Making counterfactual assumptions (2): Veltman 2005

1. FORMAL SYSTEM

Propositional logic

D1.0 (*worlds*, *situations*, and *propositions*). Let A be a finite set of atomic sentences.

- A (A-)*world* is a function w from A into $\{1, 0\}$, i.e. Dom $w = A$ & Ran $w \subseteq \{1, 0\}$
- A (A-)*situation* is a partial function s from A into $\{1, 0\}$, i.e. Dom $s \subseteq A$ & Ran $w \subseteq \{1, 0\}$
- A (A-)*proposition* is a set P of (A-)worlds, i.e. $P \subseteq A^{\{1, 0\}} =: W$

D1.1 (*propositional logic*). The *proposition expressed* by ϕ, $[\![\phi]\!]$, is defined as follows:

- $[\![\phi]\!] = \{w \in W | \langle\phi, 1\rangle \in w\}$ if $\phi \in A$
  $[\![\neg\phi]\!] = W \sim [\![\phi]\!]$
- $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$
  $[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$
  $[\![\phi \rightarrow \psi]\!] = (W \sim [\![\phi]\!]) \cup [\![\psi]\!]$

Basic update semantics

D2.0 (*cognitive states*). Let $W := A^{\{1, 0\}}$ be the set of A-worlds.

- A (*cognitive*) (A-)*state* is a pair $S = \langle U_S, F_S\rangle$ (of *S-universe* U_S and *S-fact base* F_S), where either (a) $\varnothing \neq F_S \subseteq U_S \subseteq W$ (*possible state*), or (b) $\varnothing = F_S = U_S$ (*absurd state*)
- $\mathbf{0} := \langle\varnothing, \varnothing\rangle$ (*absurd state*)
  $\mathbf{1} := \langle W, W\rangle$ (*minimal state*)

D2.1 (*update semantics*). The *update* of S with ϕ, $S[\phi]$ is defined as follows:

- $S[\phi] = \langle U_S, F_S \cap [\![\phi]\!]\rangle$ if $F_S \cap [\![\phi]\!] \neq \varnothing$
  $= \mathbf{0}$ otherwise
- $S[\Box\phi] = \langle U_S \cap [\![\phi]\!], F_S \cap [\![\phi]\!]\rangle$ if $F_S \cap [\![\phi]\!] \neq \varnothing$
  $= \mathbf{0}$ otherwise
- $S \models \phi$ iff $S[\phi] = S$
  $\phi_1, \ldots\phi_n \models \psi$ iff $S[\phi_1]\ldots[\phi_n] \models \psi$

Counterfactual updates

D3 (*basis*). Let $S = \langle U_S, F_S\rangle$ be a state.

- *s forces P* (within U_S), iff $\{w| w \in U_S \;\&\; s \subseteq w\} \subseteq P$
- *s determines w* (within U_S), iff s forces $\{w\}$ within U_S
- s is a *basis* for w (within U_S), iff $s \in \mathbf{min}_{\subseteq}\{s' | s'$ determines w within $U_S\}$

D4 (*retraction*). For any world $w \in W$, proposition $P \subseteq W$, and state $S = \langle U, F\rangle$:

- $w{\downarrow}P := \{s \subseteq w| \exists s'$: s' is a basis for w & $s \in \mathbf{max}_{\subseteq}\{s'' \subseteq s' | s''$ does not force $P\}$
  $S{\downarrow}P := \langle U, F'\rangle$, where $F' = \{w \in U| \exists w' \in F \exists s \subseteq w: s \in w'{\downarrow}P\}$

D2.2 (*counterfactual updates*).

- $S[$***if.hb*** $\phi] = (S{\downarrow}[\![\neg\phi]\!])[\phi]$ (***if.hb*** := *if it had been the case that*)
- $S \models$ ***if.hb*** ϕ, ***w.hb*** ψ iff $S[$***if.hb*** $\phi] \models \psi$ (***w.hb*** := *it would've been the case that*)

2. ADAMS EXAMPLES REVISITED

(1) Adams 1975: *real* modality
i. Kennedy was assassinated.
ii. So if Oswald didn't do it, someone else *did*.

Analysis a la Veltman 2005
Let $A = \{p, q, r\}$, where
p: Oswald assassinates Kennedy
q: someone other than Oswald assassinates Kennedy

(1′) English to propositional logic:

0.	$\Box\neg(p \wedge q)$	*world knowledge*: nobody dies twice
i.	$p \vee q$	
ii.	$\neg p \rightarrow q$	*so* presupposes: the output state of (1′i) supports (1′ii)

Cognitive A-states:

	1				$S_0 := \mathbf{1}[\Box\neg(p \wedge q)]$				$S_1 := S_0[p \vee q]$				$S_2 := S_1[\neg p \rightarrow q]$		
	p	q	r		p	q	r		p	q	r		p	q	r
$\boldsymbol{w_0}$	**0**	**0**	**0**	$\boldsymbol{w_0}$	**0**	**0**	**0**	w_0	0	0	0	w_0	0	0	0
$\boldsymbol{w_1}$	**0**	**0**	**1**	$\boldsymbol{w_1}$	**0**	**0**	**1**	w_1	0	0	1	w_1	0	0	1
$\boldsymbol{w_2}$	**0**	**1**	**0**	$\boldsymbol{w_2}$	**0**	**1**	**0**	$\boldsymbol{w_2}$	**0**	**1**	**0**	$\boldsymbol{w_2}$	**0**	**1**	**0**
$\boldsymbol{w_3}$	**0**	**1**	**1**	$\boldsymbol{w_3}$	**0**	**1**	**1**	$\boldsymbol{w_3}$	**0**	**1**	**1**	$\boldsymbol{w_3}$	**0**	**1**	**1**
$\boldsymbol{w_4}$	**1**	**0**	**0**	$\boldsymbol{w_4}$	**1**	**0**	**0**	$\boldsymbol{w_4}$	**1**	**0**	**0**	$\boldsymbol{w_4}$	**1**	**0**	**0**
$\boldsymbol{w_5}$	**1**	**0**	**1**	$\boldsymbol{w_5}$	**1**	**0**	**1**	$\boldsymbol{w_5}$	**1**	**0**	**1**	$\boldsymbol{w_5}$	**1**	**0**	**1**
$\boldsymbol{w_6}$	**1**	**1**	**0**	~~w_6~~	~~1~~	~~1~~	~~0~~	~~w_6~~	~~1~~	~~1~~	~~0~~	~~w_6~~	~~1~~	~~1~~	~~0~~
$\boldsymbol{w_7}$	**1**	**1**	**1**	~~w_7~~	~~1~~	~~1~~	~~1~~	~~w_7~~	~~1~~	~~1~~	~~1~~	~~w_7~~	~~1~~	~~1~~	~~1~~

✓*so*: $S_1 \models (1'\text{ii})$

Detailed computations:

$S_0 := \mathbf{1}[\Box\neg(p \wedge q)]$
$= \langle W \cap [\![\neg(p \wedge q)]\!], W \cap [\![\neg(p \wedge q)]\!]\rangle$ D2.1.$\Box$
$= \langle W \cap (W \sim [\![p \wedge q]\!]), W \cap (W \sim [\![p \wedge q]\!])\rangle$ D1.1.$\neg$
$= \langle W \sim [\![p \wedge q]\!], W \sim [\![p \wedge q]\!]\rangle$ df. $\sim, \cap$
$= \langle W \sim ([\![p]\!] \cap [\![q]\!]), W \sim ([\![p]\!] \cap [\![q]\!])\rangle$ D1.1.$\wedge$
$= \langle W \sim (\{w \in W | \langle p, 1\rangle \in w\} \cap \{w \in W | \langle q, 1\rangle \in w\}),$
$\langle W \sim (\{w \in W | \langle p, 1\rangle \in w\} \cap \{w \in W | \langle q, 1\rangle \in w\})\rangle$ D1.1.A
$= \langle W \sim (\{w \in W | \langle p, 1\rangle \in w \;\&\; \langle q, 1\rangle \in w\}),$
$\langle W \sim (\{w \in W | \langle p, 1\rangle \in w \;\&\; \langle q, 1\rangle \in w\})\rangle$ df. $\cap$
$= \langle \{w_0, w_1, w_2, w_3, w_4, w_5\}, \{w_0, w_1, w_2, w_3, w_4, w_5\}\rangle$ df. $w_0, \ldots w_7$ above
$=: \langle U_0, F_0\rangle$

$S_1 := S_0[p \vee q]$
$= \langle U_0, F_0 \cap [\![p \vee q]\!]\rangle$ D2.1.ϕ
$= \langle U_0, F_0 \cap ([\![p]\!] \cup [\![q]\!])\rangle$ D1.1.$\vee$
$= \langle U_0, \{w \in F_0 | \langle p, 1\rangle \in w \text{ or } \langle q, 1\rangle \in w\}\rangle$ D1.1.A, df. $\cap, \cup$
$= \langle U_0, \{w_2, w_3, w_4, w_5\}\rangle$ df. F_0, $w_0, \ldots w_7$ above
$=: \langle U_0, F_1\rangle$

S_2	$:= S_1[\neg p \to q]$	
	$= \langle U_0, F_1 \cap [\![\neg p \to q]\!]\rangle$	D2.1.ϕ
	$= \langle U_0, F_1 \cap ((W \sim [\![\mid \neg p]\!]) \cup [\![q]\!])\rangle$	D1.1.$\to$
	$= \langle U_0, F_1 \cap ((W \sim (W \sim [\![p]\!])) \cup [\![q]\!])\rangle$	D1.1.$\neg$
	$= \langle U_0, F_1 \cap ([\![p]\!] \cup [\![q]\!])\rangle$	df. ~
	$= \langle U_0, (F_0 \cap ([\![p]\!] \cup [\![q]\!])) \cap ([\![p]\!] \cup [\![q]\!])\rangle$	df. F_1 above
	$= \langle U_0, F_0 \cap (([\![p]\!] \cup [\![q]\!]) \cap ([\![p]\!] \cup [\![q]\!]))\rangle$	df. $\cap$ (associativity)
	$= \langle U_0, (F_0 \cap ([\![p]\!] \cup [\![q]\!]))\rangle$	df. $\cap$ (idempotence)
	$= \langle U_0, F_1\rangle$	df. F_1 above
	$= S_1$	df. S_1 above
Hence:	$S_1 \models \neg p \to q$	D2.1. $\models$

(2) <u>Adams 1975: *remote* modality (wrt lone gunman)</u>

i. Kennedy was assassinated.
ii. So if Oswald didn't do it, someone else *must*'ve.
[after reading some history books]
iii. Oswald did assassinate Kennedy.
iv. It was a one-man plot,
v. so, if he hadn't done it, then Kennedy *would*n't have been assassinated

Analysis a la Veltman 2005
Let $A = \{p, q, r\}$, where
p: Oswald assassinates Kennedy
q: someone other than Oswald assassinates Kennedy
r: Oswald plots alone.

(2′) English to propositional logic:

0.	$\Box\neg(p \wedge q)$	*world knowledge*: nobody dies twice
i.	$p \vee q$	
ii.	$\neg p \to q$	in (2ii) *so* ϕ presupposes: $S_1 \models \phi$
iii.	p	
iv.	r	
iv$^+$.	$\Box(r \wedge \neg p \to \neg p \wedge \neg q)$	in (2v) *so* ψ presupposes: $S_4[\Box(r \wedge \neg p \to \neg p \wedge \neg q)] \models \psi$
v.	***if.hd*** $\neg p$, ***w.hb*** $(\neg p \wedge \neg q)$	

Cognitive A-states: **1**, S_0, S_1, S_2 as above

$S_2 := S_1$[ii]

	p	q	r
w_0	0	0	0
w_1	0	0	1
w_2	**0**	**1**	**0**
w_3	**0**	**1**	**1**
w_4	**1**	**0**	**0**
w_5	**1**	**0**	**1**
~~w_6~~	~~1~~	~~1~~	~~0~~
~~w_7~~	~~1~~	~~1~~	~~1~~

$S_3 := S_2[p]$

	p	q	r
w_0	0	0	0
w_1	0	0	1
w_2	0	1	0
w_3	0	1	1
w_4	**1**	**0**	**0**
w_5	**1**	**0**	**1**
~~w_6~~	~~1~~	~~1~~	~~0~~
~~w_7~~	~~1~~	~~1~~	~~1~~

$S_4 := S_3[r]$

	p	q	r
w_0	0	0	0
w_1	0	0	1
w_2	0	1	0
w_3	0	1	1
w_4	1	0	0
w_5	**1**	**0**	**1**
~~w_6~~	~~1~~	~~1~~	~~0~~
~~w_7~~	~~1~~	~~1~~	~~1~~

$S_{4+} := S_4[\text{iv}^+]$

	p	q	r
w_0	0	0	0
w_1	0	0	1
w_2	0	1	0
~~w_3~~	~~0~~	~~1~~	~~1~~
w_4	1	0	0
w_5	**1**	**0**	**1**
~~w_6~~	~~1~~	~~1~~	~~0~~
~~w_7~~	~~1~~	~~1~~	~~1~~

$S_5 := S_{4+}$[***if.hb*** $\neg p$]

	p	q	r
w_0	0	0	0
w_1	**0**	**0**	**1**
w_2	0	1	0
~~w_3~~	~~0~~	~~1~~	~~1~~
w_4	1	0	0
w_5	1	0	1
~~w_6~~	~~1~~	~~1~~	~~0~~
~~w_7~~	~~1~~	~~1~~	~~1~~

✓*so*: $S_{4+} \models$ (2′v)

EXERCISE for (2):
- Compute S_3 through S_5 using the formal system of Veltman 2005.
- Show that $S_{4+} \models (2'\text{v})$.

(3) <u>Adams 1975: *remote* modality (wrt conspiracy)</u>
(i) Kennedy was assassinated.
(ii) So if Oswald didn't do it, someone else *did*.
[after reading some history books]
(iii) Oswald did assassinate Kennedy.
(iv) He was just the first of five gunmen lined up for the job
(v) so, even if he had failed, Kennedy *would* still have been assassinated.

Analysis a la Veltman 2005
Let $A = \{p, q, r\}$, where
p: Oswald assassinates Kennedy
q: someone other than Oswald assassinates Kennedy
r': Oswald is the first of five gunmen lined up to assassinate Kennedy.

(3′) English to propositional logic:

(0)	$\Box\neg(p \wedge q)$	*world knowledge*: nobody dies twice
(i)	$p \vee q$	
(ii)	$\neg p \rightarrow q$	in (3ii) *so* ϕ presupposes: $S_1 \models \phi$
(iii)	p	
(iv)	r	
(iv$^+$)	$\Box(r' \wedge \neg p \rightarrow q)$	in (3v) *so* ψ presupposes: $S_4[\Box(r \wedge \neg p \rightarrow q)] \models \psi$
(v)	***if.hd*** $\neg p$, ***w.hb*** q	

Cognitive A-states: **1**, S_0, S_1, S_2 as above

$S_2 := S_1[\text{ii}]$

	p	q	r
w_0	0	0	0
w_1	0	0	1
$\boldsymbol{w_2}$	**0**	**1**	**0**
$\boldsymbol{w_3}$	**0**	**1**	**1**
$\boldsymbol{w_4}$	**1**	**0**	**0**
$\boldsymbol{w_5}$	**1**	**0**	**1**
~~w_6~~	~~1~~	~~1~~	~~0~~
~~w_7~~	~~1~~	~~1~~	~~1~~

$S_3 := S_2[p]$

	p	q	r
w_0	0	0	0
w_1	0	0	1
w_2	0	1	0
w_3	0	1	1
$\boldsymbol{w_4}$	**1**	**0**	**0**
$\boldsymbol{w_5}$	**1**	**0**	**1**
~~w_6~~	~~1~~	~~1~~	~~0~~
~~w_7~~	~~1~~	~~1~~	~~1~~

$S_4 := S_3[r]$

	p	q	r
w_0	0	0	0
w_1	0	0	1
w_2	0	1	0
w_3	0	1	1
w_4	1	0	0
$\boldsymbol{w_5}$	**1**	**0**	**1**
~~w_6~~	~~1~~	~~1~~	~~0~~
~~w_7~~	~~1~~	~~1~~	~~1~~

$S_{4+} := S_4[\text{iv}^+]$

	p	q	r
w_0	0	0	0
~~w_1~~	~~0~~	~~0~~	~~1~~
w_2	0	1	0
w_3	0	1	1
w_4	1	0	0
$\boldsymbol{w_5}$	**1**	**0**	**1**
~~w_6~~	~~1~~	~~1~~	~~0~~
~~w_7~~	~~1~~	~~1~~	~~1~~

$S_5 := S_{4+}[$***if.hb*** $\neg p]$

	p	q	r
w_0	0	0	0
~~w_1~~	~~0~~	~~0~~	~~1~~
w_2	0	1	0
$\boldsymbol{w_3}$	**0**	**1**	**1**
w_4	1	0	0
w_5	1	0	1
~~w_6~~	~~1~~	~~1~~	~~0~~
~~w_7~~	~~1~~	~~1~~	~~1~~

✓*so*: $S_{4+} \models (3'\text{v})$

EXERCISE for (3):
- Compute S_{4+} and S_5 using the formal system of Veltman 2005.
- Show that $S_{4+} \models (3'\text{v})$.

3. ANTECEDENT STRENGTHENING REVISITED

(4) <u>*Vivid* vs. *remote* modality</u>

i. This match is dry.
ii. So if I strike it, it *will* light.
iii. But if it were wet and I struck it, then it *would*n't light.

Analysis a la Veltman 2005
Let $A = \{p, q, r\}$, where
p: match m is dry
q: match m is struck
r: match m lights

(4′) English to propositional logic:

0.	$\Box(p \wedge q \leftrightarrow r)$	*world knowledge*: a match lights iff it is dry and is struck
i.	p	
ii.	$q \rightarrow r$	in (4ii), *so* ϕ presupposes: $S_1 \models \phi$
iii.	***if.hb*** $\neg p \wedge q$, ***w.hb*** $\neg r$	

Cognitive A-states:

1

	p	q	r
$\boldsymbol{w_0}$	**0**	**0**	**0**
$\boldsymbol{w_1}$	**0**	**0**	**1**
$\boldsymbol{w_2}$	**0**	**1**	**0**
$\boldsymbol{w_3}$	**0**	**1**	**1**
$\boldsymbol{w_4}$	**1**	**0**	**0**
$\boldsymbol{w_5}$	**1**	**0**	**1**
$\boldsymbol{w_6}$	**1**	**1**	**0**
$\boldsymbol{w_7}$	**1**	**1**	**1**

$S_0 := \mathbf{1}[\Box\ldots]$

	p	q	r
$\boldsymbol{w_0}$	**0**	**0**	**0**
~~w_1~~	~~0~~	~~0~~	~~1~~
$\boldsymbol{w_2}$	**0**	**1**	**0**
~~w_3~~	~~0~~	~~1~~	~~1~~
$\boldsymbol{w_4}$	**1**	**0**	**0**
~~w_5~~	~~1~~	~~0~~	~~1~~
~~w_6~~	~~1~~	~~1~~	~~0~~
$\boldsymbol{w_7}$	**1**	**1**	**1**

$S_1 := S_0[p]$

	p	q	r
w_0	0	0	0
~~w_1~~	~~0~~	~~0~~	~~1~~
w_2	0	1	0
~~w_3~~	~~0~~	~~1~~	~~1~~
$\boldsymbol{w_4}$	**1**	**0**	**0**
~~$\boldsymbol{w_5}$~~	~~**1**~~	~~**0**~~	~~**1**~~
~~w_6~~	~~1~~	~~1~~	~~0~~
$\boldsymbol{w_7}$	**1**	**1**	**1**

$S_2 := S_1[q \rightarrow r]$

	p	q	r
w_0	0	0	0
~~w_1~~	~~0~~	~~0~~	~~1~~
w_2	0	1	0
~~w_3~~	~~0~~	~~1~~	~~1~~
$\boldsymbol{w_4}$	**1**	**0**	**0**
~~w_5~~	~~1~~	~~0~~	~~1~~
~~w_6~~	~~1~~	~~1~~	~~0~~
$\boldsymbol{w_7}$	**1**	**1**	**1**

✓*so*: $S_1 \models$ (4′ii)

$S_3 := S_2[$***if.hb*** $\neg p \wedge q]$

	p	q	r
w_0	0	0	0
w_1	0	0	1
w_2	0	1	0
w_3	0	1	1
w_4	1	0	0
w_5	1	0	1
w_6	1	1	0
w_7	1	1	1

EXERCISE for (4):

- Compute S_3 using the formal system of Veltman 2005, and complete the rightmost diagram accordingly
- Use the formal system of Veltman 2005 to determine whether $S_3 \models$ ***w.hb*** $\neg r$

4. TICHY EXAMPLE

(5) Tichy 1976, version 1
i. If it is raining, Jones wears his hat.
(If it is not raining, he wears his hat, or not, at random.)
ii. Today it's raining and so Jones is wearing his hat.
iii.# But, even if it had not been raining, he would've been wearing his hat.

Veltman 2005 *analysis* (with simpler *A*)
Let $A = \{p, q\}$, where
p: it's raining
q: Jones is wearing his hat

(5′) English to propositional logic:
i. $\Box(p \to q)$
ii. $p \wedge q$
iii.# ***if.hd*** $\neg p$, ***w.hb*** q Veltman's account: $S_2 \not\models$ (5′iii)

Cognitive *A*-states:

1			$S_1 := \mathbf{1}$[i]			$S_2 := S_1$[ii]			$S_{3.1} := S_2 \downarrow [\![\neg\neg p]\!]$			$S_{3.2} := S_2$[***if.hb*** $\neg p$]		
	p	*q*		*p*	*q*		*p*	*q*		*p*	*q*		*p*	*q*
$\boldsymbol{w_0}$	**0**	**0**	$\boldsymbol{w_0}$	**0**	**0**	w_0	0	0	$\boldsymbol{w_0}$	**0**	**0**	$\boldsymbol{w_0}$	**0**	**0**
$\boldsymbol{w_1}$	**0**	**1**	$\boldsymbol{w_1}$	**0**	**1**	w_1	0	1	$\boldsymbol{w_1}$	**0**	**1**	$\boldsymbol{w_1}$	**0**	**1**
$\boldsymbol{w_2}$	**1**	**0**	~~w_2~~	~~1~~	~~0~~	~~w_2~~	~~1~~	~~0~~	~~w_2~~	~~1~~	~~1~~	~~w_2~~	~~1~~	~~1~~
$\boldsymbol{w_3}$	**1**	**1**	$\boldsymbol{w_3}$	**1**	**1**	$\boldsymbol{w_3}$	**1**	**1**	$\boldsymbol{w_3}$	**1**	**1**	w_3	1	1

Detailed computation of S_3
FACT 1:
$\{\langle p, 1\rangle\}$ is the (unique) basis for w_3 (within U_{S2})

Proof:
Let *s* be any *A*-situation. Then (1) iff (6)

1.	s is a basis for w_3 within U_{S2}	
2.	$s \in \mathbf{min}_{\subseteq}\{s' \mid s'$ determines w_3 within $U_{S2}\}$	D3.*basis*
3.	$s \in \mathbf{min}_{\subseteq}\{s' \mid s'$ forces $\{w_3\}$ within $U_{S2}\}$	D3.*determine*
4.	$s \in \mathbf{min}_{\subseteq}\{s' \mid \{w \in U_{S2} \mid s' \subseteq w\} \subseteq \{w_3\}\}$	D3.*force*
5.	$s \in \mathbf{min}_{\subseteq}\{\{\langle p, 1\rangle\}, \{\langle p, 1\rangle, \langle q, 1\rangle\}\}$	df. w_3 & $S_2 = \langle U_{S2}, F_{S2}\rangle$ above
6.	$s \in \{\{\langle p, 1\rangle\}\}$	df. $\mathbf{min}_{\subseteq} X$

□

FACT 2:
$\varnothing$ does not force $[\![p]\!]$ within U_{S2}

Proof: (1) iff (5)

1.	$\varnothing$ forces $[\![p]\!]$ within U_{S2}	
2.	$\{w \mid w \in U_{S2} \,\&\, \varnothing \subseteq w\} \subseteq [\![p]\!]$	D3.*force*
3.	$U_{S2} \subseteq [\![p]\!]$	df. $\{-:-\}, \subseteq$
4.	$U_{S2} \subseteq \{w \in W \mid \langle p, 1\rangle \in w\}$	D1.1.$\phi \in A$
5.	$\{w_0, w_1, w_3\} \subseteq \{w_2, w_3\}$	df. U_{S2}, W above

□

FACT 3:
$\{\langle p, 1\rangle\}$ forces $[\![p]\!]$ within U_{S2}

Proof: (1) iff (5)

1.	$\{\langle p, 1\rangle\}$ forces $[\![p]\!]$ within U_{S2}	
2.	$\{w \mid w \in U_{S2} \,\&\, \{\langle p, 1\rangle\} \subseteq w\} \subseteq [\![p]\!]$	D3.*force*
3.	$\{w_2, w_3\} \subseteq [\![p]\!]$	df. U_{S2}, {–:–}, $\subseteq$
4.	$\{w_2, w_3\} \subseteq \{w \in W \mid \langle p, 1\rangle \in w\}$	D1.1.$\phi \in A$
5.	$\{w_2, w_3\} \subseteq \{w_2, w_3\}$	df. W above

□

FACT 4:
For S_2 defined as above,
$w_3{\downarrow}[\![\neg\neg p]\!] = \{\varnothing\}$

Proof:
For S_2 defined as above,
$w_3{\downarrow}[\![\neg\neg p]\!]$

=	$\{s \subseteq w_3 \mid \exists s'\!: s'$ is a basis for w_3 $\;\&\; s \in \mathbf{max}_{\subseteq}\{s'' \subseteq s \mid s''$ does not force $[\![\neg\neg p]\!]$ within $U_{S2}\}\}$	D4.$w{\downarrow}P$
=	$\{s \subseteq w_3 \mid \exists s'\!: s' = \{\langle p, 1\rangle\}$ $\;\&\; s \in \mathbf{max}_{\subseteq}\{s'' \subseteq s \mid s''$ does not force $[\![\neg\neg p]\!]$ within $U_{S2}\}\}$	F1
=	$\{s \subseteq w_3 \mid s \in \mathbf{max}_{\subseteq}\{s'' \subseteq \{\langle p, 1\rangle\} \mid s''$ does not force $[\![\neg\neg p]\!]$ within $U_{S2}\}\}$	elim. $\exists s'$
=	$\{s \subseteq w_3 \mid s \in \mathbf{max}_{\subseteq}\{s'' \subseteq \{\langle p, 1\rangle\} \mid s''$ does not force $(W \sim (W \sim [\![p]\!]))$ within $U_{S2}\}\}$	D1.1.$\neg$
=	$\{s \subseteq w_3 \mid s \in \mathbf{max}_{\subseteq}\{s' \subseteq \{\langle p, 1\rangle\} \mid s'$ does not force $[\![p]\!]$ within $U_{S2}\}\}$	df. $\sim$
=	$\{s \subseteq w_3 \mid s \in \mathbf{max}_{\subseteq}\{\varnothing\}\}$	F2, F3
=	$\{s \subseteq w_3 \mid s \in \{\varnothing\}\}$	df. $\mathbf{max}_{\subseteq}$
=	$\{\varnothing\}$	df. $\subseteq$

□

FACT 5:
$S_2{\downarrow}[\![\neg\neg p]\!] = \langle U_{S2}, U_{S2}\rangle$

Proof:
For S_2 defined as above,
$S_2{\downarrow}[\![\neg\neg p]\!]$

=	$\langle U_{S2}, \{w \in U_{S2} \mid \exists w' \in F_{S2} \exists s \subseteq w\!: s \in w'{\downarrow}[\![\neg\neg p]\!]\}\rangle$	
=	$\langle U_{S2}, \{w \in U_{S2} \mid \exists w' \in \{w_3\} \exists s \subseteq w\!: s \in w'{\downarrow}[\![\neg\neg p]\!]\}\rangle$	df. S_2 above
=	$\langle U_{S2}, \{w \in U_{S2} \mid \exists s \subseteq w\!: s \in w_3{\downarrow}[\![\neg\neg p]\!]\}\rangle$	eliminate $\exists w'$
=	$\langle U_{S2}, \{w \in U_{S2} \mid \exists s \subseteq w\!: s \in \{\varnothing\}\}\rangle$	F4
=	$\langle U_{S2}, \{w \in U_{S2} \mid \varnothing \subseteq w\}\rangle$	eliminate $\exists s$
=	$\langle U_{S2}, U_{S2}\rangle$	df. {–:–}, $\subseteq$

□

FACT 6:
$S_2[\textbf{\textit{if.hb}}\ \neg p] = \langle U_{S2}, \{w_0, w_1\}\rangle$

Proof:
$S_2[\textbf{\textit{if.hb}}\ \neg p]$

=	$(S_2 \downarrow [\![\neg\neg p]\!])[\neg p]$	D2.2.S[***if.hb*** ϕ]
=	$\langle U_{S2}, U_{S2}\rangle[\neg p]$	F5
=	$\langle U_{S2}, U_{S2} \cap [\![\neg p]\!]\rangle$	D2.1.$S[\phi]$
=	$\langle U_{S2}, U_{S2} \cap (W \sim [\![p]\!])\rangle$	D1.1.$\neg$
=	$\langle U_{S2}, U_{S2} \cap \{w \in W \mid w \notin [\![p]\!]\}\rangle$	df. ~, {–|–}
=	$\langle U_{S2}, U_{S2} \cap \{w \in W \mid \langle p, 1\rangle \notin w\}\rangle$	D1.1.$\phi \in A$
=	$\langle U_{S2}, \{w_0, w_1\}\rangle$	set theory, df. U_{S2}, W

□

FACT 7:
$S_2 \not\models$ ***if.hb*** $\neg p$, ***w.hb*** q

Proof: (1) iff (9)

1.	$S_2 \models$ ***if.hb*** $\neg p$, ***w.hb*** q	
2.	$S_2[$ ***if.hb*** $\neg p] \models q$	D2.2
3.	$\langle U_{S2}, \{w_0, w_1\}\rangle \models q$	F6
4.	$\langle U_{S2}, \{w_0, w_1\}\rangle[q] = \langle U_{S2}, \{w_0, w_1\}\rangle$	D2.1.$\models$
5.	$\langle U_{S2}, \{w_0, w_1\} \cap [\![q]\!]\rangle = \langle U_{S2}, \{w_0, w_1\}\rangle$	D2.1.$S[\phi]$
6.	$\{w_0, w_1\} \cap [\![q]\!] = \{w_0, w_1\}$	df. $\langle a, b\rangle = \langle a', b'\rangle$
7.	$\{w_0, w_1\} \cap \{w \in W \mid \langle q, 1\rangle \in w\} = \{w_0, w_1\}$	D1.1.$\phi \in A$
8.	$\{w_0, w_1\} \cap \{w_1\} = \{w_0, w_1\}$	df. W above
9.	$\{w_1\} = \{w_0, w_1\}$	df. $\cap$

□