DYNAMIC PREDICATE LOGIC

1. ANAPHORIC PRONOUNS

- 1st try: Free variables in PL (Predicate Logic)

(1) *Jim*$_1$ came in. *He*$_1$ sat down. (antecedent Jim$_1$ … anaphoric he$_1$)
$\models_{M,g}$ $cm\ \iota x(x = z_1 \wedge z_1 = jim) \wedge sit\ z_1$
iff $g(z_1) \in [\![cm]\!]$ & $g(z_1) = [\![jim]\!]$ & $g(z_1) \in [\![sit]\!]$

oops! *predict*: order of antecedent & anaphor immaterial, e.g. (1) = (2)
intuition: antecedent must precede the anaphor, e.g. (1) ≠ (2)

(2) *He*$_1$ sat down. *Jim*$_1$ came in. (anaphoric he$_1$ … antecedent Jim$_1$)
$\models_{M,g}$ $sit\ z_1 \wedge cm\ \iota x(x = z_1 \wedge z_1 = jim)$
iff $g(z_1) \in [\![sit]\!]$ & $g(z_1) = [\![jim]\!]$ & $g(z_1) \in [\![cm]\!]$

- 2nd try: Free variables in DPL (Dynamic Predicate Logic, G&S 1991)

basic idea: antecedent *updates* the context for interpreting the anaphor (see steps (a)–(c))

a. antecedent [np^n…] updates the *input* value of z_n to an *output* value that satisfies the antec. (‘$g[u_1…u_n]h$’ abbreviates ‘(output) h agrees with (input) g on all var’s other than $u_1…u_n$’)

(1^1) *Jim*^{1} came in.
$\models_{M,g}$ $\exists z_1[z_1 = jim \wedge cm\ z_1]$
iff $\exists h$: $g[\![\exists z_1[z_1 = jim \wedge cm\ z_1]]\!]h$ DPL: $\models_{M,g}$
iff $\exists h$: $g[z_1]h$ & $h(z_1) = [\![jim]\!]$ & $h(z_1) \in [\![cm]\!]$ DPL: $\exists$, $\wedge$, ...
iff $[\![jim]\!] \in [\![cm]\!]$ simplify

b. an anaphoric pronoun, pn$_n$, refers to the *input* value of z_n.

(1^2) *He*$_1$ sat down.
$\models_{M,g}$ $sit\ z_1$
iff $\exists h$: $g[\![sit\ z_1]\!]h$ DPL: $\models_{M,g}$
iff $\exists h$: $g = h$ & $h(z_1) \in [\![sit]\!]$ DPL: R
iff $g(z_1) \in [\![sit]\!]$ simplify

c. the *output* context of sentence 1 serves as the *input* for sentence 2.

(1) Jim^1 came in. He$_1$ sat down.
$\models_{M,g}$ $\exists z_1[z_1 = jim \wedge cm\ z_1] \wedge sit\ z_1$
iff $\exists h$: $g[\![\exists z_1[z_1 = jim \wedge cm\ z_1] \wedge sit\ z_1]\!]h$ DPL: $\models_{M,g}$
iff $\exists h \exists k(g[\![\exists z_1[z_1 = jim \wedge cm\ z_1]]\!]k$ & $k[\![sit\ z_1]\!]h)$ sem: $\wedge$
iff $\exists h \exists k(g[z_1]k$ & $k(z_1) = [\![jim]\!]$ & $k(z_1) \in [\![cm]\!]$ & $k = h$ & $h(z_1) \in [\![sit]\!])$ (1^1), (1^2)
iff $[\![jim]\!] \in [\![cm]\!]$ & $[\![jim]\!] \in [\![sit]\!]$ simplify

(2) He$_1$ sat down. Jim^1 came in.
$\models_{M,g}$ $sit\ z_1 \wedge \exists z_1[z_1 = jim \wedge cm\ z_1]$
iff $\exists h$: $g[\![sit\ z_1 \wedge \exists z_1[z_1 = jim \wedge cm\ z_1]]\!]h$ DPL: $\models_{M,g}$
iff $\exists h \exists k(g[\![sit\ z_1]\!]k \wedge k[\![\exists z_1[z_1 = jim \wedge cm\ z_1]]\!]h)$ sem: $\wedge$
iff $\exists h \exists k(g = k$ & $k(z_1) \in [\![sit]\!] \wedge k[z_1]h$ & $h(z_1) = [\![jim]\!]$ & $h(z_1) \in [\![cm]\!])$ (1^1), (1^2)
iff $g(z_1) \in [\![sit]\!]$ & $[\![jim]\!] \in [\![cm]\!]$ simplify

2. DYNAMIC PREDICATE LOGIC (DPL)

D1.1 (DPL basic expressions)

• **Var**	=	$\{z_1, z_2, \ldots\}$	(entity) variables
• **Con**	=	$\{jim, ann, \ldots\}$	(entity) constants
$\mathbf{Prd}^1$	=	$\{man, wm, cm, sit, \ldots\}$	1-place predicates
$\mathbf{Prd}^2$	=	$\{hsb, wife, see, spk, \ldots\}$	2-place predicates
$\mathbf{Prd}^3$	=	$\{gvt, \ldots\}$	3-place predicates

D1.2 (DPL syntax)

0. $t \in \mathbf{Trm}$ if $t \in \mathbf{Con} \cup \mathbf{Var}$
i. $Rt_1 \ldots t_n \in \mathbf{For}$ if $R \in \mathbf{Prd}^n$ & $t_1, \ldots t_n \in \mathbf{Trm}$
ii. $t_1 = t_2 \in \mathbf{For}$ if $t_1, t_2 \in \mathbf{Trm}$
iii. $\neg\phi \in \mathbf{For}$ if $\phi \in \mathbf{For}$
iv. $[\phi \wedge \psi] \in \mathbf{For}$ if $\phi, \psi \in \mathbf{For}$
v. $\exists u\phi \in \mathbf{For}$ if $u \in \mathbf{Var}$ & $\phi \in \mathbf{For}$

D2.1 (DPL models and assignments)

- A DPL *model* is a pair $M = \langle D, [\![\cdot]\!]\rangle$ such that (i) D is a non-empty set (of *M-entities*), (ii) $[\![\cdot]\!]$ is an *interpretation function* that assigns to each $A \in \mathbf{Con}$ an M-entity $[\![A]\!] \in D$, and to each $B \in \mathbf{Prd}^n$, an n-place relation $[\![B]\!] \subseteq (D)^n$.
- An *M-assignment* is a function g that assigns to each $u \in \mathbf{Var}$ an M-entity $g(u) \in D$.
- For any M-assignments g, h and any variables $u_1, \ldots, u_n \in \mathbf{Var}$:
  $g[u_1 \ldots u_n]h$ iff $\forall v \in \mathbf{Var}\backslash\{u_1, \ldots, u_n\}$: $g(v) = h(v)$.

D2.2 (DPL semantics). For any DPL-model $M = \langle D, [\![\cdot]\!]\rangle$ and M-assignments g and h, the *h-denotation* $[\![\cdot]\!]^h$ and *g-h relation* $g[\![\cdot]\!]h$ are defined as follows:

0.	$[\![t]\!]^h$	=	$[\![t]\!]$	if $t \in \mathbf{Con}$
		=	$h(t)$	if $t \in \mathbf{Var}$
i.	$g[\![Rt_1 \ldots t_n]\!]h$	iff	$g = h$ & $\langle [\![t_1]\!]^h, \ldots, [\![t_n]\!]^h\rangle \in [\![R]\!]$	
ii.	$g[\![t_1 = t_2]\!]h$	iff	$g = h$ & $[\![t_1]\!]^h = [\![t_2]\!]^h$	
iii.	$g[\![\neg\phi]\!]h$	iff	$g = h$ & $\neg\exists k$: $h[\![\phi]\!]k$	
iv.	$g[\![[\phi \wedge \psi]]\!]h$	iff	$\exists k(g[\![\phi]\!]k$ & $k[\![\psi]\!]h)$	
v.	$g[\![\exists u\phi]\!]h$	iff	$\exists k(g[u]k$ & $k[\![\phi]\!]h)$	

D3 (Truth, entailment, and equivalence).

i. ϕ is *true* in M given input g, written $\models_{M,g} \phi$, iff $\exists h$: $g[\![\phi]\!]h$
ii. ϕ (DPL-)*entails* ψ, written $\phi \models \psi$, iff $\forall M \forall g \forall h(g[\![\phi]\!]h \Rightarrow \exists k$: $h[\![\psi]\!]k)$
iii. ϕ and ψ are (DPL-)*equivalent*, written $\phi \simeq \psi$, iff $\forall M \forall g \forall h$: $g[\![\phi]\!]h$ iff $g[\![\psi]\!]h$

ABBREVIATIONS

- $[\phi \rightarrow \psi]$:= $\neg[\phi \wedge \neg\psi]$
- $\forall u\phi$:= $\neg\exists u\neg\phi$

APPENDIX: DRT (Kamp 1981) ~ DPL (G&S 1991)

(3) Jim[1] has a[2] wife$_1$. She$_2$ is a doctor.

DRT (Kamp 1981)

- DRT syntax

```
 _______________
| z1   z2        |
| z1 = jim       |
| z2 wife z1     |  (=: K1)
| dr z2          |
|________________|
```

- DRT semantics
  - DRT-models: as for PL
  - DRT-contexts: ***partial*** assignments
    $F^M = \{f | \text{Dom } f \subseteq \mathbf{Var} \ \& \ \text{Ran } f \subseteq D\}$
  - M-truth

$\models_M K_1$
iff $\exists f \in F^M$:
  Dom f is the *universe* of K_1
  & f *verifies* every *condition* of K_1
iff $\exists f \in F^M$:
  Dom $f = \{z_1, z_2\}$
  & $f(z_1) = [\![jim]\!]$
  & $\langle f(z_2), f(z_1)\rangle \in [\![wife]\!]$
  & $f(z_2) \in [\![dr]\!]$
iff $\exists d \in D$:
  $\langle d, [\![jim]\!]\rangle \in [\![wife]\!]$
  & $d \in [\![dr]\!]$

DPL (G&S 1991)

- DPL syntax

$[[\exists z_1[z_1 = jim \wedge \exists z_2[wife\, z_2 z_1]] \wedge dr\, z_2]$
$\simeq \quad \exists z_1 \exists z_2[z_1 = jim \wedge wife\, z_2 z_1 \wedge dr\, z_2]$
$=: \quad K_1'$

- DPL semantics
  - DPL-models: as for PL
  - DPL-contexts: ***total*** assignments
    $G^M = \{g | \text{Dom } g = \mathbf{Var} \ \& \ \text{Ran } g \subseteq D\}$
  - M, g-truth

$\models_{M,g} K_1'$
iff $\exists h \in G^M$:
  $g[\![K_1']\!]h$

iff $\exists h \in G^M$:
  $g[z_1\, z_2]h$
  & $h(z_1) = [\![jim]\!]$
  & $\langle h(z_1), h(z_2)\rangle \in [\![wife]\!]$
  & $h(z_2) \in [\![dr]\!]$
iff $\exists d \in D$:
  $\langle d, [\![jim]\!]\rangle \in [\![wife]\!]$
  & $d \in [\![dr]\!]$

DPL: TRUTH AND ANAPHORA

1. Relational Test

(1) *She*[1] is a doctor.
$dr\ z_2$

FACT 1: $g[\![dr\ z_2]\!]h$ iff $g = h$ & $g(z_2) \in [\![dr]\!]$ — *ccp* (context change potential): *test*

FACT 2: $\models_{M,g} dr\ z_2$ iff $g(z_2) \in [\![dr]\!]$ — *tc* (truth condition)

PROOF:

	$\models_{M,g} dr\ z_2$	
iff	$\exists h$: $g[\![dr\ z_2]\!]h$	D3: $\models_{M,g}$
iff	$\exists h$: $g = h$ & $h(z_2) \in [\![dr]\!]$	D2.2: R, t
iff	$g(z_2) \in [\![dr]\!]$	elim. h

2. Existential Update

(2) *Jim*[1] has a[2] wife.

FACT 3: $g[\![\exists z_1[z_1 = jim \wedge \exists z_2\ wife\ z_2\ z_1]]\!]h$ iff $g[z_1\ z_2]h$ & $h(z_1) = [\![jim]\!]$ & $\langle h(z_2), h(z_1)\rangle \in [\![wife]\!]$ — *ccp*: *update*

PROOF:

1.	$g[\![\exists z_1[z_1 = jim \wedge \exists z_2\ wife\ z_2\ z_1]]\!]h$	
2.	$\exists k(g[z_1]k$ & $k[\![z_1 = jim \wedge \exists z_2\ wife\ z_2\ z_1]\!]h)$	D2.2: $\exists$
3.	$\exists k(g[z_1]k$ & $\exists j(k[\![z_1 = jim]\!]j$ & $j[\![\exists z_2\ wife\ z_2\ z_1]\!]h))$	D2.2: $\wedge$
4.	$\exists k(g[z_1]k$ & $\exists j(k = j$ & $j(z_1) = [\![jim]\!]$ & $j[\![\exists z_2\ wife\ z_2\ z_1]\!]h))$	D2.2: =, t
5.	$\exists k(g[z_1]k$ & $k(z_1) = [\![jim]\!]$ & $k[\![\exists z_2\ wife\ z_2\ z_1]\!]h)$	elim. j
6.	$\exists k(g[z_1]k$ & $k(z_1) = [\![jim]\!]$ & $\exists j(k[z_2]j$ & $j[\![wife\ z_2\ z_1]\!]h))$	D2.2: $\exists$
7.	$\exists k(g[z_1]k$ & $k(z_1) = [\![jim]\!]$ & $\exists j(k[z_2]j$ & $j = h$ & $\langle h(z_2), h(z_1)\rangle \in [\![wife]\!]))$	D2.2: R, t
8.	$\exists k(g[z_1]k$ & $k(z_1) = [\![jim]\!]$ & $k[z_2]h$ & $\langle h(z_2), h(z_1)\rangle \in [\![wife]\!])$	elim. j
9.	$\exists k(g[z_1]k$ & $h(z_1) = [\![jim]\!]$ & $k[z_2]h$ & $\langle h(z_2), h(z_1)\rangle \in [\![wife]\!])$	$g[\ldots]h$
10.	$\exists k(g[z_1]k$ & $k[z_2]h)$ & $h(z_1) = [\![jim]\!]$ & $\langle h(z_2), h(z_1)\rangle \in [\![wife]\!]$	$\exists$, &
11.	$g[z_1\ z_2]h$ & $h(z_1) = [\![jim]\!]$ & $\langle h(z_2), h(z_1)\rangle \in [\![wife]\!]$	$g[\ldots]h$

FACT 4: $\models_{M,g} \exists z_1[z_1 = jim \wedge \exists z_2\ wife\ z_2\ z_1]$ iff $\exists d(\langle d, [\![jim]\!]\rangle \in [\![wife]\!])$ — *tc*

PROOF:

1.	$\models_{M,g} \exists z_1[z_1 = jim \wedge \exists z_2\ wife\ z_2\ z_1]$	
2.	$\exists h$: $g[\![\exists z_1[z_1 = jim \wedge \exists z_2\ wife\ z_2\ z_1]]\!]h$	D3: $\models_{M,g}$
3.	$\exists h(g[z_1\ z_2]h$ & $h(z_1) = [\![jim]\!]$ & $\langle h(z_2), h(z_1)\rangle \in [\![wife]\!])$	F3
4.	$\exists d_1, d_2(d_1 = [\![jim]\!]$ & $\langle d_2, d_1\rangle \in [\![wife]\!])$	D2.1: g, $g[\ldots]h$
5.	$\exists d_2(\langle d_2, [\![jim]\!]\rangle \in [\![wife]\!])$	elim. d_1

3. CONJUNCTION AS RELATION COMPOSITION

(3) Jim[1] has a[2] wife. She$_2$ is a doctor.

FACT 5: $g[\![\exists z_1[z_1 = jim \wedge \exists z_2\ wife\ z_2\ z_1] \wedge dr\ z_2]\!]h$ *ccp*: *update*

iff $g[z_1\ z_2]h$ & $h(z_1) = [\![jim]\!]$ & $\langle h(z_2), h(z_1)\rangle \in [\![wife]\!]$ & $h(z_2) \in [\![dr]\!]$

PROOF:

1. $g[\![\exists z_1[z_1 = jim \wedge \exists z_2\ wife\ z_2\ z_1] \wedge dr\ z_2]\!]h$
2. $\exists k(g[\![\exists z_1[z_1 = jim \wedge \exists z_2\ wife\ z_2\ z_1]]\!]k$ & $k[\![dr\ z_2]\!]h)$ D2.2: $\wedge$
3. $\exists k(g[z_1\ z_2]k$ & $k(z_1) = [\![jim]\!]$ & $\langle k(z_2), k(z_1)\rangle \in [\![wife]\!]$ F1, F3
   & $k = h$ & $h(z_2) \in [\![dr]\!])$
4. $g[z_1\ z_2]h$ & $h(z_1) = [\![jim]\!]$ & $\langle h(z_2), h(z_1)\rangle \in [\![wife]\!]$ & $h(z_2) \in [\![dr]\!]$ elim. k

FACT 6: $\models_{M,g} \exists z_1[z_1 = jim \wedge \exists z_2\ wife\ z_2\ z_1] \wedge dr\ z_2$ *tc*

iff $\exists d(\langle d, [\![jim]\!]\rangle \in [\![wife]\!]$ & $d \in [\![dr]\!])$

PROOF:

1. $\models_{M,g} \exists z_1[z_1 = jim \wedge \exists z_2\ wife\ z_2\ z_1] \wedge dr\ z_2$
2. $\exists h$: $g[\![\exists z_1[z_1 = jim \wedge \exists z_2\ wife\ z_2\ z_1] \wedge dr\ z_2]\!]h$ D3: $\models_{M,g}$
3. $\exists h(g[z_1\ z_2]h$ & $h(z_1) = [\![jim]\!]$ & $\langle h(z_2), h(z_1)\rangle \in [\![wife]\!]$ & $h(z_2) \in [\![dr]\!])$ F5
4. $\exists d_1, d_2(d_1 = [\![jim]\!]$ & $\langle d_2, d_1\rangle \in [\![wife]\!]$ & $d_2 \in [\![dr]\!])$ D2.1: g, $g[\ldots]h$
5. $\exists d_2(\langle d_2, [\![jim]\!]\rangle \in [\![wife]\!]$ & $d_2 \in [\![dr]\!])$ elim. d_1

4. NEGATION TESTS

(4) *Jim*[1] doesn't have a[2] wife.

FACT 7: $g[\![\exists z_1[z_1 = jim \wedge \neg\exists z_2\ wife\ z_2\ z_1]]\!]h$ *ccp*

iff $g[z_1]h$ & $h(z_1) = [\![jim]\!]$ & $\neg\exists d(\langle d, h(z_1)\rangle \in [\![wife]\!])$

PROOF:

1. $g[\![\exists z_1[z_1 = jim \wedge \neg\exists z_2\ wife\ z_2\ z_1]]\!]h$

⋮

FACT 8: $\models_{M,g} \exists z_1[z_1 = jim \wedge \neg\exists z_2\ wife\ z_2\ z_1]$ *tc*

iff $\neg\exists d(\langle d, [\![jim]\!]\rangle \in [\![wife]\!])$

PROOF:

1. $\models_{M,g} \exists z_1[z_1 = jim \wedge \neg\exists z_2\ wife\ z_2\ z_1]$

⋮

FACT 9: $g[\![\phi \rightarrow \psi]\!]h$

iff $g = h$ & $\forall k(h[\![\phi]\!]k \Rightarrow \exists j$: $k[\![\psi]\!]j)$

PROOF:

1. $g[\![\phi \rightarrow \psi]\!]h$

⋮

FACT 10: $g[\![\forall u\phi]\!]h$

iff $g = h$ & $\forall k(h[u]k \Rightarrow \exists j$: $k[\![\phi]\!]j)$

PROOF:

⋮

APPENDIX: PROOFS OF F7–10

FACT 7: $g\llbracket\exists z_1[z_1 = jim \wedge \neg\exists z_2\ wife\ z_2\ z_1]\rrbracket h$	*ccp*
iff $g[z_1]h$ & $h(z_1) = \llbracket jim\rrbracket$ & $\neg\exists d(\langle d, h(z_1)\rangle \in \llbracket wife\rrbracket)$	

PROOF:

1. $g\llbracket\exists z_1[z_1 = jim \wedge \neg\exists z_2\ wife\ z_2\ z_1]\rrbracket h$	
2. $\exists k(g[z_1]k\ \&\ k\llbracket z_1 = jim \wedge \neg\exists z_2\ wife\ z_2\ z_1\rrbracket h)$	D2.2:$\exists$
3. $\exists k(g[z_1]k\ \&\ \exists j(k\llbracket z_1 = jim\rrbracket j\ \&\ j\llbracket\neg\exists z_2\ wife\ z_2\ z_1\rrbracket h))$	D2.2:$\wedge$
4. $\exists k(g[z_1]k\ \&\ \exists j(k = j\ \&\ j(z_1) = \llbracket jim\rrbracket\ \&\ j\llbracket\neg\exists z_2\ wife\ z_2\ z_1\rrbracket h))$	D2.2:=, t
5. $\exists k(g[z_1]k\ \&\ k(z_1) = \llbracket jim\rrbracket\ \&\ k\llbracket\neg\exists z_2\ wife\ z_2\ z_1\rrbracket h)$	elim. j
6. $\exists k(g[z_1]k\ \&\ k(z_1) = \llbracket jim\rrbracket\ \&\ (k = h\ \&\ \neg\exists j\text{: } h\llbracket\exists z_2\ wife\ z_2\ z_1\rrbracket j))$	D2.2:$\neg$
7. $g[z_1]h\ \&\ h(z_1) = \llbracket jim\rrbracket\ \&\ \neg\exists j\text{: } h\llbracket\exists z_2\ wife\ z_2\ z_1\rrbracket j)$	elim. k
8. $g[z_1]h\ \&\ h(z_1) = \llbracket jim\rrbracket\ \&\ \neg\exists j\exists k(h[z_2]k\ \&\ k\llbracket wife\ z_2\ z_1\rrbracket j)$	D2.2:$\exists$
9. $g[z_1]h\ \&\ h(z_1) = \llbracket jim\rrbracket\ \&\ \neg\exists j\exists k(h[z_2]k\ \&\ k = j\ \&\ \langle j(z_2), j(z_1)\rangle \in \llbracket wife\rrbracket)$	D2.2:R, t
10. $g[z_1]h\ \&\ h(z_1) = \llbracket jim\rrbracket\ \&\ \neg\exists k(h[z_2]k\ \&\ \langle k(z_2), k(z_1)\rangle \in \llbracket wife\rrbracket)$	elim. j
11. $g[z_1]h\ \&\ h(z_1) = \llbracket jim\rrbracket\ \&\ \neg\exists d(\langle d, h(z_1)\rangle \in \llbracket wife\rrbracket)$	D2.1:g, $g[\ldots]h$

FACT 8: $\models_{M,g} \exists z_1[z_1 = jim \wedge \neg\exists z_2\ wife\ z_2\ z_1]$	*tc*
iff $\neg\exists d(\langle d, \llbracket jim\rrbracket\rangle \in \llbracket wife\rrbracket)$	

PROOF:

1. $\models_{M,g} \exists z_1[z_1 = jim \wedge \neg\exists z_2\ wife\ z_2\ z_1]$	
2. $\exists h\text{: } g\llbracket\exists z_1[z_1 = jim \wedge \neg\exists z_2\ wife\ z_2\ z_1]\rrbracket h$	D3:$\models$
3. $\exists h(g[z_1]h\ \&\ h(z_1) = \llbracket jim\rrbracket\ \&\ \neg\exists d(\langle d, h(z_1)\rangle \in \llbracket wife\rrbracket))$	F7
4. $\exists c(c = \llbracket jim\rrbracket\ \&\ \neg\exists d(\langle d, c\rangle \in \llbracket wife\rrbracket))$	D2.1:g, $g[\ldots]h$
5. $\neg\exists d(\langle d, \llbracket jim\rrbracket\rangle \in \llbracket wife\rrbracket)$	elim. c

FACT 9: $g\llbracket\phi \rightarrow \psi\rrbracket h$
iff $g = h\ \&\ \forall k(h\llbracket\phi\rrbracket k \Rightarrow \exists j\text{: } k\llbracket\psi\rrbracket j)$

PROOF:

1. $g\llbracket\phi \rightarrow \psi\rrbracket h$	
2. $g\llbracket\neg(\phi \wedge \neg\psi)\rrbracket h$	DPL: $(\phi \rightarrow \psi) := \neg(\phi \wedge \psi)$
3. $g = h\ \&\ \neg\exists j\text{: } h\llbracket\phi \wedge \neg\psi\rrbracket j$	D2.2: $\neg$
4. $g = h\ \&\ \neg\exists j\exists k(h\llbracket\phi\rrbracket k\ \&\ k\llbracket\neg\psi\rrbracket j)$	D2.2: $\wedge$
5. $g = h\ \&\ \neg\exists k\exists j(h\llbracket\phi\rrbracket k\ \&\ k\llbracket\neg\psi\rrbracket j)$	PL: $\exists x\exists y\phi = \exists y\exists x\phi$
6. $g = h\ \&\ \neg\exists k(h\llbracket\phi\rrbracket k\ \&\ \exists j\text{: } k\llbracket\neg\psi\rrbracket j)$	PL: $\exists x(\phi \wedge Px) = (\phi \wedge \exists xPx)$
7. $g = h\ \&\ \forall k\neg(h\llbracket\phi\rrbracket k\ \&\ \exists j\text{: } k\llbracket\neg\psi\rrbracket j)$	PL: $\neg\exists x\phi = \forall x\neg\phi$
8. $g = h\ \&\ \forall k(h\llbracket\phi\rrbracket k \Rightarrow \neg\exists j\text{: } k\llbracket\neg\psi\rrbracket j)$	PL: $(\phi \Rightarrow \psi) := \neg(\phi\ \&\ \neg\psi)$
9. $g = h\ \&\ \forall k(h\llbracket\phi\rrbracket k \Rightarrow \neg\exists j(k = j\ \&\ \neg\exists j'\text{: } j\llbracket\psi\rrbracket j'))$	D2.2: $\neg$
10. $g = h\ \&\ \forall k(h\llbracket\phi\rrbracket k \Rightarrow \neg\neg\exists j'\text{: } k\llbracket\psi\rrbracket j')$	elim. j
11. $g = h\ \&\ \forall k(h\llbracket\phi\rrbracket k \Rightarrow \exists j'\text{: } k\llbracket\psi\rrbracket j')$	PL: $\neg\neg\phi = \phi$
12. $g = h\ \&\ \forall k(h\llbracket\phi\rrbracket k \Rightarrow \exists j\text{: } k\llbracket\psi\rrbracket j)$	rename j'

FACT 10: $g[\![\forall u\phi]\!]h$
iff $g = h \ \&\ \forall k(h[u]k \Rightarrow \exists j\!: k[\![\phi]\!]j)$

PROOF:

1. $g[\![\forall u\phi]\!]h$	
2. $g[\![\neg\exists u\neg\phi]\!]h$	DPL: $\forall u\phi := \neg\exists u\phi$
3. $g = h \ \&\ \neg\exists j\!: h[\![\exists u\neg\phi]\!]j$	D2.2: $\neg$
4. $g = h \ \&\ \neg\exists j\exists k(h[u]k \ \&\ k[\![\neg\phi]\!]j)$	D2.2: $\exists$
5. $g = h \ \&\ \neg\exists k\exists j(h[u]k \ \&\ k[\![\neg\phi]\!]j)$	PL: $\exists x\exists y\phi = \exists y\exists x\phi$
⋮	⋮
12. $g = h \ \&\ \forall k(h[u]k \Rightarrow \exists j\!: k[\![\phi]\!]j)$	rename j'

PLURALITY AND ANAPHORA (K&R 93, ch. 4)

1 SOME EXAMPLES

• Quantification (by *pl*) & summation	Salient reading(s)
(1) [1]*John introduced Bill to Mary.* [2]They *had been friends for a long time.*	rdg1: *they* = John + Bill rdg2: *they* = John + Mary
(2) [1]*John introduced Bill to Mary.* [2]They *became friends.*	*they* = Bill + Mary
(3) [1]*John introduced Bill to Mary.* [2]*Now* they *are* all *inseparable.*	*they all* = John + Bill + Mary
(4) [1](*Last month*) *Fred bought three donkeys.* [2](*Now*) they *are* all *unhappy*.	rdg1: *they all* = the 3 donkeys rdg2: *they all* = Fred + the 3 donk's

• Collective & distributive predicates

(5) [1]*John and Mary are a nice couple.* [2]They *met in Alaska.*	collective
(6) [1](*Last year*) *John and Mary bought a house.* [2]They *both like* it *a lot.*	collective rdg
(7) [1](*Last year*) *John and Mary* (*both*) *bought a house.* [2]They *both like* them *a lot.*	distributive rdg
(8) [1](*Today*) *Bill's students worked in groups.* [2]They *all worked on different problems.* [3]*The* group with *the hardest* problem came up with *the best solution.*	distribution down to pluralities quantification over pluralities superlative quantification

• Dependent plurals

(9) [1]Most *of Bill's friends own* cars *with* automatic transmissions. [2]They *like* them *a lot.*	one or more cars per friend one transmission per car rdg1: *them* = their cars rdg2: *them* = their car's transmission

2 DPL WITH PLURALITIES (DPL$^+$, G&S 91, K&R 93, Brasoveanu 07:Ch. 1–2)

D1.1 (DPL$^+$ basic expressions)

- **Var** = $\{x, y, z, \ldots, z_1, z_2, \ldots\}$
- **Con** = $\{jim, ann, \ldots\}$
  Prd^1 = $\{\text{SG}, \text{PL}\} \cup \{man, dnk, \ldots, cm, sit, \ldots\}$
  Prd^2 = $\{hsb, wife, see, spk, \ldots\}$
  Prd^3 = $\{gvt, \ldots\}$

D1.2 (DPL$^+$ syntax)

0.	$t \in$ **Trm**	if	$t \in$ **Con** $\cup$ **Var**	
i.	$(t_1 + t_2) \in$ **Trm**	if	$t, t_1, \ldots, t_n \in$ **Trm**	summation
ii.	$[u] \in$ **For**	if	$u \in$ **Var**	
iii.	$(\phi; \psi) \in$ **For**	if	$\phi, \psi \in$ **For**	
iv.	$\neg\phi \in$ **For**	if	$\phi \in$ **For**	
v.	$Rt_1\ldots t_n \in$ **For**	if	$R \in$ **Prd**^{n} & $t_1, \ldots, t_n \in$ **Trm**	
vi.	$t_1 \leq t_2 \in$ **For**	if	$t_1, t_2 \in$ **Trm**	part-of
vii.	$\lvert t\rvert = n \in$ **For**	if	$t \in$ **Trm** and $n \in \{1, 2, \ldots\}$	cardinality
viii.	$t = \Sigma u.\phi \in$ **For**	if	$t \in$ **Trm**, $u \in$ **Var** and $\phi \in$ **For**	abstraction
ix.	$Q_u(\phi, \psi) \in$ **For**	if	$\phi, \psi \in$ **For**, $u \in$ **Var** and $Q \in \{\text{ALL}, \text{MOST}, \ldots\}$	selective wk. qnt.

D2.1 (DPL$^+$ models and assignments)

- A DPL$^+$ *model* is a pair $M = \langle D, [\![\cdot]\!]\rangle$ such that (i) $D = \mathcal{P}(\boldsymbol{A})\backslash\{\varnothing\}$ for some set $\boldsymbol{A} \neq \varnothing$
  (ii) $[\![\cdot]\!]$ assigns $[\![A]\!] \in \{d \in D: |d| = 1\}$ to each $A \in$ **Con**, and $[\![B]\!] \subseteq (D)^n$ to each $B \in$ **Prd**^{n}.
  (iii) $[\![\text{SG}]\!] = \{d \in D: |d| = 1\}$
  $[\![\text{PL}]\!] = \{d \in D: |d| > 1\}$
- $G := \{g|\ g\text{: } \mathbf{Var} \to D\}$ is the set of all *M-assignments*
- for all $g, h \in G$: $g[u_1\ldots u_n]h$ iff $\forall v \in \mathbf{Var}\backslash\{u_1, \ldots, u_n\}$: $g(v) = h(v)$

D2.2 (DPL$^+$ semantics). For any DPL-model $M = \langle D, [\![\cdot]\!]\rangle$ and M-assignments g and h:

0.	$[\![t]\!]^h$	=	$[\![t]\!]$	if $t \in$ **Con**
		=	$h(t)$	if $t \in$ **Var**
i.	$[\![t_1 + t_2]\!]^h$	=	$([\![t_1]\!]^h \cup [\![t_2]\!]^h)$	
ii.	$g[\![[u]]\!]h$	iff	$g[u]h$	
iii.	$g[\![(\phi; \psi)]\!]h$	iff	$\exists k(g[\![\phi]\!]k \ \&\ k[\![\psi]\!]h)$	
iv.	$g[\![\neg\phi]\!]h$	iff	$g = h \ \&\ \neg\exists k\text{: } h[\![\phi]\!]k$	
v.	$g[\![Rt_1\ldots t_n]\!]h$	iff	$g = h \ \&\ \langle [\![t_1]\!]^h, \ldots, [\![t_n]\!]^h\rangle \in [\![R]\!]$	
vi.	$g[\![t_1 \leq t_2]\!]h$	iff	$g = h \ \&\ [\![t_1]\!]^h \subseteq [\![t_2]\!]^h$	
vii.	$g[\![\lvert t\rvert = n]\!]h$	iff	$g = h \ \&\ \lvert[\![t]\!]^h\rvert = n$	
viii.	$g[\![t = \Sigma u.\phi]\!]h$	iff	$g = h \ \&\ [\![t]\!]^h = \cup\{k(u)\text{: } h[\![\phi]\!]k\}$	
ix.	$g[\![Q_u(\phi, \psi)]\!]h$	iff	$g = h \ \&\ \mathsf{Q}(\{k(u)\text{: } h[\![\phi]\!]k\}, \{k(u)\text{: } \exists j(h[\![\phi]\!]j \ \&\ j[\![\psi]\!]k)\})$	
	where $\mathsf{all}(X, Y)$	iff	$X \subseteq X$	
	$\mathsf{most}(X, Y)$	iff	$\lvert X \cap Y\rvert > \lvert X\backslash Y\rvert$	

D3 (Truth, entailment, and equivalence).

i. ϕ is *true* in M given input g, written $|=_{M,g}\ \phi$, iff $\exists h$: $g[\![\phi]\!]h$

ii. ϕ (DPL$^+$-)*entails* ψ, written $\phi \models \psi$, iff $\forall M \forall g \forall h(g[\![\phi]\!]h \Rightarrow \exists k$: $h[\![\psi]\!]k)$

iii. ϕ is (DPL$^+$-)*equivalent* to ψ, written $\phi \simeq \psi$, iff $\forall M \forall g \forall h(g[\![\phi]\!]h$ iff $g[\![\psi]\!]h)$

ABBREVIATIONS

- $\exists u\phi$:= $([u]; \phi)$
  $\forall u\phi$:= $\neg\exists u\neg\phi$
  $(\phi \rightarrow \psi)$:= $\neg(\phi; \neg\psi)$
- $[\phi]$ = ϕ if ϕ is a test (i.e. $\forall M \forall g \forall h$: $g[\![\phi]\!]h \Rightarrow g = h$)
  = undefined otherwise
  (ϕ, ψ) := $([\phi]; [\psi])$
  $[u_1 \ldots u_n|\ \phi]$:= $([u_1]; \ldots ([u_n]; [\phi]))$
  $(t_1 = t_2)$:= $(t_1 \leq t_2, t_2 \leq t_1)$
  $(t_1 \in t_2)$:= $(\text{SG}\ t_1, t_1 \leq t_2)$
  $(t_1 \varnothing t_2)$:= $\neg([x]; [x \leq t_1, x \leq t_2])$

3. USEFUL FACTS

F$\exists$: $g[\![\exists u\phi]\!]h$
iff $\exists k(g[u]k$ & $k[\![\phi]\!]h)$

F$\forall$: $g[\![\forall u\phi]\!]h$
iff $g = h$ & $\forall k(g[u]k \Rightarrow \exists i$: $k[\![\phi]\!]i)$

F$\rightarrow$: $g[\![\phi \rightarrow \psi]\!]h$
iff $g = h$ & $\forall k(g[\![\phi]\!]k \Rightarrow \exists i$: $k[\![\psi]\!]i)$

F,: $g[\![(\phi, \psi)]\!]h$
iff $g = h$ & $h[\![\phi]\!]h$ & $h[\![\psi]\!]h$

F|: $g[\![[u|\ \phi_1, \ldots, \phi_n]]\!]h$
iff $g[u]h$ & $h[\![\phi_1]\!]h$ & … $h[\![\phi_n]\!]h)$

$g[\![[u_1\ u_2|\ \phi_1, \ldots, \phi_n]]\!]h$
iff $g[u_1\ u_2]h$ & $h[\![\phi_1]\!]h$ & … $h[\![\phi_n]\!]h)$

$g[\![[u_1\ u_2\ u_3|\ \phi_1, \ldots, \phi_n]]\!]h$
iff $g[u_1\ u_2\ u_3]h$ & $h[\![\phi_1]\!]h$ & … $h[\![\phi_n]\!]h)$

F=: $g[\![(t_1 = t_2)]\!]h$
iff $g = h$ & $[\![t_1]\!]^h = [\![t_2]\!]^h$

F$\in$: $g[\![(t_1 \in t_2)]\!]h$
iff $g = h$ & $|[\![t_1]\!]^h| = 1$ & $[\![t_1]\!]^h \subseteq [\![t_2]\!]^h$

F$\varnothing$: $g[\![(t_1 \varnothing t_2)]\!]h$
iff $g = h$ & $[\![t_1]\!]^h \cap [\![t_2]\!]^h = \varnothing$

PLURAL ANAPHORA IN DPL$^+$

1 SUMMATION

(1) [1]*Al met his wife in Paris.* [2]They *are a nice couple.*

tr(1):
$[x\ y\ z|\ x = al,\ wife^{of}\ yx,\ z = paris,\ meet^{in}\ xyz];\ [x_1|\ x_1 = x + y,\ nice.couple\ x_1]$

ccp:
$g[\![[x\ y\ z|\ x = al,\ wife^{of}\ yx,\ z = paris,\ meet^{in}\ xyz];\ [x_1|\ x_1 = x + y,\ nice.couple\ x_1]]\!]h$
iff $\exists k(g[x\ y\ z]k$
$\quad \&\ k(x) = [\![al]\!]\ \&\ \langle k(y), k(x)\rangle \in [\![wife^{of}]\!]\ \&\ k(z) = [\![paris]\!]\ \&\ \langle k(x), k(y), k(z)\rangle \in [\![meet^{in}]\!]$
$\quad \&\ k[x_1]h$
$\quad \&\ h(z) = h(x) \cup h(y)\ \&\ h(z) \in [\![nice.couple]\!])$
iff $g[x\ y\ z\ x_1]h$
$\quad \&\ h(x) = [\![al]\!]\ \&\ \langle h(y), h(x)\rangle \in [\![wife^{of}]\!]\ \&\ h(x_1) = [\![paris]\!]\ \&\ \langle h(x), h(y), h(z)\rangle \in [\![meet^{in}]\!]$
$\quad \&\ h(x_1) = h(x) \cup h(y)\ \&\ h(x_1) \in [\![nice.couple]\!]$

M,*g*-truth:
$|\!=_{M,\,g}\ [x\ y\ z|\ x = al,\ wife^{of}\ yx,\ z = paris,\ meet^{in}\ xyz];\ [x_1|\ x_1 = x + y,\ nice.couple\ x_1]$
iff $\exists b(\langle b, [\![al]\!]\rangle \in [\![wife^{of}]\!]\ \&\ \langle [\![al]\!], b, [\![paris]\!]\rangle \in [\![meet^{in}]\!]\ \&\ [\![al]\!] \cup b \in [\![nice.couple]\!])$

2 DISTRIBUTION VS. ABSTRACTION

(2) [1](*Last month*) *Al bought three donkeys.* [2](*Now*) they *are* all *unhappy*.

model A:
$[\![al]\!] = \{\mathsf{a}\}$ $\quad [\![dnk]\!] = \{\{\mathsf{d}_1\}, \ldots, \{\mathsf{d}_{10}\}\}$
$\{d: d \in [\![dnk]\!]\ \&\ \langle [\![al]\!], d\rangle \in [\![buy]\!]\} = \{\{\mathsf{d}_1\}, \ldots, \{\mathsf{d}_3\}\}$ $\quad [\![unh]\!] = \{\{\mathsf{a}\}, \{\mathsf{d}_1\}, \ldots, \{\mathsf{d}_3\}\}$
MB intuition: (1) ***true***

model B:
$[\![al]\!] = \{\mathsf{a}\}$ $\quad [\![dnk]\!] = \{\{\mathsf{d}_1\}, \ldots, \{\mathsf{d}_{10}\}\}$
$\{d: d \in [\![dnk]\!]\ \&\ \langle [\![al]\!], d\rangle \in [\![buy]\!]\} = \{\{\mathsf{d}_1\}, \ldots, \{\mathsf{d}_4\}\}$ $\quad [\![unh]\!] = \{\{\mathsf{a}\}, \{\mathsf{d}_1\}, \ldots, \{\mathsf{d}_3\}\}$
MB intuition: (1) ***false***

model C:
$[\![al]\!] = \{\mathsf{a}\}$ $\quad [\![dnk]\!] = \{\{\mathsf{d}_1\}, \ldots, \{\mathsf{d}_{10}\}\}$
$\{d: d \in [\![dnk]\!]\ \&\ \langle [\![al]\!], d\rangle \in [\![buy]\!]\} = \{\{\mathsf{d}_1\}, \ldots, \{\mathsf{d}_4\}\}$ $\quad [\![unh]\!] = \{\{\mathsf{a}\}, \{\mathsf{d}_1\}, \ldots, \{\mathsf{d}_4\}\}$
MB intuition: (1) ***false***

tr(2): version 1

- $[x\ y|\ x = al,\ buy\ xy,\ |y| = 3,\ \text{ALL}_{y1}([y_1|\ y_1 \in y],\ [dnk\ y_1])];$
  $[z|\ z = x + y,\ \text{ALL}_{y1}([y_1|\ y_1 \in z],\ [unh\ y_1])]$

• ccp

$g[\![[x\ y|\ x = al,\ buy\ xy,\ |y| = 3,\ \text{ALL}_{y1}([y_1|\ y_1 \in y],\ [dnk\ y_1])];$
$\quad [z|\ z = x + y,\ \text{ALL}_{y1}([y_1|\ y_1 \in z],\ [unh\ y_1])]]\!]h$

iff $g[x\ y\ z]h$
$\quad \&\ h(x) = [\![al]\!]\ \&\ \langle h(x), h(y)\rangle \in [\![buy]\!]\ \&\ |h(y)| = 3\ \&\ \forall d \in D(|d| = 1\ \&\ d \subseteq h(y) \Rightarrow d \in [\![dnk]\!])$
$\quad \&\ h(z) = h(x) \cup h(y)\ \&\ \forall d(|d| = 1\ \&\ d \subseteq h(z) \Rightarrow d \in [\![unh]\!])$

• M, g-truth

$\models_{M, g} [x\ y|\ x = al,\ buy\ xy,\ |y| = 3,\ \text{ALL}_{y1}([y_1|\ y_1 \in y],\ [dnk\ y_1])];$
$\quad [z|\ z = x + y,\ \text{ALL}_{y1}([y_1|\ y_1 \in z],\ [unh\ y_1])]$

iff $\exists a, b(a = [\![al]\!]\ \&\ \langle a, b\rangle \in [\![buy]\!]\ \&\ |b| = 3\ \&\ \forall d \in D(|d| = 1\ \&\ d \subseteq b \Rightarrow d \in [\![dnk]\!])$
$\quad \&\ \forall d \in D(|d| = 1\ \&\ d \subseteq (a \cup b) \Rightarrow d \in [\![unh]\!]))$

• ***predict*** (1) is ***true*** in model B and C on the plausible assumption that
$\langle a, b_1\rangle \in [\![buy]\!]\ \&\ \langle a, b_2\rangle \in [\![buy]\!] \Rightarrow \langle a, b_1 \cup b_2\rangle \in [\![buy]\!]$

MB intuition:

tr(2): version 2

- $[x\ y|\ x = al,\ |y| = 3,\ y = \Sigma y_1.[y_1|\ dnk\ y_1,\ buy\ xy_1]];$
  $[z|\ z = x + y,\ \text{ALL}_{y1}([y_1|\ y_1 \in z],\ [unh\ y_1])]$

• ccp

$g[\![[x\ y|\ x = al,\ |y| = 3,\ y = \Sigma y_1.[y_1|\ dnk\ y_1,\ buy\ xy_1]];$
$\quad [z|\ z = x + y,\ \text{ALL}_{y1}([y_1|\ y_1 \in z],\ [unh\ y_1])]]\!]h$

iff $g[x\ y]h$
$\quad \&\ h(x) = [\![al]\!]\ \&\ |h(y)| = 3\ \&\ h(y) = \cup\{d: d \in [\![dnk]\!]\ \&\ \langle h(x), d\rangle \in [\![buy]\!]\}$
$\quad \&\ \forall d \in D(|d| = 1\ \&\ d \subseteq (h(x) \cup h(y)) \Rightarrow d \in [\![unh]\!])$

• M, g-truth

$\models_{M, g} [x\ y|\ x = al,\ |y| = 3,\ y = \Sigma y_1.[y_1|\ dnk\ y_1,\ buy\ xy_1]];$
$\quad [z|\ z = x + y,\ \text{ALL}_{y1}([y_1|\ y_1 \in z],\ [unh\ y_1])]$

iff $\exists a, b(a = [\![al]\!]\ \&\ |b| = 3\ \&\ b = \cup\{d: d \in [\![dnk]\!]\ \&\ \langle h(x), d\rangle \in [\![buy]\!]\}$
$\quad \&\ \forall d \in D(|d| = 1\ \&\ d \subseteq (a \cup b) \Rightarrow d \in [\![unh]\!]))$

3 PROBLEM?

(3) [1](*Last year*) *Al, Bill, and Sue each bought a house.* [2]They *like* them *a lot.*

tr(3):

APPENDIX: SAMPLE DERIVATIONS

FACT 1: $g[\![[x\ y|\ x = al,\ buy\ xy,\ |y| = 3,\ \text{ALL}_{y1}([y_1|\ y_1 \in y],\ [dnk\ y_1])]]\!]h$

iff $g[x\ y]h$

$\&\ h(x) = [\![al]\!]\ \&\ \langle h(x), h(y)\rangle \in [\![buy]\!]\ \&\ |h(y)| = 3$

$\&\ \forall d \in D(|d| = 1\ \&\ d \subseteq h(y) \Rightarrow d \in [\![dnk]\!])$

PROOF:

1. $g[\![[x\ y|\ x = al,\ buy\ xy,\ |y| = 3,\ \text{ALL}_{y1}([y_1|\ y_1 \in y],\ [dnk\ y_1])]]\!]h$
2. $g[x\ y]h\ \&\ h[\![x = al]\!]h\ \&\ h[\![buy\ xy]\!]h\ \&\ h[\![|y| = 3]\!]h\ \&\ h[\![\text{ALL}_{y1}([y_1|\ y_1 \in y],\ [dnk\ y_1])]\!]h$ F|
3. $g[x\ y]h$
   $\&\ h(x) = [\![al]\!]\ \&\ \langle h(x), h(y)\rangle \in [\![buy]\!]\ \&\ |h(y)| = 3$ F=, D2.2:*R*, |*t*|, *t*
   $\&\ \{k(y_1): h[\![[y_1|\ y_1 \in y]]\!]k\} \subseteq \{k(y_1): \exists j(k[\![[dnk\ y_1]]\!]j)\}$ D2.2:∀
4. $g[x\ y]h$
   $\&\ h(x) = [\![al]\!]\ \&\ \langle h(x), h(y)\rangle \in [\![buy]\!]\ \&\ |h(y)| = 3$
   $\&\ \{k(y_1): h[y_1]k\ \&\ |k(y_1)| = 1\ \&\ k(y_1) \subseteq k(y)\}$ F|, F∈, D2.2:*t*
   $\subseteq \{k(y_1): \exists j(k = j\ \&\ j(y_1) \in [\![dnk]\!])\}$ [ϕ], D2.2:*R*, *t*
5. $g[x\ y]h$
   $\&\ h(x) = [\![al]\!]\ \&\ \langle h(x), h(y)\rangle \in [\![buy]\!]\ \&\ |h(y)| = 3$
   $\&\ \{k(y_1): h[y_1]k\ \&\ |k(y_1)| = 1\ \&\ k(y_1) \subseteq h(y)\}$ D2.1:*g*[…]*h*
   $\subseteq \{k(y_1): k(y_1) \in [\![dnk]\!]\}$ elim. *j*
6. $g[x\ y]h$
   $\&\ h(x) = [\![al]\!]\ \&\ \langle h(x), h(y)\rangle \in [\![buy]\!]\ \&\ |h(y)| = 3$
   $\&\ \{d \in D|\ \exists k(k(y_1) = d\ \&\ h[y_1]k\ \&\ |k(y_1)| = 1\ \&\ k(y_1) \subseteq h(y))\}$ D2.1, $\{f(a): \ldots\}$
   $\subseteq \{d \in D|\ \exists k(k(y_1) = d\ \&\ k(y_1) \in [\![dnk]\!]\}$
7. $g[x\ y]h$
   $\&\ h(x) = [\![al]\!]\ \&\ \langle h(x), h(y)\rangle \in [\![buy]\!]\ \&\ |h(y)| = 3$
   $\&\ \{d \in D|\ |d| = 1\ \&\ d \subseteq h(y)\} \subseteq \{d \in D|\ d \in [\![dnk]\!]\}$ D2.1:*g*, *g*[…]*h*
8. $g[x\ y]h$
   $\&\ h(x) = [\![al]\!]\ \&\ \langle h(x), h(y)\rangle \in [\![buy]\!]\ \&\ |h(y)| = 3$
   $\&\ \forall d \in D(|d| = 1\ \&\ d \subseteq h(y) \Rightarrow d \in [\![dnk]\!])$ df. ⊆
9. $g[x\ y]h$
   $\&\ h(x) = [\![al]\!]\ \&\ \langle h(x), h(y)\rangle \in [\![buy]\!]\ \&\ |h(y)| = 3$
   $\&\ \forall d \in D(|d| = 1\ \&\ d \subseteq h(y) \Rightarrow d \in [\![dnk]\!])$ D2.1:*D*, *g*

□

FACT 2: $g[\![[x\ y|\ x = al,\ |y| = 3,\ y = \Sigma y_1.[y_1|\ dnk\ y_1,\ buy\ xy_1]]\!]h$

iff $g[x\ y]h$

$\&\ h(x) = [\![al]\!]\ \&\ |h(y)| = 3\ \&\ h(y) = \cup\{d \in D|\ d \in [\![dnk]\!]\ \&\ \langle h(x), d\rangle \in [\![buy]\!]\}$

PROOF:

1. $g[\![[x\ y|\ x = al,\ |y| = 3,\ y = \Sigma y_1.[y_1|\ buy\ xy_1,\ dnk\ y_1]]\!]h$
2. $g[x\ y]h\ \&\ h[\![x = al]\!]h\ \&\ h[\![|y| = 3]\!]h\ \&\ h[\![y = \Sigma y_1.[y_1|\ buy\ xy_1,\ dnk\ y_1]]\!]h$ — F|
3. $g[x\ y]h$
   $\&\ h(x) = [\![al]\!]\ \&\ |h(y)| = 3$ — F=, D2.2:$|t|$, t
   $\&\ h(y) = \cup\{k(y_1):\ h[\![[y_1|\ dnk\ y_1,\ buy\ xy_1]]\!]k\}$ — D2.2:Σ
4. $g[x\ y]h$ — F|
   $\&\ h(x) = [\![al]\!]\ \&\ |h(y)| = 3$
   $\&\ h(y) = \cup\{k(y_1):\ h[y_1]k\ \&\ k[\![dnk\ y_1]\!]k\ \&\ k[\![buy\ xy_1]\!]k\}$
5. $g[x\ y]h$ — D2.2:R, t
   $\&\ h(x) = [\![al]\!]\ \&\ |h(y)| = 3$
   $\&\ h(y) = \cup\{k(y_1):\ h[y_1]k\ \&\ k(y_1) \in [\![dnk]\!]\ \&\ \langle k(x), k(y_1)\rangle \in [\![buy]\!]\}$
6. $g[x\ y]h$ — D2.1, {-:-}
   $\&\ h(x) = [\![al]\!]\ \&\ |h(y)| = 3$
   $\&\ h(y) = \cup\{d \in D|\ \exists k(k(y_1) = d\ \&\ h[y_1]k\ \&\ k(y_1) \in [\![dnk]\!]\ \&\ \langle k(x), k(y_1)\rangle \in [\![buy]\!])\}$
7. $g[x\ y]h$ — D2.1:$g[\ldots]h$
   $\&\ h(x) = [\![al]\!]\ \&\ |h(y)| = 3$
   $\&\ h(y) = \cup\{d \in D|\ \exists k(k(y_1) = d\ \&\ h[y_1]k\ \&\ k(y_1) \in [\![dnk]\!]\ \&\ \langle h(x), k(y_1)\rangle \in [\![buy]\!])\}$
8. $g[x\ y]h$ — D2.1:g
   $\&\ h(x) = [\![al]\!]\ \&\ |h(y)| = 3\ \&\ h(y) = \cup\{d \in D|\ d \in [\![dnk]\!]\ \&\ \langle h(x), d\rangle \in [\![buy]\!]\}$

□