Philosophical Issues in Quantified Modal Logic
Handout 3: Variable Domain Semantics for QML
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I. De Re/De Dicto

From Graeme Forbes, The Metaphysics of Modality, p.48:

> A formula with modal or tense operators is *de re* iff it contains a modal or tense operator R which has within its scope either (1) an individual constant, or (2) a free variable, or (3) a variable bound by a quantifier not within R's scope. All other formulae with modal or tense operators are *de dicto*.

Ibid., p. 49:

> The difference between de re and de dicto formulae…is a difference between formulae which are, and formulae which are not, sensitive to the identities of objects at various worlds. In evaluating a de dicto formula A in a model, we eventually reach subformulae of A whose main connectives are modal or tense operators which have within their scope complete sentences which themselves do not contain any individual constants or modal or tense operators. '$\Box\forall x\ Fx$' is already of this form, and in '$\Diamond\forall x\ Fx \rightarrow \Box\forall x\ Gx$'…we reach such subformulae after one application of the evaluation rule for the main connective of each formula itself. Having reached such a subformula, one then applies the evaluation rule for the governing modal or tense operator, which in turn will require one to evaluate the formula within the scope of that operator at various worlds or times. This last formula is just a quantified sentence or a propositional combination of quantified sentences, and so in evaluating it at a world or time one is concerned to discover just whether some, or all, of the objects which exist at the world or time, satisfy certain conditions. And this last step can be effected without regard to the *identity* of the existents at that world or time. Note how this semantic account of the difference between the de re and the de dicto motivates our decision to count sentences with individual constants within the scope of modal or tense operators as being de re. For the interpretation of such sentences is sensitive to the identity of objects at worlds and times. For instance, in evaluating '$\Box$Jones is made of matter' at the actual world, we have to evaluate 'Jones is made of matter' at every world, which requires us to be able to identify Jones at each world.

Forbes's assimilation of formulae that contain constants in the scope of modal operators to de re formulae presupposes that constants are semantically treated like variables under an assignment; that they are, in short, "directly referential" (not like quantified expressions). If individual constants were semantically like definite descriptions, then clause (1) of Forbes's definition of de re formulae would be incorrect, because formulae that contain (as terms) only definite descriptions within the scope of modal operators are not "sensitive to the identities of objects at various worlds". They are just sensitive to which objects in those worlds satisfies those descriptions.

Paper topic (3-5 pages, due in two weeks): Consider the notion of an individual constant employed by Forbes in the above definition of a de re formula. Does this notion presuppose that individual constants are associated with “criteria of transworld identity” that allow us to identify objects across worlds? Why or why not?

Question: Are there modal systems in which all de re wff are equivalent to de dicto wffs? Not for any interesting system (pp. 251ff. of Hughes and Cresswell).

(1) $\Box\exists xFx$
(2) $\exists x\Box Fx$

Consider the following two models <W,R,D,V> and <W,R,D,V*>, with the same frame and domain <W,R,D> but different Vs:

$W = \{w_1,w_2\}$
$R = W^2$ (every world can see every other world)
$D = \{u_1,u_2\}$
$V(F) = \{\langle u_1,w_1\rangle, \langle u_1,w_2\rangle\}$
$V^*(F) = \{\langle u_1,w_1\rangle, \langle u_2,w_2\rangle\}$

The idea behind the proof is that for de dicto wff a model which ‘switches’ u_1 and u_2 in w_2 is equivalent, from the point of view of w_1, to one that does not.

Since <W,R,D,V> and <W,R,D,V*> have the same domain D, the set of assignment functions is the same. Given an assignment s, let x* denote the ‘anti-assignment’ such that for every variable x, $s(x)\neq x^*(x)$ [this is possible because there are only two objects].

Theorem 13.4: If α is *de dicto*, then $V_s(\alpha,w_1)=V_s{}^*(\alpha,w_1)$ and $V_s(\alpha,w_2) = V^*_{s^*}(\alpha,w_2)$.

(see Hughes and Cresswell for proof)

Now we show that ‘$\exists x\Box Fx$’ is not equivalent to any de dicto formula. Suppose that there were some de dicto wff α such that $\vdash\exists x\Box Fx \leftrightarrow \alpha$. Then for every s and every w,

$V_s(\exists x\Box Fx, w) = V_s(\alpha,w)$
$V^*_s(\exists x\Box Fx, w) = V^*_s(\alpha,w)$

But α is de dicto and so by Theorem 13.4,

$V_s(\alpha,w_1) = V^*_s(\alpha,w_1)$.

But it is easy to see that $V_s(\exists x\Box Fx, w_1) = 1$ and $V^*_s(\exists x\Box Fx, w_1) = 0$. So ‘$\exists x\Box Fx$’ is not equivalent to any de dicto formula.

[The proof of Theorem 13.4 exploits the notion of an *anti-assignment*, which is definable only in a model with two elements. For the more general case one needs the notion of an assignment based on a permutation of the domain (see Kit Fine, "Model Theory for Modal Logic, Part I, the *de re/de dicto* distionction").]

II. QML without the Barcan Formula

Recall that the Barcan Formula is:

BF $\forall x \Box Fx \rightarrow \Box \forall x\, Fx$

BF is controversial. Here is the counterexample we discussed last time. Suppose every existing thing is material, but suppose there are possible worlds with immaterial things. If something is material, then (following perhaps the essentiality of origins), it is necessarily material. So this is a situation in which (1) is true:

(1) $\forall x \Box \text{Material}(x)$

But there are possible worlds in which there are non-material things, i.e.:

(2) $\Diamond \exists x {\sim}(\text{Material}(x))$

But (2), by the definition of '$\exists$', is equivalent to:

(3) ${\sim}\Box \forall x(\text{Material}(x))$

In short, **BF** is inconsistent with the situation we have described. But surely it is perfectly coherent position to hold that everything actual is essentially material, though there could have been non-material things. Nevertheless, if BF is a law of modal logic, it is not coherent – it is ruled out *by logic*.

Since it is so evident that BF could be false, it would therefore be nice to work with a semantics that allowed us to represent the situation that is so easy to describe conceptually. That is, it would be nice to work with a semantics that invalidated BF.

A model is a quintuple: <W,R,D,Q,V>

W,R,D as before (set of worlds, a relation on the set of worlds, and a domain of all things that exist in any world, respectively)

Q is a function from members of W to subsets of D. Intuitively, Q takes a world to the set of things that exist at that world. So, we will write Q(w) as "D_w".

The inclusion requirement: If wRw' then $D_w \subseteq D_{w'}$

We will retain the inclusion requirement in our semantics (among other things, this ensures the validity of **CBF**).

[V∀] $V_s(\forall x\alpha, w) = 1$ if $V_{s'}(\alpha,w) = 1$ for every $s' \approx_x s$ such that $s'(x) \in D_w$ and 0 otherwise.
[V∃] $V_s(\forall x\alpha, w) = 1$ if $V_{s'}(\alpha,w) = 1$ for some $s' \approx_x s$ such that $s'(x) \in D_w$, and 0 otherwise.

What should we say about the truth-value of Fx with respect to an assignment s and a world w when s(x) is not in the domain of w?

Three options:

(1) Don't let this happen.
(2) Let Fx lack a truth-value
(3) Leave it up to the model.

Hughes and Cresswell adopt choice (3).

We adopt the following weird definition of validity:

A wff α is valid in a model <W,R,D,Q,V> iff for every world $w \in W$, $Vs(\alpha,w) = 1$ for every assignment s such that $s(x) \in D_w$ for every variable x that is free in α.

Why define validity this way? Consider Universal Instantiation:

∀1 If α is any wff and x and y any variables and α[y/x] is α with free y replacing every free x, then $\forall x\alpha \rightarrow \alpha[y/x]$ is an axiom of LPC+S

This rule would be invalid unless we defined validity in the way that we did. Consider:

(1) $\forall xFx \rightarrow Fy$

Suppose that we have a world w and an assignment s such that everything in D_w is in V(F, w), but s(y) is not in D_w. Then $V(\forall xFx,w) = 1$ but $V(Fy,w) = 0$. By defining validity in the way that we did, we guarantee that this does not invalidate **∀1**.

A wff α is *eligible at w with respect to an assignment s* iff $s(x)\in D_w$ for all variables x that are free in α.

A wff α is eligible at w iff it is eligible at w with respect to all assignments s. A wff α is valid iff α is true at every world at which α is eligible.

The inclusion requirement guarantees the validity of theorems such as:

(2) $\Box(\forall x Fx \rightarrow Fy)$

Without the inclusion requirement, necessitation is not validity preserving. (1) is a theorem, so (2) follows by necessitation. But (1) is valid, and we need the inclusion requirement to guarantee the validity of (2). For suppose that we didn't have the inclusion requirement, and let w_1 be a world in which the domain is $\{u_1, u_2\}$, w_2 a world in which the domain is $\{u_1\}$, and let the extension of F be $\{u_1, u_2\}$ in w_1, and $\{u_1\}$ in w_2. Then (1) is valid, but (2) is false in w_1, and so necessitation would not preserve validity (and so would not be a sound inference rule).

III. QML without the Barcan Formula or the Converse Barcan Formula: Kripke's system

Let's look again at the plausibility of the inclusion requirement. Couldn't it be that something around here might not have existed? For example, surely it's possible that my parents hadn't met. If so, I wouldn't have existed. But this is not consistent with the inclusion requirement.

The inclusion requirement is responsible for the validity of the Converse Barcan Formula. For suppose that the inclusion requirement is in place in some model, $<W,R,D,Q,V>$ but CBF is invalid in that model:

CBF $\Box\forall x\, Fx \rightarrow \forall x\Box Fx$

Since CBF is invalid, for some world w in W, $V(\Box\forall x\, Fx \rightarrow \forall x\Box Fx, w) = 0$. So, $V(\Box\forall x\, Fx, w) = 1$ and $V(\forall x\Box Fx, w) = 0$. So, $V(\exists x\Diamond{\sim}F(x), w) = 1$. So, for some object o in Q(w), and for some w' such that wRw', o is not in the extension of F at w', i.e. $V_s(Fx,w') = 0$ where s(x) = o. But then, by the inclusion requirement, o is in Q(w') as well. So then $V(\forall x Fx, w') = 0$, and since wRw', $V(\Box\forall x\, Fx, w) = 0$, contradiction.

Since CBF requires that everything that exists in this world exists necessarily, and presumably there are some contingently existing things around here, we do not want it to be valid. So we do not want the inclusion requirement as a condition on our models.

On the other hand, recall that CBF is provable in any normal quantified modal logic (LPC +K). Furthermore, as we saw in the previous section, the inclusion requirement allowed us to retain the validity of necessitation. So the situation is as follows. All the pressure in the logic is telling us to retain the inclusion requirement. But the metaphysics is telling us to jettison it. We must be wary here.

We will look at two ways of adjusting the logic to prevent the proof of CBF, and to preserve necessitation. Both methods involve altering the underlying logic for predicate calculus. First, we will look at Saul Kripke's system, in his paper "Semantical Considerations on Modal Logic". Secondly, we will look at what happens when we take the right logic for the predicate calculus to be Free Logic.

The semantics is as above, except we no longer have the inclusion requirement. So there are world-relative domains, and there is no inclusion requirement. Neither BF nor CBF are valid. So the problem now is to block the derivation of CBF, and to preserve necessitation.

Let's look again at the failure of necessitation we discussed in the last section, which led us to impose the inclusion requirement. The problem was that the rule **∀1** entails the following theorem as an instance:

(2) $\forall xFx \rightarrow Fy)$

One application of necessitation yields:

(3) $\Box(\forall xFx \rightarrow Fy)$

Now recall the proof of CBF (expanded from the last handout)

(A) $\forall xFx \rightarrow Fy$ (from **∀1**)
(B) $\Box(\forall xFx \rightarrow Fy)$ (necessitation)
(C) $\Box\forall xFx \rightarrow \Box Fy$ (from K plus Modus Ponens)
(D) $\Box\forall xFx \rightarrow \forall y\Box Fy$ (from **∀2**)

Note that the problematic application of necessitation is the crucial initial step in the derivation of CBF!

Kripke writes:

> Actually, the flaw [in the proof of CBF] lies in the application of necessitation to (A). In a formula like (A), we give the free variables the generality interpretation: When (A) is asserted as a theorem, it abbreviates assertion of its ordinary universal closure
>
> (A') $\forall y(\forall xFx \rightarrow Fy)$
>
> Now if we applied necessitation to (A'), we would get
>
> (B') $\Box\forall y(\forall xFx \rightarrow Fy)$
>
> On the other hand, (B) itself is interpreted as asserting:
>
> (B'') $\forall y\Box(\forall xFx \rightarrow Fy)$
>
> To infer (B'') from (B'), we would need a law of the form $\Box\forall yC(y) \rightarrow \forall y \Box C(y)$, which is just the converse Barcan formula that we are trying to prove…

> We can avoid this sort of difficulty if, following Quine [in Mathematical Logic] we formulate quantification theory so that only closed formulae are asserted. Assertion of formulae containing free variables is at best a convenience; assertion of A(x) with free x can always be replaced by assertion of $\forall xFx$.

So, Kripke proposes the *generality interpretation* of free variables in the underlying logic for predicate calculus. This blocks applying rules such as necessitation to open sentences, and therefore blocks both the derivation of CBF, as well as problematic formula such as (3) ($\Box(\forall xFx \rightarrow Fy)$).

Kripke thus proposes replacing:

∀1 If α is any wff and x and y any variables and $\alpha[y/x]$ is α with free y replacing every free x, then $\forall x\alpha \rightarrow \alpha[y/x]$ is an axiom of LPC+S

with:

∀1K [original] If α is any wff and x and y any variables and $\alpha[y/x]$ is α with free y replacing every free x, then $\forall y(\forall x\alpha \rightarrow \alpha[y/x])$ is an axiom of LPC+S

Now, it turns out for various reasons explained in Kit Fine's paper, "The Permutation Principle in Quantificational Logic" (*Journal of Philosophical Logic* 1983), Kripke's **∀1K [original]** is not fully satisfactory, and Kripke's system with **∀1K** in place of **∀1** is not complete. For the following permutation principle is valid in the semantics:

$(\forall x\forall yRxy) \rightarrow (\forall y\forall xRxy)$

But as Kit Fine proved, this principle could be false in a model in which all of Kripke's axioms are valid and rules of inference validity-preserving. So we need to adjust **∀1K [original]** so that the permutation principle is derivable. The required adjustment is as follows (see Hughes and Cresswell, p. 305):

∀1K If α is any wff and x and y any variables and $\alpha[y/x]$ is α with free y replacing every free x, then $\forall y\forall z(\forall x\alpha \rightarrow \alpha[y/x])$ is an axiom of LPC+S

Kripke's system is **∀1K** together with

Truth-functional tautologies, T, Necessitation, modus ponens, and:

$A \rightarrow \forall xA$, where x is not free in A.

$\forall x(A\rightarrow B) \rightarrow (\forall xA\rightarrow\forall xB)$

IV. Free Logic

The following system we will call Lower Predicate Calculus plus identity, or LPC=:

PC Any LPC substitution instance of a valid wff of PC is an axiom of LPC.
∀1 If α is any wff and x and y any variables and $\alpha[y/x]$ is α with free y replacing every free x, then $\forall x\alpha \rightarrow \alpha[y/x]$ is an axiom of LPC
MP If α and $\alpha \rightarrow \beta$ are theorems of LPC + S, then so is β
∀2 If $\alpha \rightarrow \beta$ is a theorem of LPC + S and x is not free in α then $\alpha \rightarrow \forall x\beta$ is a theorem of LPC + S
= t=t

As we saw last time, one theorem of CQT= is $\exists x(t=x)$.

(1) $\forall x{\sim}(t=x) \rightarrow {\sim}(t=t)$ by **∀1**
(2) ${\sim\sim}(t=t) \rightarrow {\sim}\forall x{\sim}(t=x)$ by PC (propositional calculus, contraposition)
(3) $t=t \rightarrow {\sim}\forall x{\sim}(t=x)$ by PC (double negation elimination)
(4) ${\sim}\forall x{\sim}(t=x)$ (3), =, MP
(5) $\exists x(t=x)$ Def ∃

Relatedly, another theorem involves the derived rule of existential generalization. Consider, for example, the following proof of $\vdash Fy \rightarrow \exists x\, Fx$:

(1) $\forall x{\sim}Fx \rightarrow {\sim}Fy$ by **∀1**
(2) ${\sim\sim}Fy \rightarrow {\sim}\forall x{\sim}Fx$ by PC (propositional calculus, contraposition)
(3) $Fy \rightarrow \exists xFx$ Double negation, Def ∃

Some logicians have found these consequences problematic. First, some philosophers, including Quine, have argued that logic should allow for *the empty domain*. Since $\exists x(t=x)$ is a theorem of CQT=, and it is false in the empty domain, it should not be a theorem. The logics that were developed to deliver theorems that were valid in all domains, including the empty domain, are called *inclusive logics*. Secondly, some philosophers have argued that the language of the predicate calculus should allow for non-denoting singular terms. If t is such a term, then $\exists x(t=x)$ should not be a theorem of such a logic, and existential generalization should not be a valid rule. The logics that were developed to deliver allow for non-denoting terms are called *free logics*. As Bencivenga points out, the topics are related, because in the empty domain, all terms are non-denoting.

One way to obtain a free logic is to replace **∀1** by:

∀1L If α is any wff and x and y any variables and $\alpha[y/x]$ is α with free y replacing every free x, then $\forall y(\forall x\alpha \rightarrow \alpha[y/x])$ is an axiom.

This blocks the derivation of (5).

Note also that **∀1L** is just **∀1K**. If we replace **∀1** with **∀1L** in LPC=, then what we have is basically the free logical system of Karl Lambert, in his 1963 paper “Existential Import Revisted” (*Notre Dame Journal of Formal Logic*). Lambert’s system is complete, since apparently adding identity allows the relevant permutation principles. In short, Kripke and Lambert were both describing very similar systems, but for very different reasons.

This is not the free logical system discussed in Hughes and Cresswell. But it is obviously similar to Kripke’s system in “Semantical Considerations on Modal Logic”, though devised for a different purpose.

The free logical system in Hughes and Cresswell involves the addition of an existence predicate to the language of LPC=, which has the semantics:

[VE] <u,w> is in V(E) iff u is in D_w

They then replace **∀1** by:

∀1E If α is any wff and x and y any variables and $\alpha[y/x]$ is α with free y replacing every free x, then $(\forall x\alpha \,\&\, Ey) \rightarrow \alpha[y/x])$ is an axiom.

And add:

UE $\forall xEx$

They also modify several rules (see pp. 293ff.). But the point of all this seems to be to derive their **∀1’**, which is just **∀1L** (see their page 294).

Notice that replacing **∀1** with either **∀1L** or **∀1E** preserves necessitation. For recall the worry was that “$\forall xFx \rightarrow Fy$” is a theorem, and an application of necessitation yields “$\Box(\forall xFx \rightarrow Fy)$”. But now “$(\forall xFx \rightarrow Fy)$” is no longer a theorem. What is a theorem is “$\forall y(\forall xFx \rightarrow Fy)$”. But “$\Box\forall y(\forall xFx \rightarrow Fy)$” is perfectly valid, so this raises no worries.