Lidl, R & G. Pilz "Applied Abstract Algebra" pp. 2 - 5

Let A and B be nonempty sets. A *relation* R from A to B is a subset of $A \times B$. Relations from A to A are called relations on A, for short. If $(a, b) \in R$ then we write $a\,R\,b$ and say that "a is in relation R to b". Also, if a is not in relation R to b, we write $a \not R\, b$.

A relation R on a nonempty set A may have some of the following properties:

R is *reflexive* if for all a in A we have $a\,R\,a$.

R is *symmetric* if for all a and b in A: $a\,R\,b$ implies $b\,R\,a$.

R is *antisymmetric* if for all a and b in A: $a\,R\,b$ and $b\,R\,a$ imply $a = b$.

R is *transitive* if for all a, b, c in A: $a\,R\,b$ and $b\,R\,c$ imply $a\,R\,c$.

1.1 Definition. A relation R on a set A is called a *partial order* (*relation*) if R is reflexive, antisymmetric and transitive.

In this case (A, R) is called a *partially ordered set* or *poset*.

1.2 Definition. A partial order relation $\leq$ on A is called a *total order* (or *linear order*) if for each $a, b \in A$ either $a \leq b$ or $b \leq a$. $(A, \leq)$ is then called a *chain*, or *totally ordered set*.

For example, $(\{1, 2, 3, 4, 5\}, \leq)$ is a total order, $(\mathcal{P}(\{1, 2, 3\}), \subseteq)$ is not a total order.

Let $(A, \leq)$ be a poset. We say, "a is a greatest element" if "all other elements are smaller". More precisely, $a \in A$ is called a *greatest element* of A if for all $x \in A$ we have $x \leq a$. The element b in A is called a *smallest element* of A if $b \leq x$ for all $x \in A$. The element $c \in A$ is called a *maximal element* of A if $c \leq x$ implies $c = x$ for all $x \in A$; similarly, $d \in A$ is called a *minimal element* of A if $x \leq d$ implies $x = d$ for all $x \in A$. It can be shown that $(A, \leq)$ has at most one greatest and one smallest element. However, there may be none, one, or several maximal or minimal elements. Every greatest element is maximal and every smallest element is minimal. For instance, in the poset of Figure 1.3

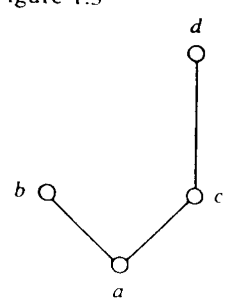


Figure 1.3

a is a minimal element and is a smallest element. b and d are maximal, but there is no greatest element.

1.3 Definition. Let $(A, \leq)$ be a poset and $B \subseteq A$.

(i) $a \in A$ is called an *upper bound* of $B :\Leftrightarrow \forall\, b \in B$: $b \leq a$.
(ii) $a \in A$ is called a *lower bound* of $B :\Leftrightarrow \forall\, b \in B$: $a \leq b$.
(iii) The greatest amongst the lower bounds, whenever it exists, is called the *infimum* of B, and is denoted by $\inf B$.
(iv) The least upper bound of B, whenever it exists, is called the *supremum* of B, and is denoted by $\sup B$.

For instance, let $(A, \leq) = (\mathbb{R}, \leq)$ and $B = [0, 3)$ then $\inf B = 0$ and $\sup B = 3$. Thus the infimum (supremum) of B may be an element of B, but does not have to be. If $B' = \mathbb{N}$ then in $(\mathbb{R}, \leq)$ we have $\inf B' = 1$, but $\sup B'$ does not exist.

1.5 Definition. A poset $(L, \leq)$ is called *lattice ordered* if for every pair x, y of elements of L the $\sup(x, y)$ and $\inf(x, y)$ exist.

1.6 Remark. (i) Every ordered set is lattice ordered.
(ii) In a lattice ordered set $(L, \leq)$ the following statements are equivalent for all x and y in L:
(a) $x \leq y$;
(b) $\sup(x, y) = y$;
(c) $\inf(x, y) = x$.

There is another (yet equivalent) approach, that does not use order relations, but algebraic operations.

1.7 Definition. An (algebraic) *lattice* $(L, \sqcap, \sqcup)$ is a nonempty set L with two binary operations $\sqcap$ (*meet*) and $\sqcup$ (*join*) (also called intersection or product and union or sum, respectively), which satisfy the following conditions for all $x, y, z \in L$:

(L1)	$x \sqcap y = y \sqcap x,$	$x \sqcup y = y \sqcup x;$
(L2)	$x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z,$	$x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z;$
(L3)	$x \sqcap (x \sqcup y) = x,$	$x \sqcup (x \sqcap y) = x.$

Two applications of (L3), namely $x \sqcap x = x \sqcap (x \sqcup (x \sqcap x)) = x$, lead to the additional condition

(L4) $\quad x \sqcap x = x, \quad x \sqcup x = x.$

(L1) is the *commutative law*, (L2) is the *associative law*, (L3) is the *absorption law*, and (L4) is the *idempotent law*.

1.8 Theorem. (i) *Let $(L, \leq)$ be a lattice ordered set. If we define*

$$x \sqcap y := \inf(x, y), \qquad x \sqcup y := \sup(x, y),$$

then $(L, \sqcap, \sqcup)$ is an algebraic lattice.
(ii) *Let $(L, \sqcap, \sqcup)$ be an algebraic lattice. If we define*

$$x \leq y :\Leftrightarrow x \sqcap y = x \quad (\text{or } x \leq y :\Leftrightarrow x \sqcup y = y),$$

then $(L, \leq)$ is a lattice ordered set.

Kamp, H. & U. Reyle 1993, ch. 4 "The plural"
pp. 401-3

Definition 4.3.1

An *upper semilattice* is a structure $\mathcal{A} = \langle A, \subset \rangle$, where A is a set of entities and '$\subset$' is a partial order of A – i.e. a binary relation such that for all a, b, c $\in$ A:

(A1) $a \subset a$ (REFlexivity)

(A2) $a \subset b \;\&\; b \subset c \rightarrow a \subset c$ (TRANSitivity)

(A3) $a \subset b \;\&\; b \subset a \rightarrow a = b$ (ANTISymmetry),

with the following additional property that for all a, b $\in$ A

(A4) $(\exists c \in A)\,(a \subset c \;\&\; b \subset c \;\&\; (\forall d \in A)\,((a \subset d \;\&\; b \subset d) \rightarrow c \subset d))$ (Least Upper Bound)

In lattice theory the object c postulated by (A4) is called *the supremum*. In the application we intend here, it corresponds to what we have already been calling 'the sum of a and b' and have denoted as '$a \oplus b$'. The use of the definite description (c.q. *the* supremum) is justified, because (A3) guarantees that the c is unique: For suppose c and c′ both satisfy the condition of (A4). Then, since $a \subset c$ and $b \subset c$ and c′ has the property $(\forall d)\,(a \subset d \;\&\; b \subset d) \rightarrow c' \subset d)$, it follows that $c' \subset c$. Similarly, $c \subset c'$, and so by (A3) $c = c'$. The uniqueness of the supremum c of a and b permits us to think of the association of c with a and b which (A4) establishes as a function, which maps pairs of elements a, b to their corresponding sums. The next definition makes this explicit.

Definition 4.3.2

Suppose $\mathcal{A} = \langle A, \subset \rangle$ is an upper semilattice. For all $a, b \in A$ the element denoted by $a \oplus b$ is the unique c such that

(i) $a \subset c$,
(ii) $b \subset c$ and
(iii) $(\forall d)\,(a \subset d \;\&\; b \subset d \rightarrow c \subset d)$

The concept of supremum is applicable not only to pairs a, b of lattice elements, but more generally to sets of them.

Definition 4.3.3

Suppose $\mathcal{A} = \langle A, \subset \rangle$ is an upper semilattice. Suppose B is a subset of A. b *is the supremum* of B (*in* $\langle A, \subset \rangle$) iff

(i) $(\forall x \in B)\,(x \subset b)$ and
(ii) $(\forall d \in A)\,(((\forall x \in B)\, x \subset d) \rightarrow b \subset d)$

By the same reasoning as above one shows that when such a b exists, it is unique. When b exists, we denote it as $\oplus B$.

It is easily shown that $\oplus B$ always exists when B is finite. But if B is infinite, this need not be so (Exercise 1, p. 417).

Definition 4.3.4

(i) An upper semilattice $\langle A, \subset \rangle$ is called *complete* if for all $X \subseteq A$ the supremum $\oplus X$ exists.

(ii) If a is the "largest" element of A – i.e. for all $x \in A$, $x \subset a$ – then a is called the *one of* A and denoted as $1_{\mathcal{A}}$. Similarly, if a is the "smallest" element of A – i.e. for all $x \in A$, $a \subset x$ – then a is called the *zero of* A and denoted as $0_{\mathcal{A}}$.

(iii) By an *atom of* $\mathcal{A}$ we understand any element $a \neq 0_{\mathcal{A}}$, such that $\forall x\,(x \subset a \rightarrow (x = a \vee x = 0_{\mathcal{A}}))$. When a is an atom, we write At(a).

(iv) $\mathcal{A}$ is said to be *atomic* if for every a, b $\in$ A such that $a \not\subset b$ there is an atom c such that $c \subset a$ and $c \not\subset b$.

(v) $\mathcal{A}$ is *free* if for all $a \in A$, $X \subseteq A$ if At(a) and $a \subset \oplus X$ then $(\exists b \in X)\; a \subset b$.

Upper semilattices with zero that are complete, atomic and free have a remarkably simple structure. In fact, each such upper semilattice $\langle A, \subset \rangle$ is isomorphic to a structure $\langle \mathcal{P}(B), \subseteq \rangle$ where $\mathcal{P}(B)$ is the set of all subsets of some given set B and '$\subseteq$' is the relation of set-theoretical inclusion.[46] In particular, we can take B to be the set of all atoms $At(\mathcal{A})$ of $\mathcal{A}$:

Theorem 1: Each upper semilattice has at most one one and at most one zero.

Theorem 2: If $\mathcal{A}$ is atomic and complete, then for each non-zero element a of $\mathcal{A}$
$$a = \oplus\{b \in A : At(b) \;\&\; b \subset a\}$$

Theorem 3: Let $\mathcal{A} = \langle A, \subset \rangle$ be a complete, atomic, free upper semilattice with zero, and let $At(\mathcal{A})$ be the set of atoms of $\mathcal{A}$. Then $\mathcal{A}$ is isomorphic to the structure $\langle \mathcal{P}(At(\mathcal{A})), \subseteq \rangle$.

[46] $\langle A, \subset \rangle$ is *isomorphic to* $\langle A', \subset' \rangle$ if there exists a 1-1 function f from A onto A′ such that for all a, b $\in$ A $a \subset b$ iff $f(a) \subset' f(b)$.

Kamp & Reyle 1993, ch. 4 "The plural"
pp. 425-7

4.3.2.2 Summary of Definitions

Definition 4.3.5

A model M for DRL_{Plur} is a triple $\langle \mathcal{U}_M, Name_M, Pred_M, Quant_M\rangle$ consisting of

(i) a complete, free, atomic upper semilattice $\mathcal{U}_M = \langle U_M, \subset\rangle$ with zero.

(ii) a function $Name_M$ mapping each name a of DRL_{Plur} to its bearer in U_M such that

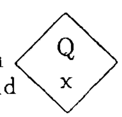
$Name_M(A) \in At(U_M)$ if a names an individual.
$Name_M(A) \in U_M \setminus (At(U_M) \cup \{0_M\})$ if a names a group.

(iii) a function $Pred_M$ mapping predicates P to their extensions in M.

Definition 4.3.6

(i) A *DRS* K *confined to* V *and* R is a pair consisting of a subset U_K (possibly empty) of R and a set Con_K of DRS-conditions confined to V and R;

(ii) A *DRS-condition confined to* V *and* R is an expression of one of the following forms:

(a) at(x), non-at(x), where x belongs to R
(b) $x = y_1 \oplus \ldots \oplus y_n$ $(n \geq 2)$
(c) $x = \Sigma z\ K$
(d) $|x| = \nu$ where $\nu \in \{1, 2, 3, \ldots\}$
(e) $x \in y$
(f) $\pi(x)$, where x belongs to R and π is a name from V
(g) $\eta(x)$, $x\eta$, where x belongs to R and η is a unary predicate from V[56]
(h) $x\xi y$, where x, y belong to R and ξ is a binary predicate from V
(i) $\neg K$, where K is a DRS confined to V and R.
(j) $K_1 \Rightarrow K_2$, $K_1 V \ldots V K_2$ where K_1 to K_n $(n \geq 2)$ are DRSs confined to V and R.
(k) K_1  K_2, where K_1 and K_2 are DRSs confined to V and R, and Q is a quantifying determiner from V.

[56]Recall our practice to write '$\eta(x)$' when η is an English common noun and '$x\eta$' when η is an English verb. For present purposes the difference is of course immaterial.

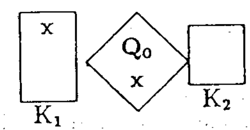
Definition 4.3.7

Let K be a DRS confined to V and R, γ a DRS-condition and f an embedding from K into M, i.e. a function whose Domain equals U_K and whose Range is included in U_M

(i) f *verifies* the DRS K in M iff f verifies each of the conditions γ belonging to Con_K in M

(ii) "f *verifies* the condition γ in M" is defined by cases, depending on the form of γ:

(a) (i) $M \models_f$ at(x) if f(x) is an atom of M
(ii) $M \models_f$ non-at(x) if f(x) is a non-atomic entity of M
(b) $M \models_f x = y_1 \oplus \ldots \oplus y_n$ iff $f(x) = f(y_1) \oplus_M \ldots \oplus_M f(y_n)$
(c) $M \models_f x = \Sigma z\ K'$ iff $f(x) = \oplus\{b : b \in U_M\ \&\ M \models_{f\cup\{\langle z,b\rangle\}} K'\}$
(d) $M \models_f |x| = \nu$ iff $|\{b \in U_M$: b is an atom of M & $b \subset_M f(x)\}| = \nu$
(e) $M \models_f x \in y$ iff f(x) is an atom of M and $f(x) \subset_M f(y)$
(f) $M \models_f \pi(x)$, where π is a name iff $\langle \pi, f(x)\rangle \in Name_M$
(g) (i) $M \models_f \eta(x)$, where η is a noun iff $f(x) \in Pred_M(\eta)$
(ii) $M \models_f x\zeta$, where ζ is an intransitive verb iff $f(x) \in Pred_M(\zeta)$
(h) $M \models_f x\xi y$, where ξ is an intransitive verb iff $\langle f(x), f(y)\rangle \in Pred_M(\xi)$
(i) $M \models_f \neg K$ iff there is no extension g of f to $U_{K'}$ such that $M \models_g K$
(j) (i) $M \models_f K_1 \Rightarrow K_2$ iff for every extension g of f to U_{K_1} such that $M \models_g K_1$ there is an extension h of g to U_{K_2} such that $M \models_h K_2$
(ii) $M \models_f K_1 V \ldots V K_n$ iff for some i with $1 \leq i \leq n$ $M \models_f K_i$

(k) $M \models_f$  iff $\langle A,B\rangle \in Quant_M(Q_0)$, where

$A = \{b : b \in U_M\ \&\ (\exists g)\ (f \cup \{\langle x,b\rangle\} \subseteq_{U_{K_1}-\{x\}} g\ \&\ M \models_g K_1)\}$

and

$B = \{b : b \in U_M\ \&\ (\exists g)\ (f \cup \{\langle x,b\rangle\} \subseteq_{U_{K_1}-\{x\}} g\ \&\ M \models_g K_1)\ \&$
$\forall g\ (f \cup \{\langle x,b\rangle\} \subseteq_{U_{K_1}-\{x\}} g \rightarrow (\exists h)\ (g \subseteq_{U_{K_2}} h\ \&\ M \models_h K_2))\}$