Notes on van den Berg 1994, 1995: Plural Predicate Logic

I. Two approaches to collectivity and distributivity

Paradigm examples:

(1)	[John and Mary] are a nice couple.	collective VP
(2)	[John and Mary] are students.	distributive VP
(3)	[John and Mary] bought a house	ambiguous VP

Plural entity approach (Link 1983, 1987)

- D_e structured (e.g. by $\subseteq$) into *atoms* (w/o proper parts) & *pluralities* (w/ proper parts)
- *Collective predicates* are predicated of pluralities
- *Distributive predicates* are distributed, by ^{D}, over atomic parts of pluralities

e.g.

(1) Adam and Beth are a nice couple. — collective VP

(1_L) $C(a + b)$

(2) Adam and Beth are students. — distributive VP

(2_L) $^D\lambda x[Sx](a + b)$

$= \forall y(y \leq (a + b) \wedge \mathbf{1}y \rightarrow Sy)$ — df. ^{D} (Link 1987)

$= Sa \wedge Sb$ — if $\mathbf{1}a$, $\mathbf{1}b$

(3) Adam and Beth bought a house

(3_L) $\exists z(Hz \wedge B(a + b)z)$ — collective reading

($3'_L$) $^D\lambda x[\exists y(Hy \wedge Bxy)](a + b)$ — distributive reading

$= \forall y(y \leq (a + b) \wedge \mathbf{1}y \rightarrow \exists z(Hz \wedge Byz))$ — df. ^{D}

$= \exists y(Hy \wedge Bay) \wedge \exists y(Hy \wedge Bby)$ — if $\mathbf{1}a$, $\mathbf{1}b$

Plural information state approach (van den Berg 1994, 1995)

- G structured (e.g. by $\subseteq$) into *atomic u-states* (w/o proper *u*-parts) & *plural states*
- *collective predicates* are predicated of the sets of values in the current plural info-state
- *distributive predicates* are distributed, by δ_u, over atomic *u*-states of the current plural state

e.g.

(1) Adam and Beth are a nice couple. — collective VP

(1_B) $C(a + b)$

(2) Adam and Beth are students. — distributive VP

(2_B) $\exists x(x = (a + b) \wedge \delta_x(Sx))$

$= \exists x(x = (a + b) \wedge \forall y(y \subseteq x \wedge \mathbf{1}y \rightarrow Sy))$ — df. δ_x (vd Berg 1994)

$= Sa \wedge Sb$

(3) Adam and Beth bought a house

(3_B) $\exists y(Hy \wedge B(b + a)y)$ — collective reading

($3'_B$) $\exists x(x = (a + b) \wedge \delta_x(\exists z(Hz \wedge Bxz)))$ — distributive reading

$= \exists x(x = (a + b) \wedge \forall y(y \subseteq x \wedge \mathbf{1}y \rightarrow \exists z(Hz \wedge Byz))$ — df. δ_x

$= \exists z(Hz \wedge Baz) \wedge \exists z(Hz \wedge Bbz)$

II Logic with Pluralities (LP$^{\leq}$, a la Link 1983)

DEFINITION 0 (LP$^{\leq}$-syntax)

t	$\alpha \in$ **Trm**	if $\alpha \in$ **Con** or $\alpha \in$ **Var**
+	$(\alpha + \beta) \in$ **Trm**	if $\alpha, \beta \in$ **Trm**
≤	$(\alpha \leq \beta) \in$ **Wff**	iff $\alpha, \beta \in$ **Trm**
R	$\alpha\beta_1\ldots\beta_n \in$ **Wff**	iff $\alpha \in$ **Prd**^{n} & $\beta_1, \ldots, \beta_n \in$ **Trm**
∧	$(\phi \wedge \psi) \in$ **Wff**	iff $\phi, \psi \in$ **Wff**
¬	$\neg\phi \in$ **Wff**	iff $\phi \in$ **Wff**
∃	$\exists u\phi \in$ **Wff**	iff $u \in$ **Var**, $\phi \in$ **Wff**

DEFINITION 1 (LP$^{\leq}$ models and assignments)

- An LP$^{\leq}$-*model* is a structure $M = \langle D^M, [\![\cdot]\!]^M \rangle$ such that:
  (i) For some non-empty set A, $D^M = \mathcal{P}(A) - \varnothing$
  (ii) For all $\alpha \in$ **Con**, $[\![\alpha]\!]^M \in D^M$, and for all $\alpha \in$ **Prd**^{n}, $[\![\alpha]\!]^M \subseteq (D^M)^n$
- An *M-assignment* is a function g such that Dom g = **Var** and Ran $g \subseteq D^M$.
  The set of M-assignments is denoted by **G**. For any $g, h \in$ **G**, $u \in$ **Var**, $d \in D^M$,
  $g[u/d] = h$ iff $h(u) = d$ & $\forall v \in$ (**Var** $- \{u\}$): $h(v) = g(v)$
  $g \approx_u h$ iff $\exists d \in D^M$: $g[u/d] = h$

DEFINITION 2 (LP$^{\leq}$-semantics)

t	$g[\![\alpha]\!]^M$	=	$[\![\alpha]\!]^M$		if $\alpha \in$ **Con**
		=	$g(\alpha)$		if $\alpha \in$ **Var**
+	$g[\![(\alpha + \beta)]\!]^M$	=	$(g[\![\alpha]\!]^M \cup g[\![\beta]\!]^M)$		
≤	$g[\![(\alpha \leq \beta)]\!]^M$	=	⊤	iff	$g[\![\alpha]\!]^M \subseteq g[\![\beta]\!]^M$
R	$g[\![\alpha\beta_1\ldots\beta_n]\!]^M$	=	⊤	iff	$\langle g[\![\beta_1]\!]^M, \ldots g[\![\beta_n]\!]^M \rangle \in [\![\alpha]\!]^M$
∧	$g[\![(\phi \wedge \psi)]\!]^M$	=	⊤,	iff	$g[\![\phi]\!]^M = \top$ & $g[\![\psi]\!]^M = \top$
¬	$g[\![\neg\phi]\!]^M$	=	⊤,	iff	$g[\![\phi]\!]^M = \bot$
∃	$g[\![\exists u\phi]\!]^M$	=	⊤,	iff	$\exists h$: $g \approx_x h$ & $h[\![\phi]\!]^M = \top$

DEFINITION 3 (LP$^{\leq}$ Truth).

- ϕ is *true* in M, $\models_M \phi$, iff $\forall g \in$ **G**: $g[\![\phi]\!]^M = \top$

ABBREVIATIONS 1 (standard)

- $(\alpha \vee \beta)$:= $\neg(\neg\alpha \wedge \neg\beta)$
  $(\alpha \rightarrow \beta)$:= $\neg(\alpha \wedge \neg\beta)$
- $\forall u\phi$:= $\neg\exists u\neg\phi$
- $\phi[u/u']$:= the wff like ϕ except that every free occurrence of u is replaced with u'

ABBREVIATIONS 2 (identity, atomicity, distributivity)

- $(\alpha = \beta)$:= $((\alpha \leq \beta) \wedge (\beta \leq \alpha))$ — $\alpha, \beta \in$ **Trm**
- $\mathbf{1}\alpha$:= $\forall x(x \leq \alpha \rightarrow x = \alpha)$ — $\alpha \in$ **Trm**
- $^{D}\lambda u[\phi]\alpha$:= $\forall u'(u' \leq \alpha \wedge \mathbf{1}u' \rightarrow \phi[u/u'])$ — $\phi \in$ **Wff**, $\alpha \in$ **Trm**, no free u' in ϕ

III Plural Predicate Logic (PPL, see van den Berg 1994, sec. 2)

DEFINITION 0 (PPL-syntax)

$\boldsymbol{t}$	$\alpha \in \mathbf{Trm}$	if $\alpha \in \mathbf{Con}$ or $\alpha \in \mathbf{Var}$
$+$	$(\alpha + \beta) \in \mathbf{Trm}$	if $\alpha, \beta \in \mathbf{Trm}$
$\subseteq$	$(\alpha \subseteq \beta) \in \mathbf{Wff}$	iff $\alpha, \beta \in \mathbf{Trm}$
$\boldsymbol{R}$	$\alpha\beta_1\ldots\beta_n \in \mathbf{Wff}$	iff $\alpha \in \mathbf{Prd}^n$ & $\beta_1, \ldots, \beta_n \in \mathbf{Trm}$
$\wedge$	$(\phi \wedge \psi) \in \mathbf{Wff}$	iff $\phi, \psi \in \mathbf{Wff}$
$\neg$	$\neg\phi \in \mathbf{Wff}$	iff $\phi \in \mathbf{Wff}$
$\exists$	$\exists u\phi \in \mathbf{Wff}$	iff $u \in \mathbf{Var}$, $\phi \in \mathbf{Wff}$

DEFINITION 1 (PPL models and assignments) **3rd try**

- A PPL-model is a structure $M = \langle D^M, [\![\cdot]\!]^M \rangle$ such that: (i) D^M is a non-empty set, and (ii) for all $\alpha \in \mathbf{Con}$, $[\![\alpha]\!]^M \in D^M$, and for all $\alpha \in \mathbf{Prd}^n$, $[\![\alpha]\!]^M \subseteq (\mathcal{P}(D^M))^n$
- An *M-assignment* is a function g such that Dom $g \subseteq \mathbf{Var}$ and Ran $g \subseteq D^M$.
  The set of M-assignments is denoted by $\mathbf{G}$. For any $g, h \in \mathbf{G}$, $u \in \mathbf{Var}$, $d \in D^M$,
  $g[u/d] = h$ iff Dom $g \cup \{u\}$ = Dom h & $h(u) = d$ & $\forall u' \in (\text{Dom } g - \{u\})$: $h(u') = g(u')$
  $g \approx_u h$ iff $\exists d \in D$: $g[u/d] = h$
- An *M-information state* is a set of M-assignments. For any $G, H \subseteq \mathbf{G}$, $u \in \mathbf{Var}$, $d \in D^M$,

$G(u)$	:=	$\{d \in D^M \mid \exists g \in G: g(u) = d\}$	cf. vd Berg 1995:79
$G[u/d]$	:=	$\{g[u/d]: g \in G\}$	"
$G \approx_u H$	iff	$\forall d \in D^M$: $G[u/d] = H[u/d]$	"

DEFINITION 2 (PPL-semantics)

$\boldsymbol{t}$	$G[\![\alpha]\!]^M$	=	$\{[\![\alpha]\!]^M\}$	if $\alpha \in \mathbf{Con}$	
		=	$G(\alpha)$	if $\alpha \in \mathbf{Var}$	
$+$	$G[\![(\alpha + \beta)]\!]^M$	=	$G[\![\alpha]\!]^M \cup G[\![\beta]\!]^M$		
$\subseteq$	$G[\![(\alpha \subseteq \beta)]\!]^M$	=	$\top$ iff	$G[\![\alpha]\!]^M \subseteq G[\![\beta]\!]^M$	vd Berg 1994:4
$\boldsymbol{R}$	$G[\![\alpha\beta_1\ldots\beta_n]\!]^M$	=	$\top$ iff	$\langle G[\![\beta_1]\!]^M, \ldots, G[\![\beta_n]\!]^M \rangle \in [\![\alpha]\!]^M$	"
$\wedge$	$G[\![(\phi \wedge \psi)]\!]^M$	=	$\top$, iff	$G[\![\phi]\!]^M = \top$ & $G[\![\psi]\!]^M = \top$	"
$\neg$	$G[\![\neg\phi]\!]^M$=	$\top$, iff	$G[\![\phi]\!]^M = \bot$		"
$\exists$	$G[\![\exists u\phi]\!]^M$	=	$\top$, iff	$\exists H$: $G \approx_u H$ & $H[\![\phi]\!]^M = \top$	"

DEFINITION 3 (Truth).

- ϕ is *true* in M, $\models_M \phi$, iff $\mathbf{G}[\![\phi]\!]^M = \top$

ABBREVIATIONS 1 (standard)

- $(\alpha \vee \beta)$:= $\neg(\neg\alpha \wedge \neg\beta)$
  $(\alpha \rightarrow \beta)$:= $\neg(\alpha \wedge \neg\beta)$
- $\forall u\phi$:= $\neg\exists u\neg\phi$
- $\phi[u/u']$:= the wff like ϕ except that every free occurrence of u is replaced with u'

ABBREVIATIONS 2 (identity, emptiness, atomicity, distributivity)

- $(\alpha = \beta)$:= $((\alpha \subseteq \beta) \wedge (\beta \subseteq \alpha))$ for $\alpha, \beta \in \mathbf{Trm}$
- $\mathbf{0}u$:= $\forall u'(u' \subseteq u \rightarrow u' = u)$
  $\mathbf{1}u$:= $(\neg\mathbf{0}u \wedge \forall u'(u' \subseteq u \wedge \neg\mathbf{0}u' \rightarrow u' = u))$ $\neq$ vd Berg 1994:(14)
- $\delta_u(\phi)$:= $\forall u'(u' \subseteq u \wedge \mathbf{1}u' \rightarrow \phi[u/u'])$ $\phi \in \mathbf{Wff}$, no u' in ϕ

IV. Some useful facts

$LP^{\leq}$ facts:

Fact A1

∨	$g[\![(\phi \vee \psi)]\!]^M$	$= \bot$,	iff	$g[\![\phi]\!]^M = \bot$ & $g[\![\psi]\!]^M = \bot$
→	$g[\![(\phi \rightarrow \psi)]\!]^M$	$= \bot$,	iff	$g[\![\phi]\!]^M = \top$ & $g[\![\psi]\!]^M = \bot$
∃	$g[\![\exists u\phi]\!]^M$	$= \top$,	iff	$\exists d \in D^M$: $g[u/d][\![\phi]\!]^M = \top$
∀	$g[\![\forall u(\phi \rightarrow \psi)]\!]^M$	$= \top$,	iff	$\forall d \in D^M$: $g[u/d][\![\phi]\!]^M = \top \rightarrow g[u/d][\![\psi]\!]^M = \top$

Fact A2

=	$g[\![(\alpha = \beta)]\!]^M$	$= \top$,	iff	$g[\![\alpha]\!]^M = g[\![\beta]\!]^M$
1	$g[\![\mathbf{1}\alpha]\!]^M$	$= \top$,	iff	$\lvert g[\![\alpha]\!]^M \rvert = 1$

Proof: Exercise

PPL facts:

Fact A1′

∨	$G[\![(\phi \vee \psi)]\!]^M$	$= \bot$,	iff	$G[\![\phi]\!]^M = \bot$ & $G[\![\psi]\!]^M = \bot$
→	$G[\![(\phi \rightarrow \psi)]\!]^M$	$= \bot$,	iff	$G[\![\phi]\!]^M = \top$ & $G[\![\psi]\!]^M = \bot$
∀	$G[\![\forall u(\phi \rightarrow \psi)]\!]^M$	$= \top$,	iff	$\forall H$: $G \approx_u H$ & $H[\![\phi]\!]^M = \top \rightarrow H[\![\psi]\!]^M = \top$

Proof:

∨, →

Exercise

∀

1.	$G[\![\forall u(\phi \rightarrow \psi)]\!]^M = \top$	
2.	$G[\![\neg\exists u\neg(\phi \rightarrow \psi)]\!]^M = \top$	A1.∀
3.	$G[\![\exists u\neg(\phi \rightarrow \psi)]\!]^M = \bot$	D2.¬
4.	$\neg\exists H$: $G \approx_u H$ & $H[\![\neg(\phi \rightarrow \psi)]\!]^M = \top$	D2.∃
5.	$\neg\exists H$: $G \approx_u H$ & $H[\![(\phi \rightarrow \psi)]\!]^M = \bot$	D2.¬
6.	$\neg\exists H$: $G \approx_u H$ & $H[\![\phi]\!]^M = \top$ & $H[\![\psi]\!]^M = \bot$	Fact A1′.→
7.	$\forall H$: $G \approx_u H$ & $H[\![\phi]\!]^M = \top \rightarrow H[\![\psi]\!]^M = \top$	rearrange

Fact A2′

=	$G[\![(\alpha = \beta)]\!]^M$	$= \top$,	iff	$G[\![\alpha]\!]^M \subseteq G[\![\beta]\!]^M$
0	$G[\![\mathbf{0}u]\!]^M$	$= \top$,	iff	$G(u) = \{\}$
1	$G[\![\mathbf{1}u]\!]^M$	$= \top$,	iff	$\lvert G(u) \rvert = 1$

Proof:

=

1.	$G[\![(\alpha = \beta)]\!]^M = \top$	
2.	$G[\![((\alpha \subseteq \beta) \wedge (\beta \subseteq \alpha))]\!]^M = \top$	A2.=
3.	$G[\![(\alpha \subseteq \beta)]\!]^M = \top$ & $[\![(\beta \subseteq \alpha)]\!]^M = \top$	D2.∧
4.	$G[\![\alpha]\!]^M \subseteq G[\![\beta]\!]^M$ & $G[\![\beta]\!]^M \subseteq G[\![\alpha]\!]^M$	D2.⊆
5.	$G[\![\alpha]\!]^M = G[\![\beta]\!]^M$	df. ⊆

0 (Assuming $u \neq u'$)

1. $G[\![\mathbf{0}u]\!]^M = \top$
2. $G[\![\forall u'(u' \subseteq u \to u' = u)]\!]^M = \top$ — A2.**0**
3. $\forall H$: $G \approx_{u'} H$ & $H[\![u' \subseteq u]\!]^M = \top \to H[\![u' = u]\!]^M = \top$ — Fact A1′.∀
4. $\forall H$: $G \approx_{u'} H$ & $H[\![u']\!]^M \subseteq H[\![u]\!]^M \to H[\![u']\!]^M = H[\![u]\!]^M$ — D2.⊆, Fact A1′.=
5. $\forall H$: $G \approx_{u'} H$ & $H(u') \subseteq H(u) \to H(u') = H(u)$ — D2.***t***
6. $\forall H$: $G \approx_{u'} H$ & $H(u') \subseteq G(u) \to H(u') = G(u)$ — D1.≈, $G[u/d]$
7. $G(u) = \{\}$ — see below

(6) ⇒ (7)
Define $G_{u'} := \{g \in G |\ u' \in \text{Dom } g\}$ and $H_{u'} := \{g - \{\langle u', g(u')\rangle\} : g \in G_{u'}\} \cup (G - G_{u'})$.
Then, $G \approx_{u'} H_{u'}$, since $\forall d \in D^M$, $G[u'/d] := \{g[u'/d]: g \in G\} = \{h[u'/d]: h \in H_{u'}\} =: H_{u'}[u'/d]$.
Moreover, $H_{u'}(u') \subseteq G(u)$, since $H_{u'}(u') := \{d \in D^M |\ \exists h \in H_{u'}: u' \in \text{Dom } h\} = \varnothing$. Given (**6**), $G \approx_{u'} H_{u'}$ & $H_{u'}(u') \subseteq G(u)$ implies $H_{u'}(u') = G(u)$. Since $H_{u'}(u') = \varnothing$, we thus get (**7**).
(7) ⇒ (6)
Assume (**7**) $G(u) = \varnothing$. Then, for all H, $H(u') \subseteq G(u)$ implies $H(u') = \varnothing$. Hence (**6**).

1[1] (Assuming $u \neq u'$)

1. $G[\![\mathbf{1}u]\!]^M = \top$
2. $G[\![\neg\mathbf{0}u \wedge \forall u'(u' \subseteq u \wedge \neg\mathbf{0}u' \to u' = u)]\!]^M = \top$ — A2.**1**
3. $G[\![\mathbf{0}u]\!]^M = \bot$ & $G[\![\forall u'(u' \subseteq u \wedge \neg\mathbf{0}u' \to u' = u)]\!]^M = \top$ — D2.∧, ¬
4. $G(u) \neq \{\}$ & $\forall H$: $G \approx_{u'} H$ & $H[\![u' \subseteq u \wedge \neg\mathbf{0}u']\!]^M = \top \to H[\![u' = u]\!]^M = \top$ — Fact A1′.∀, A2′.**0**
5. $G(u) \neq \{\}$
 & $\forall H$: $G \approx_{u'} H$ & $H[\![u' \subseteq u]\!]^M = \top$ & $H[\![\mathbf{0}u']\!]^M = \bot \to H[\![u' = u]\!]^M = \top$ — D2.∧, ¬
6. $G(u) \neq \{\}$
 & $\forall H$: $G \approx_{u'} H$ & $H[\![u']\!]^M \subseteq H[\![u]\!]^M$ & $H(u') \neq \{\} \to H[\![u']\!]^M = H[\![u]\!]^M$ — D2.⊆, Fact A2′.**0**, **=**
7. $G(u) \neq \{\}$
 & $\forall H$: $G \approx_{u'} H$ & $H(u') \subseteq G(u)$ & $H(u') \neq \{\} \to H(u') = G(u)$ — D2.***t*****,** D1.≈, $G[u/d]$
8. $|G(u)| = 1$ — see below

(7) ⇒ (8)
Consider any $d_1, d_2 \in G(u)$. Define $H_1 := \{g[u'/d_1]: g \in G\}$ and $H_2 := \{g[u'/d_2]: g \in G\}$. Then (i) $H_1(u') = \{d_1\}$ and $H_2(u') = \{d_2\}$. Moreover, (ii) $H_1 \approx_{u'} G$ & $H_1(u') \subseteq G(u)$ & $H_1(u') \neq \varnothing$ and (iii) $H_2 \approx_{u'} G$ & $H_2(u') \subseteq G(u)$ & $H_2(u') \neq \varnothing$. Given (**7**), (ii) implies (iv) $H_1(u') = G(u)$, while (iii) implies (v) $H_2(u') = G(u)$. Hence (vi) $H_1(u') = H_2(u')$, which together with (i), implies (vii) $d_1 = d_2$. Since this holds for any $d_1, d_2 \in G(u)$, we must have (**8**) $|G(u)| = 1$.
(8) ⇒ (7)
Assume (**8**) $|G(u)| = 1$. Then, for some $d \in D^M$, $G(u) = \{d\}$. So $H(u') \subseteq G(u)$ & $H(u') \neq \varnothing$ implies $H(u') = \{d\}$. Hence (**8**).

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[1] Note that '**1**u' is *not* equivalent to van den Berg's '**sing**(x)'. Given the definitions,

1x := $\neg\mathbf{0}x \wedge \forall y(y \subseteq x \wedge \neg\mathbf{0}y \to y = x)$ — A2′ above
sing x := $\forall y(y \subseteq x \to y = x \vee \mathbf{0}y)$ — vd Berg 1994:(14)

$G[\![\mathbf{1}x]\!]^M = \top$ iff $|G(x)| = 1$, whereas $G[\![\mathbf{sing}(x)]\!]^M = \top$ iff $|G(x)| \leq 1$. Given what van den Berg says about (14), he wants '**1**x', not '**sing**(x)', so that's what we'll use, e.g., in the definition of δ_x.

V. <u>Collectivity and distributivity: LP$^{\leq}$ v. PPL</u>

(1) Adam and Beth are a nice couple	collective VP

LP$^{\leq}$ analysis: (1) iff (5)	**LP$^{\leq}$** df's & facts
1. $\models_M C(a + b)$	
2. $\forall g$: $g[\![C(a + b)]\!]^M = \top$	D3
3. $\forall g$: $g[\![(a + b)]\!]^M \in [\![C]\!]^M$	D2.***R***
4. $\forall g$: $(g[\![a]\!]^M \cup g[\![b]\!]^M) \in [\![C]\!]^M$	D2.**+**
5. $([\![a]\!]^M \cup [\![b]\!]^M) \in [\![C]\!]^M$	D2.***t***

e.g. true in any LP$^{\leq}$-model M s.t.:
$[\![a]\!]^M = \{\text{a}\}$ $\quad$ $[\![b]\!]^M = \{\text{b}\}$ $\quad$ $\{\text{a, b}\} \in [\![C]\!]^M$

PPL analysis: (1) iff (5)	**PPL** df's & facts
1. $\models_M C(a + b)$	
2. $\mathbf{G}[\![C(a + b)]\!]^M = \top$	D3
3. $\mathbf{G}[\![(a + b)]\!]^M \in [\![C]\!]^M$	D2.***R***
4. $(\mathbf{G}[\![a]\!]^M \cup \mathbf{G}[\![b]\!]^M) \in [\![C]\!]^M$	D2.**+**
5. $(\{[\![a]\!]^M\} \cup \{[\![b]\!]^M\}) \in [\![C]\!]^M$	D2.***t***
6. $\{[\![a]\!]^M, [\![b]\!]^M\} \in [\![C]\!]^M$	df. $\cup$

e.g. true in any PPL-model M s.t.:
$[\![a]\!]^M = \text{a}$ $\quad$ $[\![b]\!]^M = \text{b}$ $\quad$ $\{\text{a, b}\} \in [\![C]\!]^M$

(2) John and Mary are students.	distributive VP

LP$^{\leq}$ analysis: (1) iff (9)	**LP$^{\leq}$** df's & facts
1. $\models_M {}^{\mathrm{D}}\lambda x[Sx](a + b)$	
2. $\models_M \forall y(y \leq (a + b) \wedge \mathbf{1}y \rightarrow Sy)$	A2.$^{\mathrm{D}}$
3. $\forall g$: $[\![\forall y(y \leq (a + b) \wedge \mathbf{1}y \rightarrow Sy)]\!]^M = \top$	D3
4. $\forall g$: $\forall d \in D^M$: $g[y/d][\![y \leq (a + b) \wedge \mathbf{1}y]\!]^M = \top$ $\rightarrow g[y/d][\![Sy]\!]^M = \top$	Fact A1.$\forall$
5. $\forall g$: $\forall d \in D^M$: $g[y/d][\![y \leq (a + b)]\!]^M = \top$ & $g[y/d][\![\mathbf{1}y]\!]^M = \top$ $\rightarrow g[y/d][\![Sy]\!]^M = \top$	D2.$\wedge$
6. $\forall g$: $\forall d \in D^M$: $g[y/d][\![y]\!]^M \subseteq g(y/d)[\![(a + b)]\!]^M$ & $\lvert g[y/d][\![y]\!]^M\rvert = 1$ $\rightarrow g[y/d][\![y]\!]^M \in [\![S]\!]^M$	D2.$\leq$, Fact A1.**1** D2.***R***
7. $\forall g$: $\forall d \in D^M$: $d \subseteq (g(y/d)[\![a]\!]^M \cup g(y/d)[\![b]\!]^M)$ & $\lvert d\rvert = 1$ $\rightarrow d \in [\![S]\!]^M$	D2.***t***, D1.$g[u/d]$
8. $\forall g$: $\forall d \in D^M$: $d \subseteq ([\![a]\!]^M \cup [\![b]\!]^M)$ & $\lvert d\rvert = 1$ $\rightarrow d \in [\![S]\!]^M$	D2.***t***
9. $\forall d \in D^M$: $d \subseteq ([\![a]\!]^M \cup [\![b]\!]^M)$ & $\lvert d\rvert = 1 \rightarrow d \in [\![S]\!]^M$	simplify

e.g. true in any LP$^{\leq}$-model M s.t.:
$[\![a]\!]^M = \{\text{a}\}$ $\quad$ $[\![b]\!]^M = \{\text{b}\}$ $\quad$ $\{\text{a}\}, \{\text{b}\} \in [\![S]\!]^M$

PPL analysis: (1) iff (9)	**PPL** df's
1. $\models_M \exists x(x = (a+b) \wedge \delta_x(Sx))$	
2. $\mathbf{G}\llbracket \exists x(x = (a+b) \wedge \delta_x(Sx))\rrbracket^M = \top$	D3
3. $\exists H$: $\mathbf{G} \approx_x H$ & $H\llbracket x = (a+b) \wedge \forall y(y \subseteq x \wedge \mathbf{1}y \rightarrow Sy)\rrbracket^M = \top$	D2.$\exists$, A2.δ_u
4. $\exists H$: $\mathbf{G} \approx_x H$ & $H\llbracket x = (a+b)\rrbracket^M = \top$ & $H\llbracket \forall y(y \subseteq x \wedge \mathbf{1}y \rightarrow Sy)\rrbracket^M = \top$	D2.$\wedge$
5. $\exists H$: $\mathbf{G} \approx_x H$	
& $H\llbracket x\rrbracket^M = H\llbracket a\rrbracket^M \cup H\llbracket b\rrbracket^M$	Fact A2$'$.**=**, **+**
& $\forall K$: $H \approx_y K$ & $(K\llbracket y \subseteq x \wedge \mathbf{1}y\rrbracket^M = \top \rightarrow K\llbracket Sy\rrbracket^M = \top)$	Fact A1$'$.$\forall$
6. $\exists H$: $\mathbf{G} \approx_x H$	
& $H(x) = \{\llbracket a\rrbracket^M\} \cup \{\llbracket b\rrbracket^M\}$	D2.$\boldsymbol{t}$
& $\forall K$: $H \approx_y K$ & $(K\llbracket y\rrbracket^M \subseteq K\llbracket x\rrbracket^M$ & $K\llbracket \mathbf{1}y\rrbracket^M = \top \rightarrow K\llbracket y\rrbracket^M \in \llbracket S\rrbracket^M)$	D2.$\wedge$, $\subseteq$, $\boldsymbol{R}$
7. $\exists H$: $\mathbf{G} \approx_x H$	
& $H(x) = \{\llbracket a\rrbracket^M, \llbracket b\rrbracket^M\}$	df. $\cup$
& $\forall K$: $H \approx_y K$ & $(K(y) \subseteq K(x)$ & $\lvert K(y)\rvert = 1 \rightarrow K(y) \in \llbracket S\rrbracket^M)$	D2.$\boldsymbol{t}$, Fact A2$'$.**1**
8. $\exists H$: $\mathbf{G} \approx_x H$	
& $H(x) = \{\llbracket a\rrbracket^M, \llbracket b\rrbracket^M\}$	
& $\forall K$: $H \approx_y K$ & $(K(y) \subseteq H(x)$ & $\lvert K(y)\rvert = 1 \rightarrow K(y) \in \llbracket S\rrbracket^M)$	D1.$G \approx_u H$, $G[u/d]$
9. $\{\llbracket a\rrbracket^M\} \in \llbracket S\rrbracket^M$ & $\{\llbracket b\rrbracket^M\} \in \llbracket S\rrbracket^M$	

(8) ⇒ (9)

Suppose (**8**). Then there is some H_0 such that (a) & (b) & (c):

a. $\mathbf{G} \approx_x H_0$

b. $H_0(x) = \{\llbracket a\rrbracket^M, \llbracket b\rrbracket^M\}$

c. $\forall K$: $H_0 \approx_y K$ & $(K(y) \subseteq H_0(x)$ & $\lvert K(y)\rvert = 1 \rightarrow K(y) \in \llbracket S\rrbracket^M$

Define $K_a := \{h[y/\llbracket a\rrbracket^M]: h \in H_0\}$ and $K_b := \{h[y/\llbracket b\rrbracket^M]: h \in H_0\}$. Then:

d. $H_0[y/\llbracket a\rrbracket^M] = K_a[y/\llbracket a\rrbracket^M]$ & $K_a(y) = \{\llbracket a\rrbracket^M\}$	D1.$G[u/d]$, $G(u)$, df. K_a
d$'$. $H_0[y/\llbracket b\rrbracket^M] = K_b[y/\llbracket b\rrbracket^M]$ & $K_b(y) = \{\llbracket b\rrbracket^M\}$	D1.$G[u/d]$, $G(u)$, df. K_b
e. $H_0 \approx_y K_a$ & $(K_a(y) \subseteq H_0(x)$ & $\lvert K_a(y)\rvert = 1$	D1.$G \approx_u H$, (d), (b)
e$'$. $H_0 \approx_y K_b$ & $(K_b(y) \subseteq H_0(x)$ & $\lvert K_b(y)\rvert = 1$	D1.$G \approx_u H$, (d$'$), (b)
f. $\{\llbracket a\rrbracket^M\} \in \llbracket S\rrbracket^M$	(c), (e), $\rightarrow$
f$'$. $\{\llbracket b\rrbracket^M\} \in \llbracket S\rrbracket^M$	(c), (e$'$), $\rightarrow$

Hence **(9).**

(9) ⇒ (8)

Exercise.

e.g. (9) is true in any PPL-model M s.t.:

$\llbracket a\rrbracket^M = \text{a}$ $\llbracket b\rrbracket^M = \text{b}$ $\{\text{a}\}, \{\text{b}\} \in \llbracket S\rrbracket^M$

Quantifiers in Plural Logic

0. Quantifiers as 2-place predicates

Extend PPL with:

D1

- For all $A, B \in \mathcal{P}(D^M)$:
  $\langle A, B\rangle \in [\![all]\!]^M$ iff $A \subseteq B$
  $\langle A, B\rangle \in [\![sm]\!]^M$ iff $A \cap B \neq \{\}$
  $\langle A, B\rangle \in [\![most]\!]^M$ iff $|A \cap B| > |A - B|$
  $\langle A, B\rangle \in [\![n]\!]^M$ iff $|A \cap B| = \text{n}$ for n $\in \{0, 1, \ldots\}$
  ⋮

ABBREVIATIONS 3 (proper part, maximization, induced quantifiers)

- $(\alpha \subset \beta) := ((\alpha \subseteq \beta) \wedge \neg(\beta = \alpha))$ for $\alpha, \beta \in$ **Trm**
- $\mathbf{M}_u(\phi) := (\neg\exists u'(u \subset u' \wedge \phi[u/u']) \wedge \phi)$ $\phi \in$ **Wff**, no u' in ϕ
- $\mathbf{Q}u(\phi, \psi) := \exists u\exists u'(\mathbf{M}_u(\phi[u/u']) \wedge \mathbf{M}_u(u \subseteq u' \wedge \psi) \wedge Qu'u)$ $Q \in \{all, sm, most, \ldots\}$
  $\mathbf{Q}^\delta u(\phi, \psi) := \mathbf{Q}u(\delta_u(\phi), \delta_u(\psi))$

Then:

FACT 3′

- $G[\![(\alpha \subset \beta)]\!]^M = \top$ iff $G[\![\alpha]\!]^M \subset G[\![\beta]\!]^M$
- $G[\![\mathbf{M}_u(\phi)]\!]^M = \top$ iff $\neg(\exists H: G \approx_{u'} H \;\&\; H[\![u]\!]^M \subset H[\![u']\!]^M \;\&\; H[\![\phi[u/u']]\!]^M = \top) \;\&\; G[\![\phi]\!]^M = \top$

Some paradigm examples:

(1) All the women gathered in the square. ↓MON↑, collective VP

(1′) $\mathbf{all}\; x(\delta_x(Wx), Gx)$
$:= \exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge x' \subseteq x)$

(2) Every child is asleep. ↓MON↑, distributive VP

(2′) $\mathbf{all}^\delta\; x(Cx, Sx)$
$:= \exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(Cx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge \delta_x(Sx)) \wedge x' \subseteq x)$

(3) Most women gathered in the square. MON↑, collective VP

(3′) $\mathbf{most}\; x(\delta_x(Wx), Gx)$
$:= \exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge most\; x'x)$

(4) None of the women gathered in the square. ↓MON↓, collective VP

(4′) $\neg\; \mathbf{sm}\; x(\delta_x(Wx), Gx)$
$:= \neg\exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge sm\; x'x)$

(5) (Last year) two scientists (*jointly*) wrote an important paper. collective

(5′) $\mathbf{2}x(\delta_x(Sx), \exists y(Py \wedge \mathbf{1}y \wedge Wxy))$
$:= \exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(Sx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge \exists y(Py \wedge \mathbf{1}y \wedge Wxy) \wedge 2\; x'x)$

(6) (Last year) two scientists (*each*) wrote an important paper. distributive

(6′) $\mathbf{2}x(\delta_x(Sx), \delta_x(\exists y(Py \wedge \mathbf{1}y \wedge Wxy)))$
$:= \exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(Sx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge \delta_x(\exists y(Py \wedge \mathbf{1}y \wedge Wxy)) \wedge 2\; x'x)$

1. Details of (1)

(1) All the women gathered in the square. $\downarrow$MON$\uparrow$, collective VP

(1′) $\textbf{all}\ x(\delta_x(Wx), Gx)$

$:=\ \exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge x' \subseteq x)$

Truth condition:

1. $\models_M \exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge x' \subseteq x)$
2. $\mathbf{G}\llbracket \exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge x' \subseteq x)\rrbracket^M = \top$ — D3
3. $\exists H$: $\mathbf{G} \approx_x H$ & $H\llbracket \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge x' \subseteq x)\rrbracket^M = \top$ — D2.$\exists$
4. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — D2.$\exists$, rearr.
   & $K\llbracket \mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge x' \subseteq x\rrbracket^M = \top$
5. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — D2.$\wedge$
   & $K\llbracket \mathbf{M}_{x'}(\delta_{x'}(Wx'))\rrbracket^M = \top$ & $K\llbracket \mathbf{M}_x(x \subseteq x' \wedge Gx)\rrbracket^M = \top$
   & $K\llbracket x' \subseteq x\rrbracket^M = \top$
6. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — F3′.$\mathbf{M}$, D2.$\wedge$, $\subseteq$
   & $\neg(\exists L'$: $K \approx_y L'$ & $L'\llbracket x'\rrbracket^M \subset L'\llbracket y\rrbracket^M$ & $L'\llbracket \delta_y(Wy)\rrbracket^M = \top)$
   & $K\llbracket \delta_{x'}(Wx')\rrbracket^M = \top$
   & $\neg(\exists L$: $K \approx_y L$ & $L\llbracket x\rrbracket^M \subset L\llbracket y\rrbracket^M$ & $L\llbracket y\rrbracket^M \subseteq L\llbracket x'\rrbracket^M$ & $L\llbracket Gy\rrbracket^M = \top)$
   & $K\llbracket x\rrbracket^M \subseteq K\llbracket x'\rrbracket^M$ & $K\llbracket Gx\rrbracket^M = \top$
   & $K\llbracket x'\rrbracket^M \subseteq K\llbracket x\rrbracket^M$
7. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — simplify
   & $\neg(\exists L'$: $K \approx_y L'$ & $L'\llbracket x'\rrbracket^M \subset L'\llbracket y\rrbracket^M$ & $L'\llbracket \delta_y(Wy)\rrbracket^M = \top)$
   & $K\llbracket \delta_{x'}(Wx')\rrbracket^M = \top$
   & $\neg(\exists L$: $K \approx_y L$ & $L\llbracket x\rrbracket^M \subset L\llbracket y\rrbracket^M \subseteq L\llbracket x'\rrbracket^M$ & $L\llbracket Gy\rrbracket^M = \top)$
   & $K\llbracket x\rrbracket^M = K\llbracket x'\rrbracket^M$ & $K\llbracket Gx\rrbracket^M = \top$
8. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — D2.$\boldsymbol{R}$, $\boldsymbol{t}$, D1.$\approx_u$
   & $\neg(\exists L'$: $K \approx_y L'$ & $K(x') \subset L'(y)$ — A2.δ_u, F1′.$\forall$
   & $\forall L''$: $L' \approx_z L''$ & $L''\llbracket z \subseteq y \wedge \mathbf{1}z\rrbracket^M = \top \rightarrow L''\llbracket Wz\rrbracket^M = \top)$
   & $\forall K'$: $K \approx_z K'$ & $K'\llbracket z \subseteq x' \wedge \mathbf{1}z\rrbracket^M = \top \rightarrow K'\llbracket Wz\rrbracket^M = \top$
   & $\neg(\exists L$: $K \approx_y L$ & $K(x) \subset L(y) \subseteq K(x')$ & $L(y) \in \llbracket G\rrbracket^M)$
   & $K(x) = K(x')$ & $K(x) \in \llbracket G\rrbracket^M$
9. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — D2.$\wedge$, $\subseteq$, $\boldsymbol{R}$, $\boldsymbol{t}$
   & $\neg(\exists L'$: $K \approx_y L'$ & $K(x') \subset L'(y)$ — F2′.$\mathbf{1}$
   & $\forall L''$: $L' \approx_z L''$ & $L''(z) \subseteq L'(y)$ & $|L''(z)| = 1 \rightarrow L''(z) \in \llbracket W\rrbracket^M)$
   & $\forall K'$: $K \approx_z K'$ & $K'(z) \subseteq K(x')$ & $|K'(z)| = 1 \rightarrow K'(z) \in \llbracket W\rrbracket^M$
   & $\neg(\exists L$: $K \approx_y L$ & $K(x) \subset L(y) \subseteq K(x')$ & $L(y) \in \llbracket G\rrbracket^M)$
   & $K(x) = K(x')$ & $K(x) \in \llbracket G\rrbracket^M$
10. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — rearrange
   & $\forall K'$: $K \approx_z K'$ & $K'(z) \subseteq K(x')$ & $|K'(z)| = 1 \rightarrow K'(z) \in \llbracket W\rrbracket^M$
   & $K(x) = K(x')$ & $K(x) \in \llbracket G\rrbracket^M$
   & $\neg(\exists L'$: $K \approx_y L'$ & $K(x') \subset L'(y)$
   & $\forall L''$: $L' \approx_z L''$ & $L''(z) \subseteq L'(y)$ & $|L''(z)| = 1 \rightarrow L''(z) \in \llbracket W\rrbracket^M)$
   & $\neg(\exists L$: $K \approx_y L$ & $K(x) \subset L(y) \subseteq K(x')$ & $L(y) \in \llbracket G\rrbracket^M)$

11. $\exists H, K$: $\mathbf{G} \approx_x H \;\&\; H \approx_{x'} K$
$\&\; \forall d \in D^M$: $\{d\} \subseteq K(x') \rightarrow \{d\} \in [\![W]\!]^M$ — eliminate $\forall K'$
$\&\; K(x) = K(x') \;\&\; K(x) \in [\![G]\!]^M$
$\&\; \neg(\exists L'$: $K \approx_y L' \;\&\; K(x') \subset L'(y)$
$\&\; \forall d \in D^M$: $\{d\} \subseteq L'(y) \rightarrow \{d\} \in [\![W]\!]^M)$ — eliminate $\forall L''$
$\&\; \neg(\exists L$: $K \approx_y L \;\&\; K(x) \subset L(y) \subseteq K(x') \;\&\; L(y) \in [\![G]\!]^M)$

12. $\exists H, K$: $\mathbf{G} \approx_x H \;\&\; H \approx_{x'} K$
$\&\; \forall d$: $d \in K(x') \rightarrow \{d\} \in [\![W]\!]^M$ — simplify
$\&\; K(x) = K(x') \;\&\; K(x) \in [\![G]\!]^M$
$\&\; \neg(\exists L'$: $K \approx_y L' \;\&\; K(x') \subset L'(y)$
$\&\; \forall d$: $d \in L'(y) \rightarrow \{d\} \in [\![W]\!]^M)$ — simplify
$\&\; \neg(\exists L$: $K \approx_y L \;\&\; K(x) \subset L(y) \subseteq K(x') \;\&\; L(y) \in [\![G]\!]^M)$

13. $\exists A, A' \subseteq D^M$: — eliminate $\exists H, K$
$\forall d$: $d \in A' \rightarrow \{d\} \in [\![W]\!]^M$
$\&\; A = A' \;\&\; A \in [\![G]\!]^M$
$\&\; \neg(\exists B' \subseteq D^M$: $A' \subset B' \;\&\; \forall d$: $d \in B' \rightarrow \{d\} \in [\![W]\!]^M)$ — eliminate $\exists L', L$
$\&\; \neg(\exists B \subseteq D^M$: $A \subset B \subseteq A' \;\&\; B \in [\![G]\!]^M)$

14. $\exists A \subseteq D^M$: — eliminate A'
$\forall d$: $d \in A \rightarrow \{d\} \in [\![W]\!]^M$
$\&\; A \in [\![G]\!]^M$
$\&\; \neg(\exists B'$: $A \subset B' \subseteq D^M \;\&\; \forall d$: $d \in B' \rightarrow \{d\} \in [\![W]\!]^M)$ — simplify

Sample models:	Witness for A	Truth values	
• $M_1 = \langle\{a, b, c, d\}, [\![\cdot]\!]^M\rangle$	$A := \{a, b, c\}$	predicted:	$\models_{M1}$ (1′)
$[\![W]\!]^{M1} = \{\{a\}, \{b\}, \{c\}\}$		MB intuition:	$\models_{M1}$ (1)
$[\![G]\!]^{M1} = \{X \subseteq \{a, b, c\}: \lvert X\rvert \geq 2\}$			✓
• $M_2 = \langle\{a, b, c, d\}, [\![\cdot]\!]^M\rangle$	none	predicted:	$\not\models_{M2}$ (1′)
$[\![W]\!]^{M2} = \{\{a\}, \{b\}, \{c\}\}$		MB intuition:	$\not\models_{M2}$ (1)
$[\![G]\!]^{M2} = \{X \subseteq \{a, b, d\}: \lvert X\rvert \geq 2\}$			✓
• $M_3 = \langle\{a, b, c, d\}, [\![\cdot]\!]^M\rangle$	$A := \{a, b, c\}$	predicted:	$\models_{M3}$ (1′)
$[\![W]\!]^{M3} = \{\{a\}, \{b\}, \{c\}\}$		MB intuition:	$\models_{M3}$ (1′)
$[\![G]\!]^{M3} = \{X \subseteq \{a, b, c, d\}: \lvert X\rvert \geq 2\}$			✓
• $M_4 = \langle\{a, b, c, d\}, [\![\cdot]\!]^M\rangle$	none	predicted:	$\not\models_{M4}$ (1′)
$[\![W]\!]^{M4} = \{\{a\}, \{b\}, \{c\}\}$		MB intuition:	$\models_{M4}$ (1′)
$[\![G]\!]^{M4} = \{\{a, b, c, d\}\}$			**Oops!**

Solution: constrain $[\![\cdot]\!]^M$ for collective predicates

2. Details of (2)

(2) Every child is asleep. ↓MON↑, distributive VP

(2′) $\mathbf{all}^{\delta}\, x(Cx, Sx)$

$:= \exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Cx')) \wedge \mathbf{M}_{x}(x \subseteq x' \wedge \delta_{x}(Sx)) \wedge x' \subseteq x)$

Truth condition:

1. $\models_M \exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Cx')) \wedge \mathbf{M}_{x}(x \subseteq x' \wedge \delta_{x}(Sx)) \wedge x' \subseteq x)$
2. $\mathbf{G}\llbracket \exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Cx')) \wedge \mathbf{M}_{x}(x \subseteq x' \wedge \delta_{x}(Sx)) \wedge x' \subseteq x)\rrbracket^M = \top$ — D3
3. $\exists H$: $\mathbf{G} \approx_x H$ & $H\llbracket \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Cx')) \wedge \mathbf{M}_{x}(x \subseteq x' \wedge \delta_{x}(Sx)) \wedge x' \subseteq x)\rrbracket^M = \top$ — D2.∃
4. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — D2.∃, rearr.
   & $K\llbracket \mathbf{M}_{x'}(\delta_{x'}(Cx')) \wedge \mathbf{M}_{x}(x \subseteq x' \wedge \delta_{x}(Sx)) \wedge x' \subseteq x\rrbracket^M = \top$
5. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — D2.∧
   & $K\llbracket \mathbf{M}_{x'}(\delta_{x'}(Cx'))\rrbracket^M = \top$ & $K\llbracket \mathbf{M}_{x}(x \subseteq x' \wedge \delta_{x}(Sx))\rrbracket^M = \top$
   & $K\llbracket x' \subseteq x\rrbracket^M = \top$
6. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — F3′.**M**, D2.∧, ⊆
   & $\neg(\exists L'$: $K \approx_y L'$ & $L'\llbracket x'\rrbracket^M \subset L'\llbracket y\rrbracket^M$ & $L'\llbracket \delta_y(Cy)\rrbracket^M = \top)$
   & $K\llbracket \delta_{x'}(Cx')\rrbracket^M = \top$
   & $\neg(\exists L$: $K \approx_y L$ & $L\llbracket x\rrbracket^M \subset L\llbracket y\rrbracket^M$ & $L\llbracket y\rrbracket^M \subseteq L\llbracket x'\rrbracket^M$ & $L\llbracket \delta_y(Sy)\rrbracket^M = \top)$
   & $K\llbracket x\rrbracket^M \subseteq K\llbracket x'\rrbracket^M$ & $K\llbracket \delta_x(Sx)\rrbracket^M = \top$
   & $K\llbracket x'\rrbracket^M \subseteq K\llbracket x\rrbracket^M$
7. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — simplify, rearr.
   & $K\llbracket x\rrbracket^M = K\llbracket x'\rrbracket^M$
   & $K\llbracket \delta_{x'}(Cx')\rrbracket^M = \top$
   & $K\llbracket \delta_x(Sx)\rrbracket^M = \top$
   & $\neg(\exists L'$: $K \approx_y L'$ & $L'\llbracket x'\rrbracket^M \subset L'\llbracket y\rrbracket^M$ & $L'\llbracket \delta_y(Cy)\rrbracket^M = \top)$
   & $\neg(\exists L$: $K \approx_y L$ & $L\llbracket x\rrbracket^M \subset L\llbracket y\rrbracket^M \subseteq L\llbracket x'\rrbracket^M$ & $L\llbracket \delta_y(Sy)\rrbracket^M = \top)$
8. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — D2.$\boldsymbol{R}$, $\boldsymbol{t}$, D1.$\approx_u$
   & $K(x) = K(x')$ — A2.δ_u, F1′.∀
   & $\forall L'$: $K \approx_z L'$ & $L'\llbracket z \subseteq x' \wedge \mathbf{1}z\rrbracket^M = \top \rightarrow L'\llbracket Cz\rrbracket^M = \top$
   & $\forall L$: $K \approx_z L$ & $L\llbracket z \subseteq x \wedge \mathbf{1}z\rrbracket^M = \top \rightarrow L\llbracket Sz\rrbracket^M = \top$
   & $\neg(\exists M'$: $K \approx_y M'$ & $K(x') \subset M'(y)$
   & $\forall M''$: $M' \approx_z M''$ & $M''\llbracket z \subseteq y \wedge \mathbf{1}z\rrbracket^M = \top \rightarrow M''\llbracket Cz\rrbracket^M = \top)$
   & $\neg(\exists M$: $K \approx_y M$ & $K(x) \subset M(y) \subseteq K(x')$
   & $\forall M''$: $M \approx_z M''$ & $M''\llbracket z \subseteq y \wedge \mathbf{1}z\rrbracket^M = \top \rightarrow M''\llbracket Sz\rrbracket^M = \top)$
9. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — simplify
   & $K(x) = K(x')$
   & $\forall L'$: $K \approx_z L'$ & $L'\llbracket z \subseteq x' \wedge \mathbf{1}z\rrbracket^M = \top \rightarrow L'\llbracket Cz\rrbracket^M = \top$
   & $\forall L$: $K \approx_z L$ & $L\llbracket z \subseteq x \wedge \mathbf{1}z\rrbracket^M = \top \rightarrow L\llbracket Sz\rrbracket^M = \top$
   & $\neg(\exists M'$: $K \approx_y M'$ & $K(x') \subset M'(y)$
   & $\forall M''$: $M' \approx_z M''$ & $M''\llbracket z \subseteq y \wedge \mathbf{1}z\rrbracket^M = \top \rightarrow M''\llbracket Cz\rrbracket^M = \top)$

10. $\exists H, K$: $\mathbf{G} \approx_x H \;\&\; H \approx_{x'} K$ D2.$\wedge$, $\subseteq$, $\boldsymbol{R}$, $\boldsymbol{t}$

$\quad \& \; K(x) = K(x')$ F2$'$.**1**

$\quad \& \; \forall L'$: $K \approx_z L' \;\&\; L'(z) \subseteq L'(x') \;\&\; |L'(z)| = 1 \rightarrow L'(z) \in [\![C]\!]^M$

$\quad \& \; \forall L$: $K \approx_z L \;\&\; L(z) \subseteq L(x) \;\&\; |L(z)| = 1 \rightarrow L(z) \in [\![S]\!]^M$

$\quad \& \; \neg(\exists M'$: $K \approx_y M' \;\&\; K(x') \subset M'(y)$

$\quad\quad \& \; \forall M''$: $M' \approx_z M'' \;\&\; M''(z) \subseteq M''(y) \;\&\; |M''(z)| = 1 \rightarrow M''(z) \in [\![C]\!]^M)$

11. $\exists A, A' \subseteq D^M$: simplify

$\quad A = A'$

$\quad \& \; \forall d$: $d \in A' \rightarrow \{d\} \in [\![C]\!]^M$

$\quad \& \; \forall d$: $d \in A \rightarrow \{d\} \in [\![S]\!]^M$

$\quad \& \; \neg(\exists B'$: $A' \subset B' \subseteq D^M \;\&\; \forall d$: $d \in B' \rightarrow \{d\} \in [\![C]\!]^M)$

12. $\exists A \subseteq D^M$: eliminate A'

$\quad \forall d$: $d \in A \rightarrow \{d\} \in [\![C]\!]^M \;\&\; \{d\} \in [\![S]\!]^M$

$\quad \& \; \neg(\exists B'$: $A \subset B' \subseteq D^M \;\&\; \forall d$: $d \in B' \rightarrow \{d\} \in [\![C]\!]^M)$

Sample models:	Witness for A	Truth values	
• $M_1 = \langle \{a, b, c, d\}, [\![\cdot]\!]^M \rangle$	$A := \{a, b, c\}$	predicted:	$\models_{M1}$ (2$'$)
$[\![C]\!]^{M1} = \{\{a\}, \{b\}, \{c\}\}$		MB intuition:	$\models_{M1}$ (2)
$[\![S]\!]^{M1} = \{\{a\}, \{b\}, \{c\}\}$			✓
• $M_2 = \langle \{a, b, c, d\}, [\![\cdot]\!]^M \rangle$	none	predicted:	$\not\models_{M2}$ (2$'$)
$[\![C]\!]^{M2} = \{\{a\}, \{b\}, \{c\}\}$		MB intuition:	$\not\models_{M2}$ (2$'$)
$[\![S]\!]^{M2} = \{\{a\}, \{b\}, \{d\}\}$			✓
• $M_3 = \langle \{a, b, c, d\}, [\![\cdot]\!]^M \rangle$	$A := \{a, b, c\}$	predicted:	$\models_{M3}$ (2$'$)
$[\![C]\!]^{M3} = \{\{a\}, \{b\}, \{c\}\}$		MB intuition:	$\models_{M2}$ (2$'$)
$[\![S]\!]^{M3} = \{\{a\}, \{b\}, \{c\}, \{d\}\}$			✓

3. Details of (3)

(3) Most of the women gathered in the square. MON↑, collective VP

(3′) **most** $x(\delta_x(Wx), Gx)$

$:= \exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge most\ x'x)$

Truth condition:

1. $\models_M \exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge most\ x'x)$
2. $\mathbf{G}\llbracket \exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge most\ x'x)\rrbracket^M = \top$ — D3
3. $\exists H$: $\mathbf{G} \approx_x H$ & $H\llbracket \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge most\ x'x)\rrbracket^M = \top$ — D2.∃
4. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — D2.∃, rearr.
   & $K\llbracket \mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge most\ x'x\rrbracket^M = \top$
5. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — D2.∧
   & $K\llbracket \mathbf{M}_{x'}(\delta_{x'}(Wx'))\rrbracket^M = \top$ & $K\llbracket \mathbf{M}_x(x \subseteq x' \wedge Gx)\rrbracket^M = \top$
   & $K\llbracket most\ x'x\rrbracket^M = \top$
6. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — F3′.**M**, D2.∧, ⊆
   & $\neg(\exists L'$: $K \approx_y L'$ & $L'\llbracket x'\rrbracket^M \subset L'\llbracket y\rrbracket^M$ & $L'\llbracket \delta_y(Wy)\rrbracket^M = \top)$
   & $K\llbracket \delta_{x'}(Wx')\rrbracket^M = \top$
   & $\neg(\exists L$: $K \approx_y L$ & $L\llbracket x\rrbracket^M \subset L\llbracket y\rrbracket^M$ & $L\llbracket y\rrbracket^M \subseteq L\llbracket x'\rrbracket^M$ & $L\llbracket Gy\rrbracket^M = \top)$
   & $K\llbracket x\rrbracket^M \subseteq K\llbracket x'\rrbracket^M$ & $K\llbracket Gx\rrbracket^M = \top$
   & $K\llbracket most\ x'x\rrbracket^M = \top$
7. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — simplify, rearr.
   & $K\llbracket most\ x'x\rrbracket^M = \top$
   & $K\llbracket \delta_{x'}(Wx')\rrbracket^M = \top$
   & $K\llbracket x\rrbracket^M \subseteq K\llbracket x'\rrbracket^M$ & $K\llbracket Gx\rrbracket^M = \top$
   & $\neg(\exists L'$: $K \approx_y L'$ & $L'\llbracket x'\rrbracket^M \subset L'\llbracket y\rrbracket^M$ & $L'\llbracket \delta_y(Wy)\rrbracket^M = \top)$
   & $\neg(\exists L$: $K \approx_y L$ & $L\llbracket x\rrbracket^M \subset L\llbracket y\rrbracket^M \subseteq L\llbracket x'\rrbracket^M$ & $L\llbracket Gy\rrbracket^M = \top)$
8. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — D2.***R***, ***t***, D1.$\approx_u$
   & $\langle K\llbracket x'\rrbracket^M, K\llbracket x\rrbracket^M\rangle \in \llbracket most\rrbracket^M$ — A2.δ_u, F1′.∀
   & $\forall K'$: $K \approx_z K'$ & $K'\llbracket z \subseteq x' \wedge \mathbf{1}z\rrbracket^M = \top \rightarrow K'\llbracket Wz\rrbracket^M = \top$
   & $K(x) = K(x')$ & $K(x) \in \llbracket G\rrbracket^M$
   & $\neg(\exists L'$: $K \approx_y L'$ & $K(x') \subset L'(y)$
      & $\forall L''$: $L' \approx_z L''$ & $L''\llbracket z \subseteq y \wedge \mathbf{1}z\rrbracket^M = \top \rightarrow L''\llbracket Wz\rrbracket^M = \top)$
   & $\neg(\exists L$: $K \approx_y L$ & $K(x) \subset L(y) \subseteq K(x')$ & $L(y) \in \llbracket G\rrbracket^M)$
9. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ — D2.∧, ⊆, ***R*** , ***t***
   & $\langle K(x'), K(x)\rangle \in \llbracket most\rrbracket^M$ — F2′.**1**, D1.$\approx_u$
   & $\forall K'$: $K \approx_z K'$ & $K'(z) \subseteq K(x')$ & $|K'(z)| = 1 \rightarrow K'(z) \in \llbracket W\rrbracket^M$
   & $K(x) \subseteq K(x')$ & $K(x) \in \llbracket G\rrbracket^M$
   & $\neg(\exists L'$: $K \approx_y L'$ & $K(x') \subset L'(y)$
      & $\forall L''$: $L' \approx_z L''$ & $L''(z) \subseteq L'(y)$ & $|L''(z)| = 1 \rightarrow L''(z) \in \llbracket W\rrbracket^M)$
   & $\neg(\exists L$: $K \approx_y L$ & $K(x) \subset L(y) \subseteq K(x')$ & $L(y) \in \llbracket G\rrbracket^M)$

10. $\exists A, A' \subseteq D^M$: $[\![most]\!]^M$, simplify

$$|A' \cap A| > |A' - A|$$
$$\& \ \forall d: d \in A' \to \{d\} \in [\![W]\!]^M$$
$$\& \ A \subseteq A' \ \& \ A \in [\![G]\!]^M$$
$$\& \ \neg(\exists B': A' \subset B' \subseteq D^M \ \& \ \forall d: d \in B' \to \{d\} \in [\![W]\!]^M)$$
$$\& \ \neg(\exists B: A \subset B \subseteq A' \ \& \ B \in [\![G]\!]^M)$$

Sample models:	Witnesses for A', A	Truth values	
• $M_1 = \langle\{a, b, c, d\}, [\![\cdot]\!]^M\rangle$	$A' := \{a, b, c\}$	predicted:	$\models_{M1}$ (3′)
$[\![W]\!]^{M1} = \{\{a\}, \{b\}, \{c\}\}$	$A := \{a, b, c\}$	MB intuition:	$\models_{M1}$ (3)
$[\![G]\!]^{M1} = \{X \subseteq \{a, b, c\}: \lvert X\rvert \geq 2\}$			✓
• $M_2 = \langle\{a, b, c, d\}, [\![\cdot]\!]^M\rangle$	$A' := \{a, b, c\}$	predicted:	$\models_{M2}$ (3′)
$[\![W]\!]^{M2} = \{\{a\}, \{b\}, \{c\}\}$	$A := \{a, b\}$	MB intuition:	$\models_{M2}$ (3)
$[\![G]\!]^{M2} = \{X \subseteq \{a, b, d\}: \lvert X\rvert \geq 2\}$			✓
• $M_5 = \langle\{a, b, c, d, e\}, [\![\cdot]\!]^M\rangle$	$A' := \{a, b, c\}$	predicted:	$\not\models_{M5}$ (3′)
$[\![W]\!]^{M5} = \{\{a\}, \{b\}, \{c\}\}$	none for A	MB intuition:	$\not\models_{M5}$ (3)
$[\![G]\!]^{M5} = \{X \subseteq \{a, d, e\}: \lvert X\rvert \geq 2\}$			✓

4. Details of (4)

(4) None of the women gathered in the square. $\downarrow$MON$\downarrow$, collective VP

(4′) $\neg\ \mathbf{sm}\ x(\delta_x(Wx), Gx)$

$:= \neg\exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge sm\ x'x)$

Truth condition:

1. $\models_M \neg\exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge sm\ x'x)$
2. $\mathbf{G}\llbracket \exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge sm\ x'x)\rrbracket^M = \bot$ D3, D2.¬
3. $\neg\exists H$: $\mathbf{G} \approx_x H\ \&\ H\llbracket \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge sm\ x'x)\rrbracket^M = \top$ D2.∃
4. $\neg\exists H, K$: $\mathbf{G} \approx_x H\ \&\ H \approx_{x'} K$ D2.∃, rearr.
   $\&\ K\llbracket \mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge sm\ x'x\rrbracket^M = \top$
5. $\neg\exists H, K$: $\mathbf{G} \approx_x H\ \&\ H \approx_{x'} K$ D2.∧
   $\&\ K\llbracket \mathbf{M}_{x'}(\delta_{x'}(Wx'))\rrbracket^M = \top\ \&\ K\llbracket \mathbf{M}_x(x \subseteq x' \wedge Gx)\rrbracket^M = \top$
   $\&\ K\llbracket sm\ x'x\rrbracket^M = \top$
6. $\neg\exists H, K$: $\mathbf{G} \approx_x H\ \&\ H \approx_{x'} K$ F3′.**M**, D2.∧, ⊆
   $\&\ \neg(\exists L'$: $K \approx_y L'\ \&\ L'\llbracket x'\rrbracket^M \subset L'\llbracket y\rrbracket^M\ \&\ L'\llbracket\ \delta_y(Wy)\rrbracket^M = \top)$
   $\&\ K\llbracket \delta_{x'}(Wx')\rrbracket^M = \top$
   $\&\ \neg(\exists L$: $K \approx_y L\ \&\ L\llbracket x\rrbracket^M \subset L\llbracket y\rrbracket^M\ \&\ L\llbracket y\rrbracket^M \subseteq L\llbracket x'\rrbracket^M\ \&\ L\llbracket Gy\rrbracket^M = \top)$
   $\&\ K\llbracket x\rrbracket^M \subseteq K\llbracket x'\rrbracket^M\ \&\ K\llbracket Gx\rrbracket^M = \top$
   $\&\ K\llbracket sm\ x'x\rrbracket^M = \top$
7. $\neg\exists H, K$: $\mathbf{G} \approx_x H\ \&\ H \approx_{x'} K$ simplify, rearr.
   $\&\ K\llbracket sm\ x'x\rrbracket^M = \top$
   $\&\ K\llbracket \delta_{x'}(Wx')\rrbracket^M = \top$
   $\&\ K\llbracket x\rrbracket^M \subseteq K\llbracket x'\rrbracket^M\ \&\ K\llbracket Gx\rrbracket^M = \top$
   $\&\ \neg(\exists L'$: $K \approx_y L'\ \&\ L'\llbracket x'\rrbracket^M \subset L'\llbracket y\rrbracket^M\ \&\ L'\llbracket\ \delta_y(Wy)\rrbracket^M = \top)$
   $\&\ \neg(\exists L$: $K \approx_y L\ \&\ L\llbracket x\rrbracket^M \subset L\llbracket y\rrbracket^M \subseteq L\llbracket x'\rrbracket^M\ \&\ L\llbracket Gy\rrbracket^M = \top)$
8. $\neg\exists H, K$: $\mathbf{G} \approx_x H\ \&\ H \approx_{x'} K$ D2.***R***, ***t***, D1.$\approx_u$
   $\&\ \langle K\llbracket x'\rrbracket^M, K\llbracket x\rrbracket^M\rangle \in \llbracket sm\rrbracket^M$ A2.δ_u, F1′.∀
   $\&\ \forall K'$: $K \approx_z K'\ \&\ K'\llbracket z \subseteq x' \wedge \mathbf{1}z\rrbracket^M = \top \rightarrow K\llbracket Wz\rrbracket^M = \top$
   $\&\ K(x) = K(x')\ \&\ K(x) \in \llbracket G\rrbracket^M$
   $\&\ \neg(\exists L'$: $K \approx_y L'\ \&\ K(x') \subset L'(y)$
   $\&\ \forall L''$: $L' \approx_z L''\ \&\ L''\llbracket z \subseteq y \wedge \mathbf{1}z\rrbracket^M = \top \rightarrow L''\llbracket Wz\rrbracket^M = \top)$
   $\&\ \neg(\exists L$: $K \approx_y L\ \&\ K(x) \subset L(y) \subseteq K(x')\ \&\ L(y) \in \llbracket G\rrbracket^M)$
9. $\neg\exists H, K$: $\mathbf{G} \approx_x H\ \&\ H \approx_{x'} K$ D2.∧, ⊆, ***R*** , ***t***
   $\&\ \langle K(x'), K(x)\rangle \in \llbracket sm\rrbracket^M$ F2′.**1**, D1.$\approx_u$
   $\&\ \forall K'$: $K \approx_z K'\ \&\ K'(z) \subseteq K(x')\ \&\ |K'(z)| = 1 \rightarrow K'(z) \in \llbracket W\rrbracket^M$
   $\&\ K(x) \subseteq K(x')\ \&\ K(x) \in \llbracket G\rrbracket^M$
   $\&\ \neg(\exists L'$: $K \approx_y L'\ \&\ K(x') \subset L'(y)$
   $\&\ \forall L''$: $L' \approx_z L''\ \&\ L''(z) \subseteq L'(y)\ \&\ |L''(z)| = 1 \rightarrow L''(z) \in \llbracket W\rrbracket^M)$
   $\&\ \neg(\exists L$: $K \approx_y L\ \&\ K(x) \subset L(y) \subseteq K(x')\ \&\ L(y) \in \llbracket G\rrbracket^M)$

10. $\neg\exists A, A' \subseteq D^M$: $[\![sm]\!]^M$, simplify

$A' \cap A \neq \{\}$
$\& \; \forall d: d \in A' \rightarrow \{d\} \in [\![W]\!]^M$
$\& \; A \subseteq A' \; \& \; A \in [\![G]\!]^M$
$\& \; \neg(\exists B': A' \subset B' \subseteq D^M \; \& \; \forall d: d \in B' \rightarrow \{d\} \in [\![W]\!]^M)$
$\& \; \neg(\exists B: A \subset B \subseteq A' \; \& \; B \in [\![G]\!]^M)$

<u>Sample models</u>:	<u>Counterexamples for A', A</u>	<u>Truth values</u>	
• $M_1 = \langle\{a, b, c, d\}, [\![\cdot]\!]^M\rangle$	$A' := \{a, b, c\}$	predicted:	$\not\models_{M1}$ (4′)
$[\![W]\!]^{M1} = \{\{a\}, \{b\}, \{c\}\}$	$A := \{a, b, c\}$	MB intuition:	$\not\models_{M1}$ (4)
$[\![G]\!]^{M1} = \{X \subseteq \{a, b, c\}: \lvert X\rvert \geq 2\}$			✓
• $M_2 = \langle\{a, b, c, d\}, [\![\cdot]\!]^M\rangle$	$A' := \{a, b, c\}$	predicted:	$\not\models_{M2}$ (4′)
$[\![W]\!]^{M2} = \{\{a\}, \{b\}, \{c\}\}$	$A := \{a, b\}$	MB intuition:	$\not\models_{M2}$ (4)
$[\![G]\!]^{M2} = \{X \subseteq \{a, b, d\}: \lvert X\rvert \geq 2\}$			✓
• $M_3 = \langle\{a, b, c, d\}, [\![\cdot]\!]^M\rangle$	$A' := \{a, b, c\}$	predicted:	$\not\models_{M3}$ (4′)
$[\![W]\!]^{M3} = \{\{a\}, \{b\}, \{c\}\}$	$A := \{a, b, c\}$	MB intuition:	$\not\models_{M3}$ (4)
$[\![G]\!]^{M3} = \{X \subseteq \{a, b, c, d\}: \lvert X\rvert \geq 2\}$			✓
• $M_4 = \langle\{a, b, c, d, e\}, [\![\cdot]\!]^M\rangle$	$A' := \{a, b, c\}$	predicted:	$\models_{M6}$ (4′)
$[\![W]\!]^{M4} = \{\{a\}, \{b\}, \{c\}\}$	none for A	MB intuition:	$\models_{M6}$ (4)
$[\![G]\!]^{M4} = \{X \subseteq \{d, e\}: \lvert X\rvert \geq 2\}$			✓
• $M_5 = \langle\{a, b, c, d, e\}, [\![\cdot]\!]^M\rangle$	$A' := \{a, b, c\}$	predicted:	$\models_{M5}$ (4′)
$[\![W]\!]^{M5} = \{\{a\}, \{b\}, \{c\}\}$	none for A	MB intuition:	Logician Speak
$[\![G]\!]^{M5} = \{X \subseteq \{a, d, e\}: \lvert X\rvert \geq 2\}$		see ***Remark*** below	

Remark:
In the situation of M_5 (4), with *gather*, sounds odd to me--more like LogicianSpeak than English.

(4) None of the women *gathered* in the square.

(4′) $\neg\, \mathbf{sm}\, x(\delta_x(Wx), Gx)$ predicted: $\models_{M5}$ (4′)
$:= \; \neg\exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge sm\; x'x)$ MB intuition: Logician Speak

To the extent that (4) is acceptable in this situation, I am inclined to interpret it distributively, as Intuitively, (4δ) is clearly false in M_5—which is what van den Berg would predict if (4^D) is assigned the distributive translation (4δ′):

(4δ) None of the women *came to the gathering* in the square.

(4δ′) $\neg\, \mathbf{sm}\, x(\delta_x(Wx), \delta_x(CGx))$ predicted: $\not\models_{M5}$ (4δ′)
$:= \; \neg\exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge \delta_x(CGx)) \wedge sm\; x'x)$ MB intuition: $\not\models_{M5}$ (4δ)

Plural Dynamic Logic

0. Plural Predicate Logic (PPL)

DEFINITION 0 (PPL models)
A PPL-model is a structure $M = \langle D^M, [\![\cdot]\!]^M\rangle$ such that D^M is a non-empty set and

- for all $\alpha \in \mathbf{Prd}^n$, $[\![\alpha]\!]^M \subseteq (\mathcal{P}(D^M))^n$
- for all $A, B \in \mathcal{P}(D^M)$:
  $\langle A, B\rangle \in [\![all]\!]^M$ iff $A \subseteq B$ $\quad$ $\langle A, B\rangle \in [\![sm]\!]^M$ iff $A \cap B \neq \{\}$
  $\langle A, B\rangle \in [\![most]\!]^M$ iff $|A \cap B| > |A - B|$ $\quad$ $\langle A, B\rangle \in [\![\mathbf{n}]\!]^M$ iff $|A \cap B| = \text{n}$, for n $\in \{2, \ldots\}$

DEFINITION 1 (PPL assignments)

- An *M-assignment* is a function g such that Dom $g \subseteq \mathbf{Var}$ and Ran $g \subseteq D^M$.
  The set of M-assignments is denoted by $\mathbf{G}$. For any $g, h \in \mathbf{G}$, $u \in \mathbf{Var}$, $d \in D^M$,
  $g[u/d] = h$ iff Dom $g \cup \{u\}$ = Dom h & $h(u) = d$ & $\forall u' \in (\text{Dom } g - \{u\})$: $h(u') = g(u')$
  $g \approx_u h$ iff $\exists d \in D$: $g[u/d] = h$
- An *M-information state* is a set of M-assignments. For any $G, H \subseteq \mathbf{G}$, $u \in \mathbf{Var}$, $d \in D^M$,
  $G(u)$:= $\{d \in D^M | \exists g \in G: g(u) = d\}$
  $G[u/d]$:= $\{g[u/d]: g \in G\}$
  $G \approx_u H$ iff $\forall d \in D^M$: $G[u/d] = H[u/d]$

DEFINITION 2 (PPL-semantics)

$\subseteq$	$G[\![(x \subseteq y)]\!]^M$	$= \top$	iff	$G(x) \subseteq G(y)$
$\boldsymbol{R}$	$G[\![\alpha x_1 \ldots x_n]\!]^M$	$= \top$	iff	$\langle G(x_1), \ldots, G(x_n)\rangle \in [\![\alpha]\!]^M$
$\wedge$	$G[\![(\phi \wedge \psi)]\!]^M$	$= \top$,	iff	$G[\![\phi]\!]^M = \top$ & $G[\![\psi]\!]^M = \top$
$\neg$	$G[\![\neg\phi]\!]^M$	$= \top$,	iff	$G[\![\phi]\!]^M = \bot$
$\exists$	$G[\![\exists u\phi]\!]^M$	$= \top$,	iff	$\exists H$: $G \approx_u H$ & $H[\![\phi]\!]^M = \top$

DEFINITION 3 (PPL truth).

- ϕ is *true* in M, $\models_M \phi$, iff $\mathbf{G}[\![\phi]\!]^M = \top$

ABBREVIATIONS 1 (standard)

- $(\alpha \vee \beta)$:= $\neg(\neg\alpha \wedge \neg\beta)$
  $(\alpha \rightarrow \beta)$:= $\neg(\alpha \wedge \neg\beta)$
- $\forall u\phi$:= $\neg\exists u\neg\phi$
- $(\alpha = \beta)$:= $((\alpha \subseteq \beta) \wedge (\beta \subseteq \alpha))$
  $(\alpha \subset \beta)$:= $((\alpha \subseteq \beta) \wedge \neg(\alpha = \beta))$

ABBREVIATIONS 2 (zero, unit, distributivity, maximization, induced quantifiers)

- $\mathbf{0}u$:= $\forall u'(u' \subseteq u \rightarrow u' = u)$
  $\mathbf{1}u$:= $(\neg\mathbf{0}u \wedge \forall u'(u' \subseteq u \wedge \neg\mathbf{0}u' \rightarrow u' = u))$
- $\delta_u(\phi)$:= $\forall u'(u' \subseteq u \wedge \mathbf{1}u' \rightarrow \phi[u/u'])$
  $\mathbf{M}_u(\phi)$:= $(\neg\exists u'(u \subset u' \wedge \phi[u/u']) \wedge \phi)$
  $\mathbf{Q}u(\phi, \psi)$:= $\exists u\exists u'(\mathbf{M}_{u'}(\phi[u/u']) \wedge \mathbf{M}_u(u \subseteq u' \wedge \psi) \wedge Quu')$

1. Plural Dynamic Logic (PDL, van den Berg 1994)

DEFINITION 0 (PDL models)
A PDL-model is a structure $M = \langle D^M, \llbracket\cdot\rrbracket^+, \llbracket\cdot\rrbracket^-\rangle$ such that D^M is a non-empty set and

- for all $\alpha \in \mathbf{Prd}^n$, $\llbracket\alpha\rrbracket^+ \subseteq (\mathcal{P}(D^M) - \{\emptyset\})^n$ & $\llbracket\alpha\rrbracket^- = (\mathcal{P}(D^M) - \{\emptyset\})^n - \llbracket\alpha\rrbracket^+$
- for all $A, B \in (\mathcal{P}(D^M) - \{\emptyset\})^n$:
  $\langle A, B\rangle \in \llbracket all\rrbracket^+$ iff $A \subseteq B$ $\langle A, B\rangle \in \llbracket sm\rrbracket^+$ iff $A \cap B \neq \{\}$
  $\langle A, B\rangle \in \llbracket most\rrbracket^+$ iff $|A \cap B| > |A - B|$ $\langle A, B\rangle \in \llbracket \mathbf{n}\rrbracket^+$ iff $|A \cap B| = \mathrm{n}$, for $\mathrm{n} \in \{2, \ldots\}$

DEFINITION 1 (PDL assignments)

- An *M-assignment* is a function g such that Dom $g \subseteq \mathbf{Var}$ and Ran $g \subseteq D^M$.
  The set of M-assignments is denoted by $\mathbf{G}$. For any $g, h \in \mathbf{G}$, $u \in \mathbf{Var}$, $d \in D^M$,
  $g[u/d] = h$ iff Dom $g \cup \{u\} =$ Dom h & $h(u) = d$ & $\forall u' \in (\text{Dom } g - \{u\})$: $h(u') = g(u')$
  $g \approx_u h$ iff $\exists d \in D$: $g[u/d] = h$
- An *M-information state* is a set of M-assignments. For any $G, H \subseteq \mathbf{G}$, $u \in \mathbf{Var}$, $d \in D^M$,
  $G(u)$:= $\{d \in D^M | \exists g \in G: g(u) = d\}$ $G_{u=\star}$:= $\{g \in G | u \notin \text{Dom } g\}$
  $G[u/d]$:= $\{g[u/d]: g \in G\}$ $G_{u=d}$:= $\{g \in G | g(u) = d\}$
  $G \approx_u H$ iff $\forall d \in D^M$: $G[u/d] = H[u/d]$ Λ := the $g \in \mathbf{G}$ s.t. Dom $g = \{\}$

DEFINITION 2 (PDL-semantics)

$\subseteq$ $G\llbracket(x \subseteq y)\rrbracket H$ $= \top$ iff $G = H$ & $G(x) \subseteq G(y)$
$= \bot$ iff $G = H$ & $\neg[G(x) \subseteq G(y)]$
$= \star$ otherwise

R $G\llbracket\alpha x_1 \ldots x_n\rrbracket H$ $= \top$ iff $G = H$ & $\langle G(x_1), \ldots, G(x_n)\rangle \in \llbracket\alpha\rrbracket^+$
$= \bot$ iff $G = H$ & $\langle G(x_1), \ldots, G(x_n)\rangle \in \llbracket\alpha\rrbracket^-$
$= \star$ otherwise

$\wedge$ $G\llbracket(\phi \wedge \psi)\rrbracket H$ $= \star$ iff $\forall K$: $G\llbracket\phi\rrbracket K = \star$ or $K\llbracket\psi\rrbracket H = \star$
$= \top$ iff $\exists K$: $G\llbracket\phi\rrbracket K = \top$ & $K\llbracket\psi\rrbracket H = \top$
$= \bot$ otherwise

$\neg$ $G\llbracket\neg\phi\rrbracket H$ $= \top$ iff $G\llbracket\phi\rrbracket H = \bot$ & $\neg\exists K$: $G\llbracket\phi\rrbracket K = \top$
$= \bot$ iff $G\llbracket\phi\rrbracket H = \top$
$= \star$ otherwise

ε $G\llbracket\varepsilon_u\rrbracket H$ $= \top$ iff $G \approx_u H$
$= \star$ otherwise

δ $G\llbracket\delta_x(\phi)\rrbracket H$ $= \star$ iff $G_{x=\star} \neq H_{x=\star}$ or $\exists d \in G(x)$: $G_{x=d}\llbracket\phi\rrbracket H_{x=d} = \star$
$= \top$ iff $G_{x=\star} = H_{x=\star}$ & $\forall d \in G(x)$: $G_{x=d}\llbracket\phi\rrbracket H_{x=d} = \top$
$= \bot$ otherwise

M $G\llbracket\mathbf{M}_x(\phi)\rrbracket H$ $= \top$ iff $G\llbracket\phi\rrbracket H = \top$ & $\neg\exists G', H'$: $G \approx_x G'$ & $G(x) \subset G'(x)$ & $G'\llbracket\phi\rrbracket H' = \top$
$= \bot$ iff $G\llbracket\phi\rrbracket H = \bot$ & $\neg\exists G', H'$: $G \approx_x G'$ & $G(x) \subset G'(x)$ & $G'\llbracket\phi\rrbracket H' = \bot$
$= \star$ otherwise

DEFINITION 3 (PDL truth).

- ϕ is *true* in M, $\models_M \phi$, iff $\exists H$: $\{\Lambda\}\llbracket\phi\rrbracket H = \top$

2. Quantifiers in PPL and PDL

(1) Most of the women greeted each other. MON↑, collective VP

(1′) $\mathbf{most}\ x(\delta_x(Wx), Gx)$ PPL
$:=\ \exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge most\ x'x)$

(1″) $\mathbf{most}\ x(\delta_x(Wx), Gx)$ PDL
$:=\ \varepsilon_x \wedge \varepsilon_{x'} \wedge\ \mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge most\ x'x$

Truth condition for PPL-translation (1′):

1. $\models_M \exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge most\ x'x)$
2. $\mathbf{G}\llbracket \exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge most\ x'x)\rrbracket^M = \top$ D3
3. $\exists H$: $\mathbf{G} \approx_x H$ & $H\llbracket \exists x'(\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge most\ x'x)\rrbracket^M = \top$ D2.∃
4. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ D2.∃, rearr.
   & $K\llbracket \mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge most\ x'x\rrbracket^M = \top$
5. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ D2.∧
   & $K\llbracket \mathbf{M}_{x'}(\delta_{x'}(Wx'))\rrbracket^M = \top$ & $K\llbracket \mathbf{M}_x(x \subseteq x' \wedge Gx)\rrbracket^M = \top$
   & $K\llbracket most\ x'x\rrbracket^M = \top$
6. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ F3′.**M**, D2.∧, ⊆
   & $\neg(\exists L'$: $K \approx_y L'$ & $L'\llbracket x'\rrbracket^M \subset L'\llbracket y\rrbracket^M$ & $L'\llbracket\ \delta_y(Wy)\rrbracket^M = \top)$
   & $K\llbracket \delta_{x'}(Wx')\rrbracket^M = \top$
   & $\neg(\exists L$: $K \approx_y L$ & $L\llbracket x\rrbracket^M \subset L\llbracket y\rrbracket^M$ & $L\llbracket y\rrbracket^M \subseteq L\llbracket x'\rrbracket^M$ & $L\llbracket Gy\rrbracket^M = \top)$
   & $K\llbracket x\rrbracket^M \subseteq K\llbracket x'\rrbracket^M$ & $K\llbracket Gx\rrbracket^M = \top$
   & $K\llbracket most\ x'x\rrbracket^M = \top$
7. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ simplify, rearr.
   & $K\llbracket most\ x'x\rrbracket^M = \top$
   & $K\llbracket \delta_{x'}(Wx')\rrbracket^M = \top$
   & $K\llbracket x\rrbracket^M \subseteq K\llbracket x'\rrbracket^M$ & $K\llbracket Gx\rrbracket^M = \top$
   & $\neg(\exists L'$: $K \approx_y L'$ & $L'\llbracket x'\rrbracket^M \subset L'\llbracket y\rrbracket^M$ & $L'\llbracket\ \delta_y(Wy)\rrbracket^M = \top)$
   & $\neg(\exists L$: $K \approx_y L$ & $L\llbracket x\rrbracket^M \subset L\llbracket y\rrbracket^M \subseteq L\llbracket x'\rrbracket^M$ & $L\llbracket Gy\rrbracket^M = \top)$
8. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ D2.$\boldsymbol{R}$, $\boldsymbol{t}$, D1.$\approx_u$
   & $\langle K\llbracket x'\rrbracket^M, K\llbracket x\rrbracket^M\rangle \in \llbracket most\rrbracket^M$ A2.δ_u, F1′.∀
   & $\forall K'$: $K \approx_z K'$ & $K'\llbracket z \subseteq x' \wedge \mathbf{1}z\rrbracket^M = \top \rightarrow K'\llbracket Wz\rrbracket^M = \top$
   & $K(x) = K(x')$ & $K(x) \in \llbracket G\rrbracket^M$
   & $\neg(\exists L'$: $K \approx_y L'$ & $K(x') \subset L'(y)$
      & $\forall L''$: $L' \approx_z L''$ & $L''\llbracket z \subseteq y \wedge \mathbf{1}z\rrbracket^M = \top \rightarrow L''\llbracket Wz\rrbracket^M = \top)$
   & $\neg(\exists L$: $K \approx_y L$ & $K(x) \subset L(y) \subseteq K(x')$ & $L(y) \in \llbracket G\rrbracket^M)$
9. $\exists H, K$: $\mathbf{G} \approx_x H$ & $H \approx_{x'} K$ D2.∧, ⊆, $\boldsymbol{R}$, $\boldsymbol{t}$
   & $\langle K(x'), K(x)\rangle \in \llbracket most\rrbracket^M$ F2′.**1**, D1.$\approx_u$
   & $\forall K'$: $K \approx_z K'$ & $K'(z) \subseteq K(x')$ & $|K'(z)| = 1 \rightarrow K'(z) \in \llbracket W\rrbracket^M$
   & $K(x) \subseteq K(x')$ & $K(x) \in \llbracket G\rrbracket^M$
   & $\neg(\exists L'$: $K \approx_y L'$ & $K(x') \subset L'(y)$
      & $\forall L''$: $L' \approx_z L''$ & $L''(z) \subseteq L'(y)$ & $|L''(z)| = 1 \rightarrow L''(z) \in \llbracket W\rrbracket^M)$
   & $\neg(\exists L$: $K \approx_y L$ & $K(x) \subset L(y) \subseteq K(x')$ & $L(y) \in \llbracket G\rrbracket^M)$

10. $\exists A, A' \subseteq D^M$: $[\![most]\!]^M$, simplify

$$|A' \cap A| > |A' - A|$$
$$\& \ \forall d{:}\ d \in A' \rightarrow \{d\} \in [\![W]\!]^M$$
$$\& \ A \subseteq A' \ \& \ A \in [\![G]\!]^M$$
$$\& \ \neg(\exists B'{:}\ A' \subset B' \subseteq D^M \ \& \ \forall d{:}\ d \in B' \rightarrow \{d\} \in [\![W]\!]^M)$$
$$\& \ \neg(\exists B{:}\ A \subset B \subseteq A' \ \& \ B \in [\![G]\!]^M)$$

Sample models:	Witnesses for A', A	Truth values	
• $M_1 = \langle\{a, b, c, d\}, [\![\cdot]\!]^M\rangle$	$A' := \{a, b, c\}$	predicted:	$\models_{M1}$ (3′)
$[\![W]\!]^{M1} = \{\{a\}, \{b\}, \{c\}\}$	$A := \{a, b, c\}$	MB intuition:	$\models_{M1}$ (3)
$[\![G]\!]^{M1} = \{X \subseteq \{a, b, c\}: \lvert X\rvert \geq 2\}$			✓
• $M_2 = \langle\{a, b, c, d\}, [\![\cdot]\!]^M\rangle$	$A' := \{a, b, c\}$	predicted:	$\models_{M2}$ (3′)
$[\![W]\!]^{M2} = \{\{a\}, \{b\}, \{c\}\}$	$A := \{a, b\}$	MB intuition:	$\models_{M2}$ (3)
$[\![G]\!]^{M2} = \{X \subseteq \{a, b, d\}: \lvert X\rvert \geq 2\}$			✓
• $M_5 = \langle\{a, b, c, d, e\}, [\![\cdot]\!]^M\rangle$	$A' := \{a, b, c\}$	predicted:	$\not\models_{M5}$ (3′)
$[\![W]\!]^{M5} = \{\{a\}, \{b\}, \{c\}\}$	none for A	MB intuition:	$\not\models_{M5}$ (3)
$[\![G]\!]^{M5} = \{X \subseteq \{a, d, e\}: \lvert X\rvert \geq 2\}$			✓

Truth condition for PDL-translation (1″):

1. $\models_M \varepsilon_x \wedge \varepsilon_{x'} \wedge \mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge \mathit{most}\ x'x$
2. $\exists H$: $\{\Lambda\}[\![\varepsilon_x \wedge \varepsilon_{x'} \wedge \mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge \mathit{most}\ x'x]\!]H = \top$ D3
3. $\exists H, I$: $\{\Lambda\}[\![\varepsilon_x]\!]I = \top$ D2.$\wedge$
   & $I[\![\varepsilon_{x'} \wedge \mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge \mathit{most}\ x'x]\!]H = \top$

3′. $\exists H, I$: $\{\Lambda\} \approx_x I$ D2.ε
   & $I[\![\varepsilon_{x'} \wedge \mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge \mathit{most}\ x'x]\!]H = \top$

4. $\exists H, I$: $\{\Lambda\} \approx_x I$ D2.$\wedge$
   & $\exists J$: $I[\![\varepsilon_x]\!]J = \top$
      & $J[\![\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge \mathit{most}\ x'x]\!]H = \top$

4′. $\exists H, I, J$: $\{\Lambda\} \approx_x I\ \&\ I \approx_{x'} J$ D2.ε, rearrange
   & $J[\![\mathbf{M}_{x'}(\delta_{x'}(Wx')) \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge \mathit{most}\ x'x]\!]H = \top$

5. $\exists H, I, J, K, L$: $\{\Lambda\} \approx_x I\ \&\ I \approx_{x'} J$ D2.$\wedge$, rearrange
   & $J[\![\mathbf{M}_{x'}(\delta_{x'}(Wx'))]\!]K = \top$
   & $K[\![\mathbf{M}_x(x \subseteq x' \wedge Gx)]\!]L = \top$
   & $L[\![\mathit{most}\ x'x]\!]H = \top$
6. $\exists H, I, J, K, L$: $\{\Lambda\} \approx_x I\ \&\ I \approx_{x'} J$ D2.**M**, ***R***
   & $J[\![\delta_{x'}(Wx')]\!]K = \top$
   & $\neg\exists J', K'$: $J \approx_{x'} J'\ \&\ J(x') \subset J'(x')\ \&\ J'[\![\delta_{x'}(Wx')]\!]K' = \top$
   & $K[\![x \subseteq x' \wedge Gx]\!]L = \top$
   & $\neg\exists K'', L'$: $K \approx_x K''\ \&\ K(x) \subset K''(x)\ \&\ K[\![x \subseteq x' \wedge\ Gx]\!]L' = \top$
   & $L = H\ \&\ \langle L(x'), H(x)\rangle \in [\![\mathit{most}]\!]^+$

6′. $\exists H, I, J, K$: $\{\Lambda\} \approx_x I\ \&\ I \approx_{x'} J$ eliminate L
   & $J[\![\delta_{x'}(Wx')]\!]K = \top$ rename L'
   & $\neg\exists J', K'$: $J \approx_{x'} J'\ \&\ J(x') \subset J'(x')\ \&\ J'[\![\delta_{x'}(Wx')]\!]K' = \top$
   & $K[\![x \subseteq x' \wedge Gx]\!]H = \top$
   & $\neg\exists K'', H'$: $K \approx_x K''\ \&\ K(x) \subset K''(x)\ \&\ K[\![x \subseteq x' \wedge\ Gx]\!]H' = \top$
   & $\langle H(x'), H(x)\rangle \in [\![\mathit{most}]\!]^+$

6″. $\exists H, I, J, K, L$: $\{\Lambda\} \approx_x I\ \&\ I \approx_{x'} J$ D2.$\wedge$, rearr.
   & $J[\![\delta_{x'}(Wx')]\!]K = \top$
   & $\neg\exists J', K'$: $J \approx_{x'} J'\ \&\ J(x') \subset J'(x')\ \&\ J'[\![\delta_{x'}(Wx')]\!]K' = \top$
   & $K[\![x \subseteq x']\!]L = \top\ \&\ L[\![Gx]\!]H = \top$
   & $\neg\exists K'', L', H'$: $K \approx_x K''\ \&\ K(x) \subset K''(x)$
      & $K[\![x \subseteq x']\!]L' = \top\ \&\ L'[\![Gx]\!]H' = \top$
   & $\langle H(x'), H(x)\rangle \in [\![\mathit{most}]\!]^+$

7. $\exists H, I, J, K, L$: $\{\Lambda\} \approx_x I\ \&\ I \approx_{x'} J$ D2.$\subseteq$, ***R***
   & $\langle H(x'), H(x)\rangle \in [\![\mathit{most}]\!]^+$ rearrange
   & $J[\![\delta_{x'}(Wx')]\!]K = \top$
   & $K = L\ \&\ K(x) \subseteq L(x')\ \&\ L = H\ \&\ H(x) \in [\![G]\!]^+$
   & $\neg\exists J', K'$: $J \approx_{x'} J'\ \&\ J(x') \subset J'(x')\ \&\ J'[\![\delta_{x'}(Wx')]\!]K' = \top$
   & $\neg\exists K'', L', H'$: $K \approx_x K''\ \&\ K(x) \subset K''(x)$
      & $K'' = L'\ \&\ K'(x) \subseteq L'(x')\ \&\ L' = H'\ \&\ H'(x) \in [\![G]\!]^+$

7′. $\exists H, I, J$: $\{\Lambda\} \approx_x I \;\&\; I \approx_{x'} J$ — eliminate K, L

 $\&\ \langle H(x'), H(x)\rangle \in [\![most]\!]^+$

 $\&\ J[\![\delta_{x'}(Wx')]\!]H = \top$

 $\&\ H(x) \subseteq H(x') \;\&\; H(x) \in [\![G]\!]^+$

 $\&\ \neg\exists J', K'$: $J \approx_{x'} J' \;\&\; J(x') \subset J'(x') \;\&\; J[\![\delta_{x'}(Wx')]\!]K' = \top$

$\neg\exists H'$: $H \approx_x H' \;\&\; H(x) \subset H'(x) \subseteq H'(x') \;\&\; H'(x) \in [\![G]\!]^+$ — elim. K'', L', smpl.

8. $\exists H, I, J$: $\{\Lambda\} \approx_x I \;\&\; I \approx_{x'} J$ — D2.δ

 $\&\ \langle H(x'), H(x)\rangle \in [\![most]\!]^+$

 $\&\ J_{x'=\star} = H_{x'=\star} \;\&\; \forall d \in J(x')$: $J_{x'=d}[\![Wx']\!]H_{x'=d} = \top$

 $\&\ H(x) \subseteq H(x') \;\&\; H(x) \in [\![G]\!]^+$

 $\&\ \neg\exists J', K'$: $J \approx_{x'} J' \;\&\; J(x') \subset J'(x')$

  $\&\ J'_{x'=\star} = K'_{x'=\star} \;\&\; \forall d \in J'(x')$: $J'_{x'=d}[\![Wx']\!]K'_{x'=d} = \top$

 $\&\ \neg\exists H'$: $H \approx_x H' \;\&\; H(x) \subset H'(x) \subseteq H'(x') \;\&\; H'(x) \in [\![G]\!]^+$

9. $\exists H, I, J$: $\{\Lambda\} \approx_x I \;\&\; I \approx_{x'} J$ — D2.***R***, D1.$G_{x=\mathrm{d}}$

 $\&\ \langle H(x'), H(x)\rangle \in [\![most]\!]^+$

 $\&\ J_{x'=\star} = H_{x'=\star} \;\&\; \forall d \in J(x')$: $J_{x'=d} = H_{x'=d} \;\&\; \{\mathrm{d}\} \in [\![W]\!]^+$

 $\&\ H(x) \subseteq H(x') \;\&\; H(x) \in [\![G]\!]^+$

 $\&\ \neg\exists J', K'$: $J \approx_{x'} J' \;\&\; J(x') \subset J'(x')$

  $\&\ J'_{x'=\star} = K'_{x'=\star} \;\&\; \forall d \in J'(x')$: $J'_{x'=d} = K'_{x'=d} \;\&\; \{\mathrm{d}\} \in [\![W]\!]^+$

 $\&\ \neg\exists H'$: $H \approx_x H' \;\&\; H(x) \subset H'(x) \subseteq H'(x') \;\&\; H'(x) \in [\![G]\!]^+$

?10. $\exists A, A' \in (\mathcal{P}(D^M) - \{\varnothing\})$: — $[\![most]\!]^+$, simplify

 $|A' \cap A| > |A' - A|$

 $\&\ \forall d$: $d \in A' \rightarrow \{d\} \in [\![W]\!]^+$

 $\&\ A \subseteq A' \;\&\; A \in [\![G]\!]^+$

 $\&\ \neg(\exists B'$: $A' \subset B' \subseteq D^M \;\&\; \forall d$: $d \in B' \rightarrow \{d\} \in [\![W]\!]^+)$

 $\&\ \neg(\exists B$: $A \subset B \subseteq A' \;\&\; B \in [\![G]\!]^+)$

Problem:

(9) is **not** equivalent to (10). More precisely

- (9) says that there is some *superset* $H(x')$ of the maximal set of women $J(x')$ such that the maximal greet-each-other-subset $H(x)$ of $H(x')$ constitutes a majority of $H(x')$.

whereas

- (10) says that there is a maximal set of women A' such that the maximal greet-each-other-subset A of A' constitutes a majority of A'.

(9) does not entail (10), as the following reasoning shows.

(9) ≠> (10). Here is a counterexample.

- **(10)** is **false** in the following model:
  $M_5 = \langle\{a, b, c, d, e\}, [\![\cdot]\!]^+, [\![\cdot]\!]^-\rangle$
  $[\![W]\!]^+ = \{\{a\}, \{b\}, \{c\}\}$
  $[\![G]\!]^+ = \{X \subseteq \{a, d, e\}: |X| \geq 2\}$

- But **(9) is true**, as the following witnesses attest:

{Λ}		I		J		H	
x	x'	x	x'	x	x'	x	x'
★	★	a	★	a	a	a	a
				a	b	a	b
				a	c	a	c
						d	d
						e	e

Then:

1. $\forall d \in D^M$: $\{\Lambda\}[x/d] = \{g[x/d]: g \in \{\Lambda\}\} = \{\{\langle x, d\rangle\}\}$
   $= \quad I[x/d] = \{g[x/d]: g \in I\} = \{\{\langle x, d\rangle\}\}$
   So $\{\Lambda\} \approx_x I$ ✓

2. $\forall d \in D^M$: $I[x'/d] = \{g[x'/d]: g \in I\} = \{\{\langle x, a\rangle, \langle x', d\rangle\}\}$
   $= \; J[x'/d] = \{g[x'/d]: g \in J\} = \{\langle x, a\rangle, \langle x', d\rangle\}$
   So $I \approx_{x'} J$ ✓

3. $\langle H(x'), H(x)\rangle = \langle\{a, b, c, d, e\}, \{a, d, e\}\rangle$
   So $\langle H(x'), H(x)\rangle \in [\![most]\!]^+$ ✓

4a. $J_{x'=\star} := \{j \in J|\, x' \notin \text{Dom } j\} = \{\}$
   $H_{x'=\star} := \{h \in H|\, x' \in \text{Dom } h\} = \{\}$
   So $J_{x'=\star} = H_{x'=\star}$ ✓

4b. $J_{x'=a} := \{j \in J|\, j(x') = a\} = \{\{\langle x, a\rangle, \langle x', a\rangle\}\}$
   $H_{x'=a} := \{h \in H|\, h(x') = a\} = \{\{\langle x, a\rangle, \langle x', a\rangle\}\}$

   $J_{x'=b} := \{j \in J|\, j(x') = b\} = \{\{\langle x, a\rangle, \langle x', b\rangle\}\}$
   $H_{x'=b} := \{h \in H|\, h(x') = b\} = \{\{\langle x, a\rangle, \langle x', b\rangle\}\}$

   $J_{x'=c} := \{j \in J|\, j(x') = c\} = \{\{\langle x, a\rangle, \langle x', c\rangle\}\}$
   $H_{x'=c} := \{h \in H|\, h(x') = c\} = \{\{\langle x, a\rangle, \langle x', c\rangle\}\}$

   So $\forall d \in J(x')$: $J_{x'=a} = H_{x'=a}$ & $J_{x'=d}(x') = \{d\} \in [\![W]\!]^+$ ✓

5. $\{a, d, e\} \subseteq \{a, b, c, d, e\}$ & $\{a, d, e\} \in [\![G]\!]^+$
   So $H(x) \subseteq H(x')$ & $H(x) \in [\![G]\!]^+$ ✓

6. $\neg\exists J', K'$: $J \approx_{x'} J'$ & $J(x') \subset J'(x')$
   & $J'_{x'=\star} = K'_{x'=\star}$ & $\forall d \in J'(x')$: $J'_{x'=d} = K'_{x'=d}$ & $\{d\} \in [\![W]\!]^+$ ✓

7. $\neg\exists H'$: $H \approx_x H'$ & $H(x) \subset H'(x) \subseteq H'(x')$ & $H'(x) \in [\![G]\!]^+$ ✓

3. Plural Dynamic Logic (PDL, **2nd try**)

DEFINITION 0 (PDL models)

A PDL-model is a structure $M = \langle D^M, [\![\cdot]\!]^+, [\![\cdot]\!]^- \rangle$ such that D^M is a non-empty set and

- for all $\alpha \in \mathbf{Prd}^n$, $[\![\alpha]\!]^+ \subseteq (\mathcal{P}(D^M) - \{\varnothing\})^n$ & $[\![\alpha]\!]^- = (\mathcal{P}(D^M) - \{\varnothing\})^n - [\![\alpha]\!]^+$
- for all $A, B \in (\mathcal{P}(D^M) - \{\varnothing\})^n$:
  $\langle A, B\rangle \in [\![all]\!]^+$ iff $A \subseteq B$ $\langle A, B\rangle \in [\![sm]\!]^+$ iff $A \cap B \neq \{\}$
  $\langle A, B\rangle \in [\![most]\!]^+$ iff $|A \cap B| > |A - B|$ $\langle A, B\rangle \in [\![\mathbf{n}]\!]^+$ iff $|A \cap B| = \mathrm{n}$, for n $\in \{2, \ldots\}$

DEFINITION 1 (PDL assignments)

- An *M-assignment* is a function g such that Dom $g \subseteq \mathbf{Var}$ and Ran $g \subseteq D^M$.
  The set of M-assignments is denoted by $\mathbf{G}$. For any $g, h \in \mathbf{G}$, $u \in \mathbf{Var}$, $d \in D^M$,
  $g[u/d] = h$ iff Dom $g \cup \{u\}$ = Dom h & $h(u) = d$ & $\forall u' \in (\text{Dom } g - \{u\})$: $h(u') = g(u')$
  $g \approx_u h$ iff $\exists d \in D$: $g[u/d] = h$
- An *M-information state* is a set of M-assignments. For any $G, H \subseteq \mathbf{G}$, $u \in \mathbf{Var}$, $d \in D^M$,
  $G(u)$:= $\{d \in D^M | \exists g \in G: g(u) = d\}$ $G_{u=\star}$:= $\{g \in G | u \notin \text{Dom } g\}$
  $G[u/d]$:= $\{g[u/d]: g \in G\}$ $G_{u=d}$:= $\{g \in G | g(u) = d\}$
  $G \approx_u H$ iff $\forall d \in D^M$: $G[u/d] = H[u/d]$ Λ := the $g \in \mathbf{G}$ s.t. Dom $g = \{\}$

DEFINITION 2 (PDL-semantics)

$\subseteq$	$G[\![(x \subseteq y)]\!]H$	=	$\top$	iff	$G = H$ & $G(x) \subseteq G(y)$
		=	$\bot$	iff	$G = H$ & $\neg[G(x) \subseteq G(y)]$
		=	$\star$	otherwise	
R	$G[\![\alpha x_1 \ldots x_n]\!]H$	=	$\top$	iff	$G = H$ & $\langle G(x_1), \ldots, G(x_n)\rangle \in [\![\alpha]\!]^+$
		=	$\bot$	iff	$G = H$ & $\langle G(x_1), \ldots, G(x_n)\rangle \in [\![\alpha]\!]^-$
		=	$\star$	otherwise	
$\wedge$	$G[\![(\phi \wedge \psi)]\!]H$	=	$\star$	iff	$\forall K$: $G[\![\phi]\!]K = \star$ or $K[\![\psi]\!]H = \star$
		=	$\top$	iff	$\exists K$: $G[\![\phi]\!]K = \top$ & $K[\![\psi]\!]H = \top$
		=	$\bot$	otherwise	
$\neg$	$G[\![\neg\phi]\!]H$	=	$\top$	iff	$G[\![\phi]\!]H = \bot$ & $\neg\exists K$: $G[\![\phi]\!]K = \top$
		=	$\bot$	iff	$G[\![\phi]\!]H = \top$
		=	$\star$	otherwise	
$\boldsymbol{\varepsilon}$	$G[\![\varepsilon_u]\!]H$	=	$\top$	iff	$G \approx_u H$
		=	$\star$	otherwise	
$\boldsymbol{\delta}$	$G[\![\delta_x(\phi)]\!]H$	=	$\star$	iff	$G_{x=\star} \neq H_{x=\star}$ or $G(x) \neq H(x)$ or $\exists d \in G(x)$: $G_{x=d}[\![\phi]\!]H_{x=d} = \star$
		=	$\top$	iff	$G_{x=\star} = H_{x=\star}$ & $G(x) = H(x)$ & $\forall d \in G(x)$: $G_{x=d}[\![\phi]\!]H_{x=d} = \top$
		=	$\bot$	otherwise	
M	$G[\![\mathbf{M}_x(\phi)]\!]H$	=	$\top$	iff	$G[\![\phi]\!]H = \top$ & $\neg\exists G', H'$: $G \approx_x G'$ & $G(x) \subset G'(x)$ & $G'[\![\phi]\!]H' = \top$
		=	$\bot$	iff	$G[\![\phi]\!]H = \bot$ & $\neg\exists G', H'$: $G \approx_x G'$ & $G(x) \subset G'(x)$ & $G'[\![\phi]\!]H' = \bot$
		=	$\star$	otherwise	

DEFINITION 3 (PDL truth).

- ϕ is *true* in M, $\models_M \phi$, iff $\exists H$: $\{\Lambda\}[\![\phi]\!]H = \top$

Applications of Plural Dynamic Logic

0. Plural Dynamic Logic (PDL, **version 3**)

DEFINITION 0 (PDL models)

A PDL-model is a structure $M = \langle D^M, [\![\cdot]\!]^+, [\![\cdot]\!]^- \rangle$ such that D^M is a non-empty set and

- for all $\alpha \in \mathbf{Con}$, $[\![\alpha]\!]^+ \in D^M$
- for all $\alpha \in \mathbf{Prd}^n$, $[\![\alpha]\!]^+ \subseteq (\mathcal{P}(D^M) - \{\varnothing\})^n$ & $[\![\alpha]\!]^- = (\mathcal{P}(D^M) - \{\varnothing\})^n - [\![\alpha]\!]^+$
- for all $A, B \in (\mathcal{P}(D^M) - \{\varnothing\})^n$:
  $\langle A, B\rangle \in [\![all]\!]^+$ iff $A \subseteq B$ $\quad$ $\langle A, B\rangle \in [\![sm]\!]^+$ iff $A \cap B \neq \{\}$
  $\langle A, B\rangle \in [\![most]\!]^+$ iff $|A \cap B| > |A - B|$ $\quad$ $\langle A, B\rangle \in [\![\mathbf{n}]\!]^+$ iff $|A \cap B| = \mathrm{n}$, for $\mathrm{n} \in \{2, \ldots\}$

DEFINITION 1 (PDL assignments)

- An *M-assignment* is a function g such that Dom $g \subseteq \mathbf{Var}$ and Ran $g \subseteq D^M$.
  The set of M-assignments is denoted by $\mathbf{G}$. For any $g, h \in \mathbf{G}$, $u \in \mathbf{Var}$, $d \in D^M$,
  $g[u/d] = h$ iff Dom $g \cup \{u\}$ = Dom h & $h(u) = d$ & $\forall u' \in (\text{Dom } g - \{u\})$: $h(u') = g(u')$
  $g \approx_u h$ iff $\exists d \in D$: $g[u/d] = h$
- An *M-information state* is a set of M-assignments. For any $G, H \subseteq \mathbf{G}$, $u \in \mathbf{Var}$, $d \in D^M$,

$G(u)$	:=	$\{d \in D^M \mid \exists g \in G: g(u) = d\}$	$G_{u=\star}$	:=	$\{g \in G \mid u \notin \text{Dom } g\}$
$G[u/d]$	:=	$\{g[u/d]: g \in G\}$	$G_{u=d}$	:=	$\{g \in G \mid g(u) = d\}$
$G \approx_u H$	iff	$\forall d \in D^M$: $G[u/d] = H[u/d]$	Λ	:=	the $g \in \mathbf{G}$ s.t. Dom $g = \{\}$

DEFINITION 2 (PDL-semantics)

t	$G[\![\alpha]\!]$	=	$\{[\![\alpha]\!]^+\}$		for $\alpha \in \mathbf{Con}$
		=	$G(\alpha)$		for $\alpha \in \mathbf{Var}$
$\oplus$	$G[\![(\alpha \oplus \beta)]\!]$	=	$G[\![\alpha]\!] \cup G[\![\beta]\!]$		
$\mathbf{n}^+$	$G[\![\mathbf{n}^+\alpha]\!]H$	=	$\top$	iff	$G = H$ & $0 < \mathrm{n}$ & $\lvert G[\![\alpha]\!]\rvert \geq \mathrm{n}$
		=	$\bot$	iff	$G = H$ & $0 < \mathrm{n}$ & $\lvert G[\![\alpha]\!]\rvert < \mathrm{n}$
		=	$\star$	otherwise	
$\mathbf{n}$	$G[\![\mathbf{n}\alpha]\!]H$	=	$\top$	iff	$G = H$ & $0 < \mathrm{n}$ & $\lvert G[\![\alpha]\!]\rvert = \mathrm{n}$
		=	$\bot$	iff	$G = H$ & $0 < \mathrm{n}$ & $\lvert G[\![\alpha]\!]\rvert \neq \mathrm{n}$
		=	$\star$	otherwise	
$\subseteq$	$G[\![(\alpha \subseteq \beta)]\!]H$	=	$\top$	iff	$G = H$ & $G[\![\alpha]\!] \subseteq G[\![\beta]\!]$
		=	$\bot$	iff	$G = H$ & $\neg[G[\![\alpha]\!] \subseteq G[\![\beta]\!]]$
		=	$\star$	otherwise	
R	$G[\![\alpha\beta_1\ldots\beta_n]\!]H$	=	$\top$	iff	$G = H$ & $\langle G[\![\beta_1]\!], \ldots, G[\![\beta_n]\!]\rangle \in [\![\alpha]\!]^+$
		=	$\bot$	iff	$G = H$ & $\langle G[\![\beta_1]\!], \ldots, G[\![\beta_n]\!]\rangle \in [\![\alpha]\!]^-$
		=	$\star$	otherwise	
$\wedge$	$G[\![(\phi \wedge \psi)]\!]H$	=	$\star$	iff	$\forall K$: $G[\![\phi]\!]K = \star$ or $K[\![\psi]\!]H = \star$
		=	$\top$	iff	$\exists K$: $G[\![\phi]\!]K = \top$ & $K[\![\psi]\!]H = \top$
		=	$\bot$	otherwise	

$\neg$	$G[\![\neg\phi]\!]H$	$=$	$\top$	iff	$G[\![\phi]\!]H = \bot \;\&\; \neg\exists K\text{: } G[\![\phi]\!]K = \top$
		$=$	$\bot$	iff	$G[\![\phi]\!]H = \top$
		$=$	$\star$	otherwise	
ε	$G[\![\varepsilon_u]\!]H$	$=$	$\top$	iff	$G \approx_u H$
		$=$	$\star$	otherwise	
$+$	$G[\![+[\phi]]\!]H$	$=$	$\top$	iff	$G[\![\phi]\!]H = \top$
		$=$	$\star$	otherwise	
δ	$G[\![\delta_x(\phi)]\!]H$	$=$	$\star$	iff	$G_{x=\star} \neq H_{x=\star}$ or $G(x) \neq H(x)$ or $\exists d \in G(x)\text{: } G_{x=d}[\![\phi]\!]H_{x=d} = \star$
		$=$	$\top$	iff	$G_{x=\star} = H_{x=\star} \;\&\; G(x) = H(x) \;\&\; \forall d \in G(x)\text{: } G_{x=d}[\![\phi]\!]H_{x=d} = \top$
		$=$	$\bot$	otherwise	
M	$G[\![\mathbf{M}_x(\phi)]\!]H$	$=$	$\top$	iff	$G[\![\phi]\!]H = \top \;\&\; \neg\exists G', H'\text{: } G \approx_x G' \;\&\; G(x) \subset G'(x) \;\&\; G'[\![\phi]\!]H' = \top$
		$=$	$\bot$	iff	$G[\![\phi]\!]H = \bot \;\&\; \neg\exists G', H'\text{: } G \approx_x G' \;\&\; G(x) \subset G'(x) \;\&\; G'[\![\phi]\!]H' = \bot$
		$=$	$\star$	otherwise	

DEFINITION 3 (PDL truth).
ϕ is *true* in M, $\models_M \phi$, iff $\exists H\text{: } \{\Lambda\}[\![\phi]\!]H = \top$

1. From English to PDL: Sample problems

(1) Most of the women greeted each other.

KEY: *Wx* '*x* is a woman'
Gx '*x* greeted each other'

(1′) $+[\varepsilon_{x'} \wedge \mathbf{2^+}x' \wedge \mathbf{M}_{x'}(\delta_{x'}(Wx'))] \wedge \varepsilon_x \wedge \mathbf{2^+}x \wedge \mathbf{M}_x(x \subseteq x' \wedge Gx) \wedge \textit{most}\ x'x$

(2) Unicycles have wheels.

KEY: *Ux* '*x* is a unicycle'
Wx '*x* is a wheel'

(2′) $+[\varepsilon_x \wedge \mathbf{2^+}x \wedge \mathbf{M}_x(\delta_x(Ux))] \wedge \varepsilon_y \wedge \mathbf{2^+}y \wedge \delta_y(Wy) \wedge \delta_x(Hxy)$

(3) [1]Ann and Bill bought a house (together). [2]They (both separately) invited their parents to their new home.

KEY:

a	'Ann'	*Bxy*	'*x* bought *y*'
b	'Bill'	*Pxy*	'*x* is a parent of *y*'
Hx	'*x* is a house'	*H′xy*	'*x* is *y*'s new home'
		Ixyz	'*x* invited *y* to *z*'

(3^1) ____

(3^2) ____

(4) ^{1}Most of Bill's friends^x have cars^y with automatic transmissions^z.
^{2y} They like them$_y$ a lot. them = the cars
^{2z} They like them$_z$ a lot. them = the transmissions

KEY:				
b	'Bill'		*Fxy*	'*x* is a friend of *y*'
Cx	'*x* is a car'		*Hxy*	'*x* has *y*'
Tx	'*x* is an automatic transmission'		*Lxy*	'*x* likes *y* a lot'

(4^1) ____

(4^{2y}) ____

(4^{2z}) ____

(5) ^{1}Most businessmen who sent a contribution to Gore sent a larger contribution to Bush.
^{2}Most of them asked Bush to spend it in their state.

KEY:				
a	'Al Gore'		*Lxy*	'*x* is larger than *y*'
b	'George Bush'		*Sxyz*	'*x* sent *y* to *z*'
Bx	'*x* is a businessman'		*Axyz*	'*x* asked *y* to spend *z* in *x*'s state'
Cx	'*x* is a contribution'			

(5^1) ____

(5^2) ____

(6) ^{1}Most men (in this town) have a gun, but ^{2}most people don't use it.
^{3}Only Adam and Bill use their guns (now and then).

KEY:				
a	'Adam'		*Uxy*	'*x* uses *y* (now and then)'
b	'Bill'		*Hxy*	'*x* has *y*'
Mx	'*x* is a man (in this town)'			
Px	'*x* is a person'			
Gx	'*x* is a gun'			

(6^1) ____

(6^2) ____

(6^3) ____

(7) ^{1}(Today in Bill's class) the students worked in groups. ^{2}They all worked on different problems. ^{3}The group with the hardest problem came up with the best solution.

KEY:				
Sx	'*x* is a student (in B.'s class today)'		*W′xy*	'*x* worked on *y*'
Gx	'*x* is a group'		*Cxy*	'*x* came up with *y*'
Wx	'*x* worked'		*S′xy*	'*x* is a solution to *y*'
Px	'*x* is a problem'		*Hxy*	'*x* is harder than *y*'
			Bxy	'*x* is better than *y*'

(7^1) ____

(7^2) ____

(7^3) ____

2. From English to PDL: Sample analyses

(3) [1]Ann and Bill bought a house together. [2]They (both separately) invited their parents to their new home.

KEY:	a	'Ann'	Bxy	'x bought y'
	b	'Bill'	Pxy	'x is a parent of y'
	Hx	'x is a house'	$H'xy$	'x is y's new home'
			$Ixyz$	'x invited y to z'

(3^1) $+[\varepsilon_x \wedge x = (a \oplus b)] \wedge \varepsilon_y \wedge \mathbf{1}y \wedge Hy \wedge Bxy \wedge +[\mathbf{2}^+x]$

(3^2) $+[\varepsilon_{x'} \wedge \mathbf{2}^+x' \wedge x' = x] \wedge \delta_{x'}(+[\varepsilon_{z'} \wedge \mathbf{2}^+z' \wedge \mathbf{M}_{z'}(\delta_{z'}(Pz'x'))] \wedge +[\mathbf{2}^+x \wedge \mathbf{1}y \wedge H'yx] \wedge Ix'z'y)$

(4) [1]Most of Bill's friends^x have cars^y with automatic transmissions^z.
[2y] They like them$_y$ a lot. them = the cars
[2z] They like them$_z$ a lot. them = the transmissions

KEY:	b	'Bill'	Fxy	'x is a friend of y'
	Cx	'x is a car'	Hxy	'x has y'
	Tx	'x is an automatic transmission'	Lxy	'x likes y a lot'

(4^1) $+[\varepsilon_{x'} \wedge \mathbf{2}^+x' \wedge \mathbf{M}_{x'}(\delta_{x'}(Fx'b))]$
$\wedge\ \varepsilon_x \wedge \mathbf{2}^+x \wedge \mathbf{M}_x(x \subseteq x' \wedge \varepsilon_y \wedge \mathbf{2}^+y \wedge \delta_y(Cy) \wedge \delta_x(Hxy) \wedge \varepsilon_z \wedge \mathbf{2}^+z \wedge \delta_z(Tz) \wedge \delta_y(Hyz))$
$\wedge\ most\ x'x$

(4^{2y}) $+[\mathbf{2}^+x] \wedge +[\mathbf{2}^+y] \wedge \delta_x(Lxy)$

(4^{2z}) $+[\mathbf{2}^+x] \wedge +[\mathbf{2}^+z] \wedge \delta_x(Lxz)$

(5) [1]Most businessmen who sent a contribution to Gore sent a larger contribution to Bush. [2]Most of them asked Bush to spend it in their state.

KEY:	a	'Al Gore'	Lxy	'x is larger than y'
	b	'George Bush'	$Sxyz$	'x sent y to z'
	Bx	'x is a businessman'	$Axyz$	'x asked y to spend z in x's state'
	Cx	'x is a contribution'		

(5^1) $+[\varepsilon_{x'} \wedge \mathbf{2}^+x' \wedge \mathbf{M}_{x'}(\delta_{x'}(Bx' \wedge \varepsilon_{y'} \wedge \mathbf{1}y' \wedge Cy' \wedge Sx'y'a))]$
$\wedge\ \varepsilon_x \wedge \mathbf{2}^+x \wedge \mathbf{M}_x(x \subseteq x' \wedge \delta_x(\varepsilon_y \wedge \mathbf{1}y \wedge Cy \wedge Lyy' \wedge Sxyb))$
$\wedge\ most\ x'x$

(5^2) $+[\mathbf{2}^+x] \wedge \varepsilon_z \wedge \mathbf{2}^+z \wedge \mathbf{M}_z(z \subseteq x \wedge \delta_z(Azby))$
$\wedge\ most\ xz$

(6) [1]Most men (in this town) have a gun, but [2]most people don't use it.
[3]Only Adam and Bill use their guns (now and then).

KEY:				
a	'Adam'		Uxy	'x uses y (now and then)'
b	'Bill'		Hxy	'x has y'
Mx	'x is a man (in this town)'			
Px	'x is a person'			
Gx	'x is a gun'			

(6^1) $+[\varepsilon_{x'} \wedge \mathbf{2}^{+}x' \wedge \mathbf{M}_{x'}(\delta_{x'}(Mx'))] \wedge \varepsilon_x \wedge \mathbf{2}^{+}x \wedge \mathbf{M}_x(x \subseteq x' \wedge \delta_x(\varepsilon_y \wedge \mathbf{1}y \wedge Gy \wedge Hxy)) \wedge most\ x'x$

(6^2) $+[\mathbf{2}^{+}x \wedge \delta_x(Px)] \wedge \varepsilon_z \wedge \mathbf{2}^{+}z \wedge \mathbf{M}_z(\delta_z(\neg Uzy)) \wedge most\ xz$

(6^3) $+[\mathbf{2}^{+}x] \wedge \varepsilon_{z'} \wedge \mathbf{M}_{z'}(z' \subseteq x \wedge \varepsilon_{y'} \wedge \delta_{z'}(Uz'y') \wedge +[\mathbf{2}^{+}y' \wedge y' \subseteq y \wedge \delta_{y'}(Gy' \wedge Hz'y')] \wedge z' = a \oplus b$

(7) [1](Today in Bill's class) the students worked in groups. [2]They all worked on different problems. [3]The group with the hardest problem came up with the best solution.

KEY:				
Sx	'x is a student (in B.'s class today)'		$W'xy$	'x worked on y'
Gx	'x is a group'		Cxy	'x came up with y'
Wx	'x worked'		$S'xy$	'x is a solution to y'
Px	'x is a problem'		Hxy	'x is harder than y'
			Bxy	'x is better than y'

(7^1) $+[\varepsilon_{x'} \wedge \mathbf{2}^{+}x' \wedge \mathbf{M}_{x'}(\delta_{x'}(Sx'))] \wedge \delta_{x'}(\varepsilon_x \wedge x' \subseteq x \wedge Gx \wedge Wx)$

(7^2) ____

(7^3) ____

Remark: Further analysis would require imposing part-whole structure on the *e*-domain, as in Link 1983 and related work. Even with this enrichment, I do not see any simple way to make the distribution down to subgroups which is introduced in sentence one available for anaphoric reference in sentences two and three. Since there is more than one way to distribute down to subgroups, the distribution from sentence one cannot be recovered simply by repeating the distributive operator—as van den Berg recovers/reintroduces the unique distribution down to atomic individuals, by repeating δ_u.