Philosophical Issues in Quantified Modal Logic
Handout 2: Quantified Modal Logic: An Introduction
Prof. J. Stanley

I. Review of Propositional Modal Logic (continued)

A well-formed formula Φ is *valid* on a frame <W,R> iff, for every model <W,R,V> based on <W,R>, and for every w in W, V(Φ,w) = 1.

A well-formed formula Φ is valid on a class of frames iff it is valid on each frame in that class.

We are thinking of a system as a collection of axioms, together with some rules of inference. Theorems of the system are the ones that are provable from those axioms, with the use of those rules of inference (it is often useful to think of the system instead as the class of its theorems).

Soundness: A system Π is *sound* with respect to a class of frames Δ if and only if all of the theorems of Π are valid on Δ.

Completeness: A system Π is *complete* with respect to a class of frames Δ if and only if all of the wffs of the language of Π that are valid on Δ are theorems of Π.

To show that Π is sound with respect to Δ it is sufficient to show that (1) every axiom of Π is valid on Δ and (2) that the rules of inference of Π preserve validity in Δ (i.e. that if they are applied to wffs which are valid on Δ, then they yield wffs that are valid on Δ).

K is sound and complete with respect to the class of all frames <W,R>

Last time, we showed that there are some frames in which the T axiom is invalid:

T: Lp→p

These frames are ones in which the accessibility relation R is not reflexive.

A frame <W,R> is reflexive iff for all w in W, wRw.

You cannot invalidate T on a reflexive frame.

Proof: Suppose that <W,R> is reflexive, and for *reductio ad absurdum*, suppose that <W,R,V> is a model based on <W,R> in which T is invalid. Then, for some w in W, V(Lp, w) = 1, and V(p, w) = 0. But by assumption, R in <W,R,V> is reflexive. So, by the definition of "reflexive", wRw. But then, by V[L], V(Lp, w) = 0, since there is some w' such that wRw' and V(p, w) = 0, namely w itself.

In fact, the system T (which is the system K + the T axiom) is sound and complete with respect to the class of all reflexive frames.

Last time, we showed that there are some frames in which the 4 axiom is invalid:

4: $Lp \rightarrow LLp$

The key feature of these frames which allow us to invalidate 4 is that they are not transitive. You cannot invalidate 4 on a transitive frame.

Proof: Suppose that <W,R,V> is transitive, and for reductio, suppose that <W,R,V> is a model based on <W,R> in which 4 is invalid. Then, for some world w in W, V(Lp, w) = 1, and V(LLp, w) = 0. By V[L], it follows that for some w' such that wRw', V(Lp, w') = 0. By V[L] it follows that for some w'' such that w'Rw'', V(p, w'') =0. But R is transitive. So, since wRw' and w'R'', it follows that wRw''. But then V(Lp, w) = 0, since there is a w* such that wRw* and V(p, w*) = 0, namely w'' (i.e. let w* = w'').

A frame <W,R> is transitive iff for all w, w', w'' in W, if wRw' and w'Rw'', then wRw''.

The system S4, which is T + 4, is sound and complete with respect to the class of transitive frames.

The system B is T together with the B axiom:

B: $p \rightarrow LMp$

A frame <W,R> is symmetric iff for all w, w' in W, if wRw', then w'Rw.

B is sound and complete with respect to the class of symmetric frames.

The system S5 is T together with the 5 axiom:

5: $Mp \rightarrow LMp$

R is an equivalence relation iff it is reflexive, transitive, and symmetric.

S5 is sound and complete with respect to the class of frames <W,R> in which R is an equivalence relation.

II. Different readings of modals

"Necessary" and "possible" can mean many different things.

Deontic modality:

“You ought not to kill”
“You are permitted to gamble”

Legal modality:

“In the United States, it’s necessary that you drive on the right hand side.”
“It’s necessary that I show up for jury duty.”
“I must show up for jury duty.”
“In Australia, one must vote.”
“In the United States, one can abstain from voting.”

Metaphysical modality:

“It’s necessary that 9 is greater than 7”
“I could have been a contender.”

Epistemic modality:

“It’s possible that it’s raining outside”
“It must be raining outside”

Standard question: what is the right logic for modality, when we interpret the modality as metaphysical/deontic/epistemic etc.

Deontic modality: “L” is interpreted as “ought”, and “M” is interpreted as “can”.

Some truisms about deontic modality:

For good or for ill, ought does not imply is. So we do not want the T axiom, $Lp \rightarrow p$. Furthermore, it is often maintained that ought implies can. So we want:

D. $Lp \rightarrow Mp$

The system D is K + D. D is typically taken to be the right logic for deontic modality.

A frame <W,R> is *serial* iff for every w in W, there is some w’ in W such that wRw’.

D is sound and complete with respect to the class of serial frames.

Most philosophers who work on the topic think that S5 is the appropriate logic for metaphysical modality. Some philosophers (e.g. Nathan Salmon, in <u>Reference and Essence</u>) have, however, argued that metaphysical possibility is not transitive, and so reject the 4 axiom (and so deny that either S4 or S5 is the proper logic for metaphysical modality).

III. Canonical Model Completeness Proofs

A system Π is complete with respect to a class of frames Δ iff every wff valid in Δ is provable in Π.

There is a very useful way of proving completeness for modal logics, which is called the *Canonical Model* Method.

"For every consistent normal modal system S, there is a special model, called the canonical model of S, which has the remarkable property that a wff Φ is valid in the canonical model of S iff $\vdash_S \Phi$."

Connection between this fact and completeness: Suppose we are trying to show that S is complete with respect to a certain class of frames Δ. If we can show that the frame of the canonical model for S is in Δ, then we will know that all the wffs valid in Δ are provable in S (because the wffs valid in Δ are a subset of the wffs valid in the canonical model, if the frame of the canonical model is in Δ).

On pp. 113-119 of Hughes and Cresswell, they show how to construct, for any system S, the canonical model for S. On pp. 119-121, they prove that K, T, B, S4, and S5 are complete with respect to the class of all frames, reflexive frames, symmetrical frames, transitive frames, and frames where the accessibility relation is an equivalence relation, respectively. As you can see from the page lengths, the most time consuming part of this is showing how to construct the canonical model for a system. Completeness just requires showing that the frame of the canonical model for S is in the class of frames for which one is trying to prove completeness.

[On Monday, in the logic section, we will discuss canonical model completeness proofs, so study these pages for Monday]

IV. Quinean Skepticism about the Coherence of Quantified Modal Logic

The language of Quantified Modal Logic (QML) is simple. It is the language of the predicate calculus, augmented with '□' and '◊' (or 'L' and 'M').

<u>Formal Language</u>

Alphabet of L:

A, B, ...,E	Name Letters (Constants)
$F^n, G^n, ..., Z^n$	n-place Predicate Letters
P, Q, ...,Z	Sentence Letters
a, b, c, ..., w, x, y, z	Variables
~, →, ↔, v, &	Sentential Connectives
∀, ∃	Quantifiers (Universal, Existential)
□, ◊	Modal Operators (Necessity, Possibility)

Grammar of L:

Termhood of L:
(i) All name letters and variables are terms.
(ii) Nothing else is a term.

Well-formed Formula (wff) of L:

(i) 0-place predicate letters are wffs.
(ii) $\varphi\alpha_1...\alpha_n$ is a wff if φ is an n-place predicate letter, and each of $\alpha_1....\alpha_n$ is a term.
(iii) ~φ is a wff if φ is a wff.
(iv) (φ → ψ) is a wff if φ is a wff and ψ is a wff.
(v) (φ ↔ ψ) is a wff if φ is a wff and ψ is a wff.
(vi) (φ v ψ) is a wff if φ is a wff and ψ is a wff.
(vii) (φ & ψ) is a wff if φ is a wff and ψ is a wff.
(viii) ∀αφ is a wff if φ is a wff and α is a variable.
(ix) ∃αφ is a wff if φ is a wff and α is a variable.
(x) □Φ is a wff if Φ is a wff
(xi) ◊Φ is a wff if Φ is a wff
(x) Nothing else is a wff.

The issues we will be discussing in this seminar involve the proper semantic interpretation of QML. Before we turn to laying down a preliminary interpretation of QML, I want to discuss some of Quine's skepticism about the project of giving an interpretation to QML which I discussed a bit last time.

In the semantics for the predicate calculus (see Tarski handout), open formulas are satisfied by sequences, or assignment functions, that assign objects to variables. Let's call the relation that holds between a sequence, or assignment function, and an open formula *objectual satisfaction*. This is the relation between the sequence s and the open formula 'Fx' in:

(OS) s satisfies 'Fx' iff s(x) is in the interpretation of 'F'.

Objectual satisfaction is the relation we need to make sense of quantification. For example, '∀xFx' is true if and only if it is satisfied by every sequence; it is satisfied by an arbitrary sequence s iff every s' that is an x variant of x is such that s' satisfies the open formula 'Fx'. So, on the Tarskian model, quantification is understood in terms of the satisfaction relation.

Let an *open modal formula* be a formula containing a free variable in the scope of a modal operator that is not bound by any quantifier within that formula. So examples of open modal formulae are '□Fx', '◊RxA', and '□∀x(Fx → Rxy)'. Quine does not think that there is a legitimate notion of objectual satisfaction that can be given for QML. This shows that it is not possible to give sense to quantification into open modal formulae, since we need the satisfaction relation to explain quantification.

Here is one kind of reason Quine gives for his skepticism that there is a legitimate notion of objectual satisfaction for QML. Quine considers pairs of sentences such as (1) and (2):

(1) Necessarily 9>4
(2) Necessarily the number of major planets > 4

Quine writes (*Word and Object*, p. 197):

> …is it more legitimate to quantify into modal positions than into quotation? For consider (1)…; surely, on any plausible interpretation, (1) is true and [(2)] is false. Since 9 = the number of major planets, we can conclude that the position of ‘9’ in (1) is not purely referential and hence that the necessity operator is opaque.

Let ‘$\Phi_{(a/b)}$’ denote the result of substituting all free occurrences of ‘a’ in Φ with occurrence of ‘b’. Substitution is the schematic principle:

(S) $a = b \rightarrow (\Phi \leftrightarrow \Phi_{(a/b)})$

Quine’s point is that modal operators give rise to failure of substitution. Quine’s (1) and (2) are instances of the failure of the following instance of the substitution principle:

(I) 9 = the number of major planets → [$\Box$(9>4) ↔ $\Box$(the number of planets > 4)]

Here, all we have done is substitute “the number of planets” for “9” in “($\Box$(9>4)”.

Quine concludes that there is no coherent relation of objectual satisfaction between sequences and open modal formulae. That is, Quine concludes that unlike the relation between sequences and open formulae in the language of the predicate calculus (the relation we see in (OS)), there is no coherent relation between sequences and open modal formulae.

Quine’s reasoning from the failure of substitution to the incoherence of objectual satisfaction for open modal formulae proceeds as follows:

(a) Modal operators give rise to failure of substitution within their scope.
(b) Therefore, positions within the scope of modal operators are referentially opaque (“not purely referential”).
(c) Therefore, there is no notion of objectual satisfaction for open modal formulae.

It is, to say the least, not obvious what the logical connection in fact is between (a), (b), and (c). Is it really the case that failure of substitution –the scheme in (S)—has these consequences?

There is one model for Quine's argument, a case in which there is failure of substitution within a certain operator, the failure of substitution is indicative of the fact that positions within that operator are not purely referential, and there is clearly no room for a notion of objectual quantification into the scope of such operators. The example is *quotation*.

(1) John uttered the sentence, "Jason is the best dressed Rutgers philosophy professor"
(2) John uttered the sentence, "Professor Stanley is the best dressed Rutgers philosophy professor"
(3) There is an x such that John uttered the sentence "x is the best dressed Rutgers philosophy professor"

The fact that (1) and (2) do not necessarily have the same truth-value demonstrates that substitution of co-referential expressions fails within quotation marks. The fact that quantification into quotation marks is dubiously coherent is shown by the fact that (3) is nonsense. So, in the case of quotation, the failure of quotation goes along with the failure of quantification into quoted open sentences, like "x is the best dressed Rutgers philosophy professor". Indeed, the whole idea of a 'quoted open sentence' doesn't seem to make sense. It is not as if "x is the best dressed Rutgers philosophy professor" is really an open sentence, one that can be interpreted relative to an assignment of a value to the variable "x". Rather, "" x is the best dressed Rutgers philosophy professor"" seems just to denote an expression containing the variable "x".

Of course, the fact that quotation is a case in which failure of substitution and failure of the coherence of quantifying in go together does not mean that Quine's argument against the coherence of objectual satisfaction for open modal formulae is sound. However, there is a special reason why Quine thought there was an analogy between quotation and modality. It is because the dominant theories of modality at the time (e.g. Rudolf Carnap, in Meaning and Necessity) treated "necessary" as synonymous with "It is analytic". The operator "It is analytic" attaches primarily to *sentences*. So if "It is necessary that" is really to be understood (as Carnap urged) as "It is analytic that", then "It is necessary that" in fact attaches to sentences as well. If so, then modality is a context akin to quotation.

But whether or not there are philosophically cogent parallels between modality and analyticity, and hence modal operators and quotation should not bear on our evaluation of Quine's argument. It really is unclear how Quine gets from the failure of substitution to the incoherence of objectual quantification. The best discussion of Quine's argument (as well as the most cogent evaluation of it) is in David Kaplan's masterful essay, "Opacity". Here is how Kaplan reconstructs Quine's argument (p. 234):

Step 1: A purely designative occurrence of a singular term in a formula is one in which the singular term is used solely to designate the object. [This is a definition]

Step 2: If an occurrence of a singular term in a formula is purely designative, then the truth value of the formula depends only on *what* the occurrence designates, not on *how* it designates. [From 1]

Step 3: Variables are devices of pure reference; a bindable occurrence of a variable must be purely designative. [By standard semantics]

Notation: Let Φ be a formula with a single free occurrence of 'x', and let Φα, Φβ, and Φγ be the results of proper substitution of the singular terms α, β, and γ for "x".

Step 4: If α and β designate the same thing, but Φα and Φβ differ in truth value, then the indicated occurrences of α in Φα and of β in Φβ are not purely designative. [From 2]

Now assume 5.1: α and β are co-designative terms, but Φα and Φβ differ in truth-value.
5.2: γ is a variable whose value is the object co-designated by α and β.

Step 6: Either Φα and Φγ differ in truth-value, or Φβ and Φγ differ in truth-value. [From 5.1, since Φα and Φβ differ in truth value]

Step 7: The indicated occurrence of γ in Φγ is not purely designative. [From 5.2, 6, and 4]

Step 8: It is semantically incoherent to claim that the indicated occurrence of γ in Φγ is bindable.

This is Kaplan's excellent reconstruction of Quine's argument.

Kaplan's diagnosis of the *failure* of Quine's argument is the standard diagnosis (though there are other alternatives to the standard view). Kaplan argues that the problematic premise is Step 4:

> All but one of these steps seem to me to be innocuous. That one is step 4 which, of course, does not follow from step 2. All that follows from Step 2 is that at least one of the two occurrences is not purely designative. When 4 is corrected in this way, 7 no longer follows. (Kaplan, "Opacity", p. 235)

In my brief reconstruction of Quine's argument, I spoke of *positions* as not being purely referential. One way to see what Quine is doing in moving from step 2 to step 4 is moving (illicitly) from the fact that an *occurrence of a term* in a position in a formula is not purely designative, to the conclusion that *that position in that formula* is not purely designative. But that is not legitimate. Something about *the term itself* might lead to the fact its occurrence is not purely designative.

For example, perhaps *definite descriptions*, such as "the number of major planets", are not purely designative terms. They are not "used solely to describe the object". If so, then an occurrence of a definite description in the scope of a modal operator is not a purely designative occurrence of that term in that position. But nothing follows about *the position itself*.

So, for example, in (2):

(2) $\Box$(the number of major planets > 4)

The term "the number of major planets" does not have a purely designative occurrence in the formula (2). But this is not because *the position it occupies* in (2) (the same position occupied by the term "9" in "$\Box$(the number of major planets > 4)") fails to be a purely designative *position*. It is rather because definite descriptions are not *purely designative terms*.

This is in fact the orthodox response to Quine's argument. It fails because it moves from the failure of substitution with terms that are not purely designative to the conclusion that there are positions in formula (under the scope of modal operators) that are not purely designative. The recognition that definite descriptions (as opposed to variables, and if Kripke is right, names) are not purely designative is what explains the failure of substitution in modal formula, and *not* that modal operators create referentially opaque contexts.

V. Quantified Modal Logic

Systems of QML (the below is from pp. 244ff. of Hughes & Cresswell)

Where S is a system of normal modal propositional logic, then LPC + S is defined as follows, where the wff are now wff of the language of QML

S' If α is an LPC substitution instance of a theorem of S then α is an axiom of LPC + S
∀1 If α is any wff and x and y any variables and $\alpha[y/x]$ is α with free y replacing every free x, then $\forall x\alpha \rightarrow \alpha[y/x]$ is an axiom of LPC+S
N If α is a theorem of LPC + S, then so is $\Box\alpha$
MP If α and $\alpha\rightarrow\beta$ are theorems of LPC + S, then so is β
∀2 If $\alpha\rightarrow\beta$ is a theorem of LPC + S and x is not free in α then $\alpha\rightarrow\forall x\beta$ is a theorem of LPC + S

Barcan Formula:

BF $\forall x\Box Fx \rightarrow \Box\forall x\, Fx$

BF is a theorem of LPC + B and LPC + S5, but it is not a theorem schema of K, T, or S4 + LPC. We shall consider it to be an additional axiom for now.

The Converse of the Barcan Formula (The Converse Barcan Formula) is provable in LPC + K:

CBF $\Box\forall x\, Fx \rightarrow \forall x\Box Fx$

Here is the proof of CBF:

(1) $\forall x\alpha \rightarrow \alpha$ (from **∀1**)
(2) $\Box\forall x\alpha \rightarrow \Box\alpha$ (from K)
(3) $\Box\forall x\alpha \rightarrow \forall x\Box\alpha$ (from **∀2**)

Since CBF is a theorem of all normal quantified modal logics, and BF is a theorem of at least B and S5, we want a semantics that validates them.

Here is the semantics (from pp. 243-4 of Hughes and Cresswell):

A model for a language L of modal LPC is a quadruple <W,R,D,V> in which W is a set (of 'worlds'), R a relation on W, D another set and V a function such that where Φ is an n-place predicate, $V(\Phi)$ is a set of n+1 tuples each of the form $<u_1,\ldots.u_n,w>$ for $u_1\ldots.u_n$ in D and w in W.

An assignment s (or *sequence*) to the variables is a function such that, for each variable x, s(x) is in D. x-alternatives to an assignment are as in the semantics for predicate calculus (see the Tarski handout).

[V$\Box$], [V~], [Vv] as in modal propositional logic

[VΦ] $V_s(\Phi x_1\ldots.x_n,w) = 1$ if $<s(x_1),\ldots, s(x_n),w>$ is in $V(\Phi)$ and 0 otherwise.
[V$\forall$] $V_s(\forall x\alpha) = 1$ if $V_{s'}(\alpha,w) = 1$ for every $s'\approx_x s$, and 0 otherwise.
[V$\exists$] $V_s(\forall x\alpha) = 1$ if $V_{s'}(\alpha,w) = 1$ for some $s'\approx_x s$, and 0 otherwise.

A wff is valid in <W,R,D,V> iff $V_s(\alpha,w) = 1$ for every w in W and every assignment s.

A wff is valid on a frame <W,R> iff it is valid in every model based on <W,R>.

Both BF and CBF are valid, given this semantics. They are valid since there is only one domain.

There are other principles that are both derivable in the logic and valid in the semantics that we should look at. Suppose that we added to the language of the predicate calculus, a symbol "=", to be understood intuitively as identity. This would be governed by the following axiom:

= t=t

Call the resulting system LPC=. Once we add a modal system to LPC=, the following is immediately a theorem:

NE $\Box\exists x(t=x)$

Here is the derivation:

(1) $\forall x{\sim}(t=x) \rightarrow {\sim}(t=t)$ by **$\forall$1**
(2) ${\sim}{\sim}(t=t) \rightarrow {\sim}\forall x{\sim}(t=x)$ by PC (propositional calculus, contraposition)
(3) $t=t \rightarrow {\sim}\forall x{\sim}(t=x)$ by PC (double negation elimination)
(4) ${\sim}\forall x{\sim}(t=x)$ (3), =, MP
(5) $\exists x(t=x)$ Def $\exists$

So **CBF** is a theorem of any normal modal quantificational logic (LPC + K). **BF** is a theorem of LPC + B and LPC + S5, and **NE** is a theorem of $LPC^{=}$ + K.

Are these happy results?

One truly terrifying instance of **NE**:

$\Box\exists x(\text{Jeff King}=x)$

Similarly, consider **BF** and **CBF**. Suppose every existing thing is material, but suppose there are possible worlds with immaterial things. If something is material, then (following perhaps the essentiality of origins), it is necessarily material. So this is a situation in which (1) is true:

(1) $\forall x\Box\text{Material}(x)$

But there are possible worlds in which there are non-material things, i.e.:

(2) $\Diamond\exists x{\sim}(\text{Material}(x))$

But (2), by the definition of '$\exists$', is equivalent to:

(3) ${\sim}\Box\forall x(\text{Material}(x))$

In short, **BF** is inconsistent with the situation we have described. But surely it is perfectly coherent position to hold that everything actual is essentially material, though there could have been non-material things. Nevertheless, if BF is a law of modal logic, it is not coherent – it is ruled out *by logic*.

Similarly, **CBF** is controversial. Suppose that in every world, everything is F. Does it really follow that everything in our world is necessarily F? Let F be the predicate "$\exists y(y=x)$". It's plausible that $\Box\forall x(\exists y(y=x))$. But it presumably isn't plausible that $\forall x\Box(\exists y(y=x))$. Some of the things that exist do so contingently. But **CBF** requires that everything that exists here necessarily exists.

Other principles that are provable by the same reasoning as CBF are similarly implausible. Consider, for example, the formula:

(1) $\Diamond\forall xFx \rightarrow \forall x\Diamond Fx$

This formula is provable using DR3 (distribution of "◊" over the conditional) rather than K in the same way that **CBF** was proved. But it's easy to imagine a situation in which it is possible that everything in that situation is F, without it being that case that everything that actually exists being possibly F. Consider a possible world in which everything is either an inanimate object or a single-celled organism, and let F be the predicate "x is either an inanimate object or a single-celled organism". Supposing that it's not possible for Jeff King to be a single-celled organism, we have just falsified the formula "$\Diamond\forall xFx \rightarrow \forall x\Diamond Fx$". But surely the situation just described is what is in fact the case. After all, at one point in the history of our world, perhaps everything was either an inanimate object or a single-celled organism.

So we need a logic in which **CBF**, **BF**, **NE** and principles such as (1) are not provable. And we need a semantics that invalidates them all. That is the purpose of Kripke's paper, "Semantical Considerations on Modal Logic".