity concept introduced by Buchert [9]. In the split rigidity concept, different equivalent thicknesses for the extensional, shearing, bending, and twisting terms are specified. For a bead-stamped shell design cell, the equivalent extensional thicknesses and shear thickness are close to the base

Fig. 2. Bead-stamped shell design cell.
structure thickness \( t \), however, due to the alteration of shell section geometry caused by stamped-bead, the bending rigidity in the direction where bead runs increases drastically and its equivalent thickness can be evaluated approximately as

\[
t_b = \left( \frac{12I_b}{I} \right)^{1/3}
\]

where \( I_b \) is the moment of inertia of the bead-stamped shell section about its centroidal axis, \( l \) is the length of shell design cell. Since \( I_b \) is much bigger than the moment of inertia of the original shell section, \( t_b \) is normally several times larger than the base structure thickness. The equivalent bending thickness in the other direction and twisting thickness can also be thought as the base thickness because stamped-bead affects them very little.

The equivalent thicknesses are then used to compute the stress resultant-strain and stress couple-curvature relations. Because the equivalent extensional and shear thicknesses remain the same as the base thickness, membrane stiffness \( D_m \) and transverse shear stiffness \( D_s \) are identical to those in the base structure which are defined in Eq. (4). The flexural rigidity \( D_b \) is obtained by superseding the base thickness \( t \) with \( t_b \) in the first term of Eq. (6):

\[
D_b = \begin{bmatrix}
\frac{E_t}{12(1-\nu^2)} & \frac{vE_t}{12(1-\nu^2)} & 0 \\
\frac{vE_t}{12(1-\nu^2)} & \frac{E_t}{12(1-\nu^2)} & 0 \\
0 & 0 & \frac{E_t}{24(1+\nu)}
\end{bmatrix}.
\]

It is obvious that bead-stamped shell design cell shows orthotropic behavior under bending loads, which is very similar to the behavior of shell/plate structure made of orthotropic materials. In the next section, it will be shown that these two systems are bending equivalent, therefore they can replace each other in the finite element computation.

2.3. Bending equivalent orthotropic shell design cell model

Two shell structures can be thought as equivalent in the element level if they have the same stress resultant-strain and stress couple-curvature relations; if both systems have the same flexural rigidity, they are bending equivalent. Consider the flexural rigidity of an orthotropic shell design cell with uniform thickness \( t \), the material properties are characterized by the following constitutive law:

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_{12}
\end{bmatrix} =
\begin{bmatrix}
\epsilon^0_{11} & \epsilon^0_{12} & 0 \\
\epsilon^0_{12} & \epsilon^0_{22} & 0 \\
0 & 0 & \epsilon^0_{33}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix} = C_0
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix},
\]

where \( \sigma_1, \sigma_2, \sigma_{12} \) are stress components, \( \varepsilon_1, \varepsilon_2, \gamma_{12} \) are strain components, \( C_0 \) is the unrotated stiffness matrix.

The flexural rigidity \( D_b \) can be represented as

\[
D_b = \begin{bmatrix}
\frac{\epsilon^0_{11}t^2}{12} & \frac{\epsilon^0_{12}t^2}{12} & 0 \\
\frac{\epsilon^0_{12}t^2}{12} & \frac{\epsilon^0_{22}t^2}{12} & 0 \\
0 & 0 & \frac{\epsilon^0_{33}t^2}{12}
\end{bmatrix}.
\]
A bending equivalent orthotropic shell design cell model can then be set up by making its flexural rigidity equal to that of bead-stamped element defined in Eq. (8). The equivalent orthotropic material coefficients can be determined as:
\[
\begin{align*}
  c_{11}^0 &= \frac{t_b^3}{t^3} \frac{E}{1 - v^2}, & c_{12}^0 &= \frac{vE}{1 - v^2}, \\
  c_{22}^0 &= \frac{E}{1 - v^2}, & c_{33}^0 &= \frac{E}{2(1 + v)},
\end{align*}
\]

The advantages of applying the orthotropic model over directly dealing with bead-stamped structure are: first, the bead-stamped structure is complicated in its section geometry which might demand more computational efforts. Second, when the bead is rotated, it is much easier to construct the finite element formulation with the orthotropic material model than the bead-stamped model. At an arbitrary angle \( \theta \), the equivalent rotated orthotropic stiffness matrix \( C \) can be obtained from:
\[
C = T^T(\theta)C_0T(\theta),
\]
where \( C_0 \) stands for the unrotated orthotropic stiffness matrix which is defined in Eq. (9), \( T(\theta) \) is a standard rotation matrix, and in the form of:
\[
T(\theta) = \begin{bmatrix}
\cos^2 \theta & \sin \theta \cos \theta & \sin^2 \theta \\
\sin^2 \theta & \cos \theta \cos \theta & -\sin \theta \sin \theta \\
-2 \cos \theta \sin \theta & 2 \cos \theta \sin \theta & \cos^2 \theta - \sin^2 \theta
\end{bmatrix}.
\]

Since the membrane stiffness \( D_m \) and transverse shear stiffness \( D_s \) are the same as those of the base structure, the base isotropic material properties are used to calculate the extensional and shear deformations of the bead-stamped structure. For flexural deformation computation, the bending equivalent orthotropic shell design cell is applied to model each bead-stamped shell element, which converts the optimal bead orientation problem to the optimal orientation problem of orthotropic materials.

### 3. Optimum problem formulation

In this section, the problem of optimal orientation of orthotropic materials is considered, the optimality conditions are derived for both static and dynamic cases, the results are then applied to the optimal bead orientation problem by using the orthotropic model introduced in the previous section.

#### 3.1. Optimality condition of static problems

Consider a linearly elastic structure subjected to the applied body force \( f \), the surface traction \( t \) in arbitrary line \( \Gamma_t \), and the displacement boundary condition \( \Gamma_d \) as shown in Fig. 3. Through the
principle of virtual displacements, the weak form of the linearly elastostatic problem can be written as

\[
\int_{\Omega} C_{ijkl} \frac{\partial u_i}{\partial x_j} \frac{\partial v_k}{\partial x_l} \, d\Omega = \int_{\Omega} f_k v_k \, d\Omega + \int_{\Gamma_t} t_k v_k \, d\Gamma, \tag{14}
\]

where \( C_{ijkl} \) is the elastic coefficient of orthotropic material which depends on both material property and orientation variable \( \theta \), \( u_i \) is the displacement that satisfies this equation of motion and \( v_k \) represents the virtual displacement in a kinematically admissible displacement set.

To design the stiffest structure under a given static loading, the mean compliance is adopted as the objective function. The stiffest structure is defined as a structure that gives the least amount of displacement under loading and thus has the minimum mean compliance. The mean compliance \( \Pi \) can be expressed as

\[
\Pi = \int_{\Omega} \sigma_{ij} \varepsilon_{ij} \, d\Omega = \int_{\Omega} C_{ijkl} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \, d\Omega, \tag{15}
\]

where \( \sigma_{ij} \) and \( \varepsilon_{ij} \) represents stress and strain components of the structure. The optimality condition of the optimal orientation \( \theta \) is derived by setting the derivative of the objective function \( \Pi \) with respect to the orientation variable \( \theta \) in Eq. (15) being zero. We have

\[
\frac{\partial \Pi}{\partial \theta} = \int_{\Omega} \left[ \frac{\partial C_{ijkl}}{\partial \theta} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} + 2C_{ijkl} \frac{\partial}{\partial \theta} \left( \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \right) \right] \, d\Omega = 0. \tag{16}
\]

Furthermore, by taking the first derivative of the weak form in Eq. (14) with respect to the orientation variable \( \theta \), and setting the virtual displacement \( v_k \) equal to \( u_k \), we have the following relation:

\[
\int_{\Omega} \frac{\partial C_{ijkl}}{\partial \theta} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \, d\Omega = - C_{ijkl} \frac{\partial}{\partial \theta} \left( \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_l} \right) \frac{\partial u_k}{\partial x_l} \, d\Omega. \tag{17}
\]
Combining the results of Eqs. (16) and (17), the optimality condition is expressed as

\[
\frac{\partial \Pi}{\partial \theta} = - \int_{\Omega} \frac{\partial C_{ijkl}}{\partial \theta} \frac{\partial \psi_k}{\partial x_j} \frac{\partial \psi_k}{\partial x_l} d\Omega = 0.
\] (18)

3.2. Optimality condition of eigenvalue problems

To prevent structure resonance with external excited forces, certain eigenfrequencies must be avoided. One practical approach to avoiding the low frequency resonance is to make the associated natural frequencies as large as possible. For a single eigenvalue optimization problem, the weak formulation can be written as

\[
\int_{\Omega} C_{ijkl} \frac{\partial \phi^n_i}{\partial x_j} \frac{\partial \phi^n_k}{\partial x_l} d\Omega - \lambda_n \int_{\Omega} \rho \phi^n_k \psi_k d\Omega = 0,
\] (19)

where \( \lambda_n \) stands for the \( n \)th system eigenvalue, \( \phi^n \) represents the \( n \)th system eigenvector, \( \rho \) is the material density, and \( \psi \) is a virtual system eigenvector in kinematically admissible set. Performing the first partial derivative of Eq. (19) with respect to \( \theta \) gives:

\[
\int_{\Omega} \left[ \frac{\partial C_{ijkl}}{\partial \theta} \frac{\partial \phi^n_i}{\partial x_j} \frac{\partial \phi^n_k}{\partial x_l} + C_{ijkl} \frac{\partial}{\partial \theta} \left( \frac{\partial \phi^n_i}{\partial x_j} \right) \frac{\partial \psi_k}{\partial x_l} - \frac{\partial \lambda_n}{\partial \theta} \rho \phi^n_k \psi_k - \lambda_n \rho \frac{\partial \phi^n_k}{\partial \theta} \psi_k \right] d\Omega = 0.
\] (20)

By setting \( \psi_k = \phi^n_k \) and using mass normalization condition:

\[
\int_{\Omega} \rho \phi^n_i \phi^n_k d\Omega = 1,
\] (21)

we can rewrite Eq. (20) as

\[
\frac{\partial \lambda_n}{\partial \theta} = \int_{\Omega} \left[ \frac{\partial C_{ijkl}}{\partial \theta} \frac{\partial \phi^n_i}{\partial x_j} \frac{\partial \phi^n_k}{\partial x_l} + C_{ijkl} \frac{\partial}{\partial \theta} \left( \frac{\partial \phi^n_i}{\partial x_j} \right) \frac{\partial \phi^n_k}{\partial x_l} - \lambda_n \rho \frac{\partial \phi^n_k}{\partial \theta} \psi_k \right] d\Omega.
\] (22)

Set \( \psi_k = \phi^n_k \) and \( \frac{\partial \phi^n_k}{\partial \theta} \) in Eq. (19), we have

\[
\int_{\Omega} \left[ C_{ijkl} \frac{\partial}{\partial \theta} \left( \frac{\partial \phi^n_i}{\partial x_j} \frac{\partial \phi^n_k}{\partial x_l} \right) - \lambda_n \rho \frac{\partial \phi^n_k}{\partial \theta} \psi_k \right] d\Omega = 0.
\] (23)

Combining the results of Eqs. (22) and (23), the optimality condition for maximizing \( \lambda_n \) is

\[
\frac{\partial \lambda_n}{\partial \theta} = \int_{\Omega} \left[ \frac{\partial C_{ijkl}}{\partial \theta} \frac{\partial \phi^n_i}{\partial x_j} \frac{\partial \phi^n_k}{\partial x_l} \right] d\Omega = 0.
\] (24)

If we define \( \Psi \) as \( \Pi \) in static problem and \( - \lambda_n \) in dynamic problem, a unified optimality condition for both cases is obtained:

\[
\frac{\partial \Psi}{\partial \theta} = - \int_{\Omega} \frac{\partial C_{ijkl}}{\partial \theta} e_{ijkl} d\Omega = 0.
\] (25)
In dynamic case, $\varepsilon_{ij}$ represents the equivalent strain vector corresponding to the $n$th eigenvector:

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial \phi_i^n}{\partial x_j} + \frac{\partial \phi_j^n}{\partial x_i} \right).$$

(26)

### 3.3. Optimality criteria for bead orientation problem

In bead-stamped shell structures, bending equivalent orthotropic material is used to calculate the bending curvature, and the base isotropic material is applied to compute the extensional and shearing deformations. If we can neglect the membrane-bending coupling, the optimality criteria for bead orientation problem can be extended from Eq. (25) as

$$\frac{\partial \Psi}{\partial \theta} = - \int_{\Omega} \left( \frac{\partial C^m_{ijkl}}{\partial \theta} \varepsilon^{m}_{ij} \varepsilon^{m}_{kl} + \frac{\partial C^b_{ijkl}}{\partial \theta} \varepsilon^{b}_{ij} \varepsilon^{b}_{kl} + \frac{\partial C^4_{ijkl}}{\partial \theta} \varepsilon^{4}_{ij} \varepsilon^{4}_{kl} \right) d\Omega = 0$$

(27)

where $C^m_{ijkl}$, $C^b_{ijkl}$, $C^4_{ijkl}$ are material coefficients used for membrane, bending and transverse shearing calculations, $\varepsilon^m_{ij}, \varepsilon^b_{ij}, \varepsilon^4_{ij}$ are the resulting strains, respectively. Since $\partial C^m_{ijkl}/\partial \theta = 0$ and $\partial C^b_{ijkl}/\partial \theta = 0$, Eq. (27) can be simplified as

$$\frac{\partial \Psi}{\partial \theta} = - \int_{\Omega} \frac{\partial C^b_{ijkl}}{\partial \theta} \varepsilon^b_{ij} \varepsilon^b_{kl} d\Omega = 0.$$  

(28)

For a bead-stamped shell structure with $m$ design cells, there are $m$ bead orientation design variables, $\theta = \{\theta_e\}, e = 1, 2, \ldots, m$, the optimality condition can be expressed in discrete form:

$$\frac{\partial \Psi}{\partial \theta_e} = - \int_{\Omega_e} \frac{\partial C^b_{ijkl}}{\partial \theta_e} \varepsilon^b_{ij} \varepsilon^b_{kl} d\Omega \varepsilon = 0,$$

(29)

where $\Omega \varepsilon$ represents the $e$th design cell.

If the finite element mesh is fine enough, the bending strain in each shell design cell can be assumed to be independent of plane coordinates, it’s only a linear function of distance from the associated point to mid-surface where the shell remains unbended, therefore Eq. (29) can be modified as

$$\frac{\partial \Psi}{\partial \theta_e} = - \frac{1}{3} \varepsilon^T_e \frac{\partial C}{\partial \theta_e} \varepsilon_e A^e = 0, \quad e = 1, 2, \ldots, m,$$

(30)

where $\varepsilon$ represents the strain vector of the $e$th design cell on upper surface, and it is an implicit function of orientation variables in all design cells. $C$ is the equivalent rotated orthotropic stiffness matrix of the $e$th design cell and it is function of material property and orientation variable $\theta_e$ as defined in Eq. (12). $A^e$ stands for the area of the $e$th design cell, $t$ is the thickness of shell element. For simplicity, $\frac{1}{3} A^e t$ will be set to be unity thereafter.

Instead of expressing the optimality condition in the strain form, Eq. (30) can also be represented in the stress form as

$$\frac{\partial \Psi}{\partial \theta_e} = \sigma^T_e \frac{\partial S}{\partial \theta_e} \sigma_e = 0, \quad e = 1, 2, \ldots, m,$$

(31)
here $\sigma_e$ is the stress vector in the $e$th shell design cell on the upper surface, $S$ is the equivalent rotated orthotropic compliance matrix which can be expressed by the unrotated orthotropic compliance matrix $S_0$ and rotation matrix $T(\theta_e)$:

$$S = T^{-1}(\theta_e)S_0T^{-T}(\theta_e),$$

and

$$S_0 = C_0^{-1} = \begin{bmatrix} s_{11}^0 & s_{12}^0 & 0 \\ s_{12}^0 & s_{22}^0 & 0 \\ 0 & 0 & s_{33}^0 \end{bmatrix},$$

where unrotated stiffness matrix $C_0$ has been determined in Eq. (11).

4. Energy-based method

From all the solutions which satisfy Eq. (30) or Eq. (31), the one that makes $\Psi$ minimum is the optimal solution, which corresponds to the optimal bead orientations. It should be noted that Eqs. (30) and (31) are two sets of coupled equations since both strain $\varepsilon_e$ and stress $\sigma_e$ are implicit functions of the orientation variables for all shell design cells. To solve these equations, further simplifications must be made. In this section, two commonly used methods, the strain- and stress-based methods are briefly reviewed, and a new method called the energy-based method will be proposed.

4.1. Strain-based method

In the strain-based method, the element strain field is assumed to be invariant with respect to the variation of orientation variables. Base upon this assumption, Eq. (30) can be modified as

$$\frac{\partial^2\Psi}{\partial \theta_e} - \frac{\partial}{\partial \theta_e}(\varepsilon_e^T C \varepsilon_e) = 0, \quad e = 1, 2, \ldots, m.$$  

It is obvious that the term $\varepsilon_e^T C \varepsilon_e$ represents the strain energy density stored in the $e$th design cell. This expression can also be cast by the principal strains and the rotational angle between the principal strain axes and the material axes. A detailed discussion about the strain-based method was found in Refs. [2–4].

4.2. Stress-based method

In the stress-based method, the element stress field is assumed to be fixed with respect to the variation of orientation variables, under this assumption, Eq. (31) can be written as

$$\frac{\partial^2\Psi}{\partial \theta_e} = \frac{\partial}{\partial \theta_e}(\sigma_e^T S \sigma_e) = 0, \quad e = 1, 2, \ldots, m.$$
here the strain energy density $\sigma_0^T S \sigma_0$ is expressed in the stress term, and it can also be expressed by the principal stresses and the angle between the principal stress axes and the material axes. A detailed description of the stress-based method was found by Diaz and Bendsøe [5], and Cheng, Kikuchi and Ma [10].

Although the strain-based method works well in kinematically determinate systems and the stress-based method works well in statically determinate systems, they are not generally applicable. As we have pointed out previously, both strain and stress fields are implicit functions of orientation variables, they will change with the variations of orientation variables. In the next section, a new energy-based method is proposed to solve the optimal orientation problem.

### 4.3. Energy-based method

Instead of assuming that the strain and stress field are fixed with respect to the orientation variables, the dependency of strain and stress fields on material orientation is explored in the energy-based method. In this new approach, an inclusion model is used to analyze the variations of the strain and stress of one design cell due to the rotation of orthotropic material in that design cell. For the convenience in solving the optimality equations, the coupling effect, i.e. the variations generated from the rotations of surrounding cells, is assumed not very significant and is therefore neglected.

Fig. 4(a) shows an initial structure under static loading, the orthotropic material in the $e$th design cell has no rotation. In this case, we calculate the strain and stress field of the structure, and denote them as $\varepsilon_0, \sigma_0$, respectively. To calculate the new strain and stress fields in the rotated $e$th design cell, as shown in Fig. 4(b), we first take the $e$th design cell out of the structure, and apply the displacement boundary condition with the magnitude of $\varepsilon_0$ on it so that the size and shape of the $e$th design cell remain unchanged. Also a negative stress with the magnitude of $\sigma_0$ is applied.

![Fig. 4](image-url) Fig. 4. (a) Initial structure before material rotation; (b) structure after material rotated through $\theta_e$ in the $e$th design cell.
to the interface which previously separated the \textit{e}th design cell from its surroundings to keep its initial state. In this stage, the strain and stress in the \textit{e}th design cell are:

\begin{equation}
\varepsilon_{e} = \varepsilon_{0}; \quad \sigma_{e} = \sigma_{0} = C_{0}\varepsilon_{0}.
\end{equation}

Rotating the \textit{e}th design cell by an angle of $\theta_{e}$ makes the material stiffness matrix change from $C_{0}$ to $C$. In this stage, the strain of the design cell will not change because the cell is under the same displacement boundary condition, whereas the stress of the design cell will change. The new strain and stress fields of the design cell are:

\begin{equation}
\varepsilon_{e}^{'} = \varepsilon_{0}; \quad \sigma_{e}^{'} = \sigma_{1} = C\varepsilon_{0}.
\end{equation}

We then insert the \textit{e}th design cell back to the cavity which it originally occupied in the structure. Because the size and shape of the design cell and its surrounding structure remain unchanged during the previous steps, the displacement field is still continuous at the interface. However, traction stress now must suffer from discontinuity at the interface. To remove this discontinuity, we apply an opposite stress $\mathbf{\tilde{\sigma}}$ with the magnitude of the stress difference, $(C - C_{0})\varepsilon_{0}$ at the interface; this would introduce an additional strain and stress fields for both design cell and its surrounding structure. We denote this additional strain and stress as $\Delta\varepsilon$ and $\Delta\sigma$. Here, the new strain and stress in the \textit{e}th design cell are:

\begin{equation}
\varepsilon_{e} = \varepsilon_{0} + \Delta\varepsilon; \quad \sigma_{e} = \sigma_{1} + \Delta\sigma.
\end{equation}

To evaluate the strain and stress field $\Delta\varepsilon$ and $\Delta\sigma$, the concept of an energy factor is introduced. During the third step of the inclusion model, the applied traction stress $\mathbf{\tilde{\sigma}}$ would do work on both the \textit{e}th design cell and its surrounding structure. We can define an energy factor $\alpha$ as the ratio of the strain energy stored in design cell to the strain energy stored in the whole structure due to the work done by $\mathbf{\tilde{\sigma}}$. Therefore, the strain and stress $\Delta\varepsilon$ and $\Delta\sigma$ can then be approximately represented by energy factor as follows:

\begin{equation}
\Delta\varepsilon = \alpha S\mathbf{\tilde{\sigma}}; \quad \Delta\sigma = \alpha \mathbf{\tilde{\sigma}}.
\end{equation}

Using the expression for $\mathbf{\tilde{\sigma}}$, and inserting the above equation to Eq. (38), we obtain:

\begin{equation}
\varepsilon_{e} = (1 - \alpha)\varepsilon_{0} + \alpha S\sigma_{0}; \quad \sigma_{e} = (1 - \alpha)\sigma_{1} + \alpha \sigma_{0}.
\end{equation}

The above equations provide very important insights of the physical assumptions in the strain- and the stress-based methods. When $\alpha = 0$, Eq. (40) gives $\varepsilon_{e} = \varepsilon_{0}$, which is the assumption made in the strain-based method. According to the definition of energy factor, $\alpha = 0$ means that the traction stress $\mathbf{\tilde{\sigma}}$ does not do any work on the design cell, this can only be true when the surrounding body is extremely stiff compared to the design cell. When $\alpha = 1$, Eq. (40) gives $\sigma_{e} = \sigma_{0}$, which is the assumption made in the stress-based method. Physically, $\alpha = 1$ means the traction stress $\mathbf{\tilde{\sigma}}$ does not do any work to the surroundings, which can only be possible when the design cell is much stiffer than its surrounding structure. From these physical insights, the limitations of the strain-based method and the stress-based method are discovered, therefore, they must be used with caution.

Substitute Eq. (40) into Eq. (30), we get one set of uncoupled equations for orientation variables $\theta_{e}$, $e = 1, 2, \ldots, m$:

\begin{equation}
\frac{\partial \Psi}{\partial \theta_{e}} = ( - 1 + 2\alpha - \alpha^{2})\varepsilon_{0}^{T} \frac{\partial C}{\partial \theta_{e}} \varepsilon_{0} + \alpha^{2} \sigma_{0}^{T} \frac{\partial S}{\partial \theta_{e}} \sigma_{0} + (2\alpha^{2} - 2\alpha)\varepsilon_{0}^{T} \frac{\partial C}{\partial \theta_{e}} \varepsilon_{0} = 0.
\end{equation}
Eq. (41) should be solved iteratively in order to get the optimal bead orientation, in each step the equivalent orthotropic materials are changed to the new orientations. Normally convergence is reached within five iterations and one finite element analysis is required for each iteration.

In solving Eq. (41), energy factor \( \alpha \) needs to be specified. In the current implementation, we dynamically set \( \alpha = \frac{\| \sigma_e - \sigma_1 \|}{\| \sigma_0 - \sigma_1 \|} \) as derived from Eq. (40) with an initial estimate value between 0 and 1.

5. Numerical examples

5.1. Example 1

In the first example, the optimal bead orientation of a square plate is studied. A plate is supported at four corners and is subjected to a unit concentrated loading at the center. The design objective is to find the stiffest bead pattern under the given loads. The whole plate is considered as the designable area. Suppose the base square plate has Young’s modulus \( E = 10 \) GPa Poisson ratio \( \nu = 0.33 \) and shell thickness 0.5 cm. The plate is discretized by \( 20 \times 20 \) finite elements and there are 400 discretized bead orientation variables in this problem. The bead in each design cell is modeled so that the equivalent bending thickness is two times that of base structure. The initial bead orientation is shown in Fig. 5 and the optimal bead orientation is shown in Fig. 6. The mean compliance is reduced from the initial \( 5.3736 \times 10^{-4} \) N m to \( 1.5214 \times 10^{-4} \) N m in five iterations. A concentric ring pattern is easily identified as the stiffest design.

Fig. 5. The initial bead pattern of a square plate that is fixed at four corners and under a unit concentrated load at the center.
5.2. Example 2

In the second example, the coupling effects of the location of the bead and the orientation of the bead are studied. The same square plate as the first example is used here. However, instead of putting beads on the whole plate, beads are placed on the selected area. As shown in Fig. 7, only the gray region is the designable area and the white elements are the non-designable area where no bead is placed. The optimal bead pattern is shown as in Fig. 8. The mean compliance is decreased from the initial $6.7881 \times 10^{-4}$ N m to $3.5610 \times 10^{-4}$ N m in five iterations. The small square area at the center in Fig. 8 shows a similar concentric ring pattern as in the first example while the bead orientation of the four legs is strictly $45^\circ$ for directly connecting the four corners for the best support. This pattern was not shown in the first example. This strongly suggests that the optimal bead pattern is very closely coupled with the bead location.

5.3. Example 3

In the third example, the same square plate model as in the first example is used for dynamic consideration. The design objective is to maximize the first natural frequency by changing the bead orientation. The material density is chosen as $3 \times 10^3$ Kg/m$^3$. Fig. 9 shows the optimal bead orientation and the corresponding first eigenvalue is 25.0013 compared to the initial 6.6204 in four iterations. Comparing with Fig. 6, which is the optimal bead pattern design from the static case, Fig. 9 exhibits significant different pattern in the central region and in the way connecting its center.
to the supporting points. However, bead orientation in the regions near the plate edges shows similarities. We also calculated the first eigenvalue of the bead pattern in Fig. 6 and it was found to be 24.2532, which is slightly lower than the current design. This is because a different bead pattern produces a different mode shape.
6. Conclusion

In this paper, the optimal bead orientation problem of 3D shell/plate structures was studied. Based on the stress resultant-strain and stress couple-curvature relations of bead-stamped shell element, a bending equivalent orthotropic model was proposed for calculating the flexural deformations, and the bead orientation problem was solved by determining the optimal orientation of bending equivalent orthotropic materials. A new energy-based method was used to obtain the optimal bead orientation, in which the variations of strain and stress due to the orientation of material were approximated by an energy factor. Numerical examples showed the optimal bead orientation is closely related to the bead location, therefore, they should be considered simultaneously.

Acknowledgements

The support of this work from the National Science Foundation (DMI-9500170) and Ford Motor Company is gratefully acknowledged.

References