PLURAL LOGIC (vdBerg 1994, 1995)

1. TWO APPROACHES TO PLURAL NPS

• Paradigm examples

(1)	Adam and Beth *lifted (a stack of) three crates (together).*	collective VP
(2)	Adam and Beth (each) *lifted (the same stack of) three crates.*	dist. VP, wide obj
(3)	Adam and Beth (each) *lifted (a different stack of) three crates.*	dist. VP, narrow obj

• Plural entity approach (Sharvy 1980, Link 1987)

(1) Adam and Beth *lifted (a stack of) three crates (together).* — collective VP

(1_L) $\exists x(x = a \oplus b \wedge \exists y(\mathbf{3}(y) \wedge {}^{D}\lambda y'[C(y')](y) \wedge L(x, y)))$

x	y
{a, b}	{c_1, c_2, c_3}

(2) Adam and Beth (each) *lifted (the same stack of) three crates.* — dist. VP, wide obj

(2_L) $\exists x(x = a \oplus b \wedge \exists y(\mathbf{3}(y) \wedge {}^{D}\lambda y'[C(y')](y) \wedge {}^{D}\lambda x'[L(x', y)](x)))$

x	y
{a}	{c_1, c_2, c_3}
{b}	{c_1, c_2, c_3}

(3) Adam and Beth (each) *lifted (a different stack of) three crates.* — dist. VP, narrow obj

(3_L) $\exists x(x = a \oplus b \wedge {}^{D}\lambda x'[\exists y(\mathbf{3}(y) \wedge {}^{D}\lambda y'[C(y')](y) \wedge L(x', y)](x)))$

x	y
{a}	{c_1, c_2, c_3}
{b}	{c_1', c_2', c_3'}

• Plural info-state approach (vdBerg 1994, 1995)

(1) Adam and Beth *lifted (a stack of) three crates (together).* — collective VP

(1_B) $\exists x(x = a \oplus b \wedge \exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge L(x, y)))$

x	y
a	c_1
a	c_2
b	c_3

(2) Adam and Beth (each) *lifted (the same stack of) three crates.* — dist. VP, wide obj

(2_B) $\exists x(x = a \oplus b \wedge \exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge \delta_x(L(x, y))))$

x	y
a	c_1
a	c_2
a	c_3
b	c_1
b	c_2
b	c_3

(3) Adam and Beth (each) *lifted (a different stack of) three crates.* — dist. VP, narrow obj

(3_B) $\exists x(x = a \oplus b \wedge \delta_x(\exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge L(x, y))))$

2 LOGIC WITH PLURALITIES (LP, see Link 1983, 1987)

DEFINITION 1 (LP-syntax)

i.	$A \in \mathbf{Trm}$	if $A \in \mathbf{Con}$ or $A \in \mathbf{Var}$	
ii.	$(A \oplus B) \in \mathbf{Trm}$	if $A, B \in \mathbf{Trm}$	(Link 1987:152)
iii.	$\boldsymbol{n}(A) \in \mathbf{For}$	if $A \in \mathbf{Trm}$	
iv.	$B(A_1, \ldots, A_n) \in \mathbf{For}$	if $B \in \mathbf{Prd}^n$ and $A_1, \ldots, A_n \in \mathbf{Trm}$	
v.	$(A \leq B) \in \mathbf{For}$	if $A, B \in \mathbf{Trm}$	
vi.	$\neg\phi \in \mathbf{For}$	if $\phi \in \mathbf{For}$	
vii.	$(\phi \wedge \psi) \in \mathbf{For}$	if $\phi, \psi \in \mathbf{For}$	
viii.	$\exists u\phi \in \mathbf{For}$	if $u \in \mathbf{Var}$ and $\phi \in \mathbf{For}$	

DEFINITION 2 (LP-models and assignments)

M. A LP-*model* is a structure $M = \langle D, [\![\cdot]\!]\rangle$ such that:

i. $D = \mathcal{P}(\mathbf{A})\backslash\{\varnothing\}$, for some non-empty set $\mathbf{A}$ (of *proto-atoms*)

ii. For any $A \in \mathbf{Con}$, $[\![A]\!] \in D$, and for any $B \in \mathbf{Prd}^n$, $[\![B]\!] \subseteq D^n$.

g. An *M-assignment* is a function g such that Dom $g = \mathbf{Var}$ and Ran $g \subseteq D$.
The set of M-assignments is denoted by $\mathcal{G}^M$. For any $g, h \in \mathcal{G}^M$, $u \in \mathbf{Var}$, $d \in D$,

$g[u/d] := (g\backslash\{\langle u, g(u)\rangle\}) \cup \{\langle u, d\rangle\}$

$g \approx_u h$ iff $\exists d \in D$: $g[u/d] = h$

DEFINITION 3 (LP-semantics)

i.	$g[\![A]\!]$	$= [\![A]\!]$	if	$A \in \mathbf{Con}$
		$= g(A)$	if	$A \in \mathbf{Var}$
ii.	$g[\![(A \oplus B)]\!]$	$= (g[\![A]\!] \cup g[\![B]\!])$		
iii.	$g[\![\boldsymbol{n}(A)]\!]$	$= \top$	iff	$\lvert g[\![A]\!]\rvert = n$
iv.	$g[\![B(A_1 \ldots A_n)]\!]$	$= \top$	iff	$\langle g[\![A_1]\!], \ldots, g[\![A_n]\!]\rangle \in [\![B]\!]$
v.	$g[\![(A \leq B)]\!]$	$= \top$	iff	$g[\![A]\!] \subseteq g[\![B]\!]$
vi.	$g[\![\neg\phi]\!]$	$= \top$	iff	$g[\![\phi]\!] \neq \top$
vii.	$g[\![(\phi \wedge \psi)]\!]$	$= \top$	iff	$g[\![\phi]\!] = \top$ & $g[\![\psi]\!] = \top$
viii.	$g[\![\exists u\phi]\!]$	$= \top$	iff	$\exists h$: $g \approx_u h$ & $h[\![\phi]\!] = \top$

DEFINITION 4 (LP truth).

ϕ is *true* in M, written $\models_M \phi$, iff $\forall g \in \mathcal{G}^M$: $g[\![\phi]\!] = \top$

ABBREVIATIONS

- $(A = B) := ((A \leq B) \wedge (B \leq A))$
- $(\phi \rightarrow \psi) := \neg(\phi \wedge \neg\psi)$
- $\forall u\phi := \neg\exists u\neg\phi$
- ${}^{\mathrm{D}}\lambda u[\phi](A) := \forall v(v \leq A \wedge \mathbf{1}(v) \rightarrow \phi[u/v])$ $\phi \in \mathbf{For}$, no v in ϕ, $A \in \mathbf{Trm}$
  where $\phi[u/v]$ is obtained from ϕ by replacing every occurrence of u with v

3 PLURAL LOGIC (PlL, see vdBerg 1994, 1995)

DEFINITION 1 (PlL-syntax)

i.	$A \in \mathbf{Trm}$	if $A \in \mathbf{Con}$ or $A \in \mathbf{Var}$	
ii.	$(A \oplus B) \in \mathbf{Trm}$	if $A, B \in \mathbf{Trm}$	(Link 1987:152)
iii.	$\boldsymbol{n}(A) \in \mathbf{For}$	if $A \in \mathbf{Trm}$	
iv.	$B(A_1, \ldots, A_n) \in \mathbf{For}$	if $B \in \mathbf{Prd}^n$ and $A_1, \ldots, A_n \in \mathbf{Trm}$	
v.	$(A \subseteq B) \in \mathbf{For}$	if $A, B \in \mathbf{Trm}$	
vi.	$\neg\phi \in \mathbf{For}$	if $\phi \in \mathbf{For}$	
vii.	$(\phi \wedge \psi) \in \mathbf{For}$	if $\phi, \psi \in \mathbf{For}$	
viii.	$\exists u\phi \in \mathbf{For}$	if $u \in \mathbf{Var}$ and $\phi \in \mathbf{For}$	

DEFINITION 2 (PlL-models and assignments)

M. A PlL-*model* is a structure $M = \langle D, [\![\cdot]\!]\rangle$ such that:

i. D is a non-empty set (of *entities*)

ii. For any $A \in \mathbf{Con}$, $[\![A]\!] \in D$, and for any $B \in \mathbf{Prd}^n$, $[\![B]\!] \subseteq (\mathcal{P}(D))^n$.

g. An *M-assignment* is a function g such that Dom $g = \mathbf{Var}$ and Ran $g \subseteq D$.
The set of M-assignments is denoted by $\mathcal{G}^M$. For any $g \in \mathcal{G}^M$, $u \in \mathbf{Var}$, $d \in D$,

$g[u/d] \quad := \quad (g\backslash\{\langle u, g(u)\rangle\}) \cup \{\langle u, d\rangle\}$

G. An *M-info state* is a set of M-assignments. For any $G, H \subseteq \mathcal{G}^M$, $u \in \mathbf{Var}$, $d \in D$,

$G(u)$	:=	$\{g(u): g \in G\}$	vdB 95:§3.2
$G[u/d]$	:=	$\{g[u/d]: g \in G\}$	vdB 95:§3.2
$G \approx_u H$	iff	$\forall d$: $G[u/d] = H[u/d]$	vdB 95:§3.2

DEFINITION 3 (PlL-semantics)

i.	$G[\![A]\!]$	$= \{[\![A]\!]\}$	if	$A \in \mathbf{Con}$	MB
		$= G(A)$	if	$A \in \mathbf{Var}$	vdB 95:§3.2
ii.	$G[\![(A \oplus B)]\!]$	$= (G[\![A]\!] \cup G[\![B]\!])$			Link 1987:152
iii.	$G[\![\boldsymbol{n}(A)]\!]$	$= \top$	iff	$\lvert G[\![A]\!]\rvert = n$	cf. vdB 94:**0**, **sing**
iv.	$G[\![B(A_1 \ldots A_n)]\!]$	$= \top$	iff	$\langle G[\![A_1]\!], \ldots, G[\![A_n]\!]\rangle \in [\![B]\!]$	vdB 94:§2
v.	$G[\![(A \subseteq B)]\!]$	$= \top$	iff	$G[\![A]\!] \subseteq G[\![B]\!]$	vdB 94:§2
vi.	$G[\![\neg\phi]\!]$	$= \top$	iff	$G[\![\phi]\!] \neq \top$	vdB 94:§2
vii.	$G[\![(\phi \wedge \psi)]\!]$	$= \top$	iff	$G[\![\phi]\!] = \top$ & $G[\![\psi]\!] = \top$	vdB 94:§2
viii.	$G[\![\exists u\phi]\!]$	$= \top$	iff	$\exists H$: $G \approx_u H$ & $H[\![\phi]\!] = \top$	vdB 95:§3.2

DEFINITION 4 (PlL-truth).

ϕ is *true* in M, written $\models_M \phi$, iff $\mathcal{G}^M[\![\phi]\!] = \top$

ABBREVIATIONS

- $(A = B) \quad := \quad ((A \subseteq B) \wedge (B \subseteq A))$
- $(\phi \rightarrow \psi) \quad := \quad \neg(\phi \wedge \neg\psi)$
- $\forall u\phi \quad := \quad \neg\exists u\neg\phi$
- $\delta_u(\phi) \quad := \quad \forall v(v \subseteq u \wedge \mathbf{1}(v) \rightarrow \phi[u/v])$ $\qquad \phi \in \mathbf{For}$, no v in ϕ
  where $\phi[u/v]$ is obtained from ϕ by replacing every occurrence of u with v

4 SOME USEFUL FACTS

LP facts:

F=	$g[\![(A = B)]\!]$	$= \top$,	iff	$g[\![A]\!] = g[\![B]\!]$
F→	$g[\![(\phi \rightarrow \psi)]\!]$	$\neq \top$,	iff	$g[\![\phi]\!] = \top \; \& \; g[\![\psi]\!] \neq \top$
F∀	$g[\![\forall u\phi]\!]$	$= \top$,	iff	________________________________
F^D	$g[\![{}^{D}\lambda u[\phi](A)]\!]$	$= \top$,	iff	________________________________

Sample proofs:
F∀
1. $g[\![\forall u\phi]\!] = \top$
2. $g[\![\neg\exists u\neg\phi]\!] = \top$ df.∀
3. $g[\![\exists u\neg\phi]\!] \neq \top$ D3:¬
4. $\neg\exists h(g \approx_u h \; \& \; h[\![\neg\phi]\!] = \top)$ D3:∃
5. $\neg\exists h(g \approx_u h \; \& \; h[\![\phi]\!] \neq \top)$ D3:¬
6. $\forall h(g \approx_u h \rightarrow h[\![\phi]\!] \neq \top)$ rearrange

Fδ
1. $g[\![{}^{D}\lambda u[\phi](A)]\!] = \top$
⋮

PlL facts:

F=	$G[\![(A = B)]\!]$	$= \top$,	iff	$G[\![A]\!] = G[\![B]\!]$
F→	$G[\![(\phi \rightarrow \psi)]\!]$	$\neq \top$,	iff	$G[\![\phi]\!] = \top \; \& \; G[\![\psi]\!] \neq \top$
F∀	$G[\![\forall u\phi]\!]$	$= \top$,	iff	________________________________
Fδ	$G[\![\delta_u(\phi)]\!]$	$= \top$,	iff	________________________________

Sample proofs:
F∀
1. $G[\![\forall u\phi]\!] = \top$
⋮

Fδ
1. $G[\![\delta_u(\phi)]\!] = \top$
⋮

APPENDIX: SOME USEFUL FACTS

LP facts:

F∀ $g[\![\forall u\phi]\!] = \top$, iff $\forall h(g \approx_u h \rightarrow h[\![\phi]\!] = \top)$

F^D $g[\![^D\lambda u[\phi](A)]\!] = \top$, iff $\forall h(g \approx_v h \ \&\ h(v) \subseteq h[\![A]\!] \ \&\ |h(v)| = 1 \rightarrow h[\![\phi[u/v]]\!] = \top)$

Proof of F∀

1. $g[\![\forall u\phi]\!] = \top$
2. $g[\![\neg\exists u\neg\phi]\!] = \top$ df.∀
3. $g[\![\exists u\neg\phi]\!] \neq \top$ D3:¬
4. $\neg\exists h(g \approx_u h \ \&\ h[\![\neg\phi]\!] = \top)$ D3:∃
5. $\neg\exists h(g \approx_u h \ \&\ h[\![\phi]\!] \neq \top)$ D3:¬
6. $\forall h(g \approx_u h \rightarrow h[\![\phi]\!] = \top)$ rearr.

□

Proof of F^D

1. $g[\![^D\lambda u[\phi](A)]\!] = \top$
2. $g[\![\forall v(v \leq A \wedge \mathbf{1}(v) \rightarrow \phi[u/v])]\!] = \top$ df. ^{D}
3. $\forall h(g \approx_v h \rightarrow h[\![(v \leq A \wedge \mathbf{1}(v) \rightarrow \phi[u/v])]\!] = \top)$ F∀
4. $\neg\exists h(g \approx_v h \ \&\ h[\![(v \leq A \wedge \mathbf{1}(v) \rightarrow \phi[u/v])]\!] \neq \top)$ rearr.
5. $\neg\exists h(g \approx_v h \ \&\ h[\![(v \leq A \wedge \mathbf{1}(v)]\!] = \top \ \&\ h[\![\phi[u/v]]\!] \neq \top)$ F→
6. $\neg\exists h(g \approx_v h \ \&\ h(v) \subseteq h[\![A]\!] \ \&\ |h(v)| = 1 \ \&\ h[\![\phi[u/v]]\!] \neq \top)$ D3:≤, **1**, A
7. $\forall h(g \approx_v h \ \&\ h(v) \subseteq h[\![A]\!] \ \&\ |h(v)| = 1 \rightarrow h[\![\phi[u/v]]\!] = \top)$ rearr.

□

PlL facts:

F∀ $G[\![\forall u\phi]\!] = \top$, iff $\forall H(G \approx_u H \rightarrow H[\![\phi]\!] = \top)$

Fδ $G[\![\delta_u(\phi)]\!] = \top$, iff $\forall H(G \approx_v H \ \&\ H(v) \subseteq H(u) \ \&\ |H(v)| = 1 \rightarrow H[\![\phi[u/v]]\!] = \top)$

Proof of F∀

1. $G[\![\forall u\phi]\!] = \top$
2. $G[\![\neg\exists u\neg\phi]\!] = \top$ df.∀
3. $G[\![\exists u\neg\phi]\!] \neq \top$ D3:¬
4. $\neg\exists H(G \approx_u H \ \&\ H[\![\neg\phi]\!] = \top)$ D3:∃
5. $\neg\exists H(G \approx_u H \ \&\ H[\![\phi]\!] \neq \top)$ D3:¬
6. $\forall H(G \approx_u H \rightarrow H[\![\phi]\!] = \top)$ rearr.

□

Proof of Fδ

1. $G[\![\delta_u(\phi)]\!] = \top$
2. $G[\![\forall v(v \subseteq u \wedge \mathbf{1}(v) \rightarrow \phi[u/v])]\!] = \top$ df. δ
3. $\forall H(G \approx_v H \rightarrow H[\![(v \subseteq u \wedge \mathbf{1}(v) \rightarrow \phi[u/v])]\!] = \top)$ F∀
4. $\neg\exists H(G \approx_v H \ \&\ H[\![(v \subseteq u \wedge \mathbf{1}(v) \rightarrow \phi[u/v])]\!] \neq \top)$ rearr.
5. $\neg\exists H(G \approx_v H \ \&\ H[\![(v \subseteq u \wedge \mathbf{1}(v)]\!] = \top \ \&\ H[\![\phi[u/v]]\!] \neq \top)$ F→
6. $\neg\exists H(G \approx_v H \ \&\ H(v) \subseteq H(u) \ \&\ |H(v)| = 1 \ \&\ H[\![\phi[u/v]]\!] \neq \top)$ D3:⊆, **1**, A
7. $\forall H(G \approx_v H \ \&\ H(v) \subseteq H(u) \ \&\ |H(v)| = 1 \rightarrow H[\![\phi[u/v]]\!] = \top)$ rearr.

□

COLLECTIVITY & DISTRIBUTIVITY IN PlL

1. COLLECTIVE VP

(1) Adam and Beth *lifted (a stack of) three crates* (*together*).

1. $\models_M \exists x(x = a \oplus b \wedge \exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge L(x, y)))$
2. $\mathcal{G}^M[\![\exists x(x = a \oplus b \wedge \exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge L(x, y)))]\!] = \top$ — D4: $\models$
3. $\exists H$: $\mathcal{G}^M \approx_x H$ & $H[\![(x = a \oplus b \wedge \exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge L(x, y)))]\!] = \top$ — D3:$\exists$
4. $\exists H$: $\mathcal{G}^M \approx_x H$ — D3:$\wedge$, F=, D3:$\oplus$, A
   & $H(x) = \{[\![a]\!]\} \cup \{[\![b]\!]\}$
   & $H[\![\exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge L(x, y))]\!] = \top$
5. $\exists H$: $\mathcal{G}^M \approx_x H$ — df. $\cup$, D3:$\exists$, $\wedge$
   & $H(x) = \{[\![a]\!], [\![b]\!]\}$
   & $\exists J(H \approx_y J$ & $J[\![\mathbf{3}(y)]\!] = \top$ & $J[\![\delta_y(C(y))]\!] = \top$ & $J[\![L(x, y)]\!] = \top)$
6. $\exists H, J$: $\mathcal{G}^M \approx_x H$ & $H \approx_y J$ — rearr.
   & $H(x) = \{[\![a]\!], [\![b]\!]\}$ & $J[\![\mathbf{3}(y)]\!] = \top$
   & $J[\![\delta_y(C(y))]\!] = \top$ & $J[\![L(x, y)]\!] = \top$
7. $\exists H, J$: $\mathcal{G}^M \approx_x H$ & $H \approx_y J$ — D3:$\boldsymbol{n}$, Fδ, D3:$B($
   & $H(x) = \{[\![a]\!], [\![b]\!]\}$ & $|J(y)| = 3$
   & $\forall K(J \approx_z K$ & $K(z) \subseteq K(y)$ & $|K(z)| = 1 \rightarrow K[\![C(z)]\!] = \top)$
   & $\langle J(x), J(y)\rangle \in [\![L]\!]$
8. $\exists H, J$: $\mathcal{G}^M \approx_x H$ & $H \approx_y J$ — D3:$B($, $\langle \mathsf{a}\rangle = \mathsf{a}$
   & $H(x) = \{[\![a]\!], [\![b]\!]\}$ & $|J(y)| = 3$
   & $\forall K(J \approx_z K$ & $K(z) \subseteq K(y)$ & $|K(z)| = 1 \rightarrow K(z) \in [\![C]\!])$
   & $\langle J(x), J(y)\rangle \in [\![L]\!]$
9. $\exists H, J \exists \mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3$: $\mathcal{G}^M \approx_x H$ & $H \approx_y J$ — D2:$\approx_u$, $G(u)$
   & $H(x) = \{[\![a]\!], [\![b]\!]\}$ & $J(y) = \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}$ & $|\{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}| = 3$
   & $\forall K(J \approx_z K$ & $K(z) \subseteq \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}$ & $|K(z)| = 1 \rightarrow K(z) \in [\![C]\!])$
   & $\langle \{[\![a]\!], [\![b]\!]\}, \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}\rangle \in [\![L]\!]$
10. $\exists H, J \exists \mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3$: $\mathcal{G}^M \approx_x H$ & $H \approx_y J$ — simplify
    & $H(x) = \{[\![a]\!], [\![b]\!]\}$ & $J(y) = \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}$ & $|\{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}| = 3$
    & $\forall \mathsf{d}(\mathsf{d} \in \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\} \rightarrow \{\mathsf{d}\} \in [\![C]\!])$
    & $\langle \{[\![a]\!], [\![b]\!]\}, \{\mathsf{c}_1, \mathsf{c}_2. \mathsf{c}_3\}\rangle \in [\![L]\!]$

e.g. J: x	y	
a	c_1	partial matrix for one witness for J,
a	c_2	leaving out irrelevant variables & other assignments (i.e. rows)
b	c_3	with other values for irrelevant variables

11. $\exists \mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3$: — simplify
    $|\{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}| = 3$ & $\{\{\mathsf{c}_1\}, \{\mathsf{c}_2\}, \{\mathsf{c}_3\}\} \subseteq [\![C]\!]$
    & $\langle \{[\![a]\!], [\![b]\!]\}, \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}\rangle \in [\![L]\!]$

2. DISTRIBUTIVE VP WITH WIDE SCOPE OBJECT

(2) Adam and Beth (each) *lifted (the same stack of) three crates.* dist. VP, wide obj

(2_B) $\exists x(x = a \oplus b \wedge \exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge \delta_x(L(x, y))))$

1. $\models_M \exists x(x = a \oplus b \wedge \exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge \delta_x(L(x, y))))$
2. $\mathcal{G}^M[\![\exists x(x = a \oplus b \wedge \exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge \delta_x(L(x, y))))]\!] = \top$ — D4: $\models$
3. $\exists H$: $\mathcal{G}^M \approx_x H$ & $H[\![(x = a \oplus b \wedge \exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge \delta_x(L(x, y))))]\!] = \top$ — D3:$\exists$
4. $\exists H$: $\mathcal{G}^M \approx_x H$ — D3:$\wedge$, F=, D3:$\oplus$, A
   & $H(x) = \{[\![a]\!]\} \cup \{[\![b]\!]\}$
   & $H[\![\exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge \delta_x(L(x, y)))]\!] = \top$
5. $\exists H$: $\mathcal{G}^M \approx_x H$ — df. $\cup$, D3:$\exists$, $\wedge$
   & $H(x) = \{[\![a]\!], [\![b]\!]\}$
   & $\exists J(H \approx_y J$ & $J[\![\mathbf{3}(y)]\!] = \top$ & $J[\![\delta_y(C(y))]\!] = \top$ & $J[\![\delta_x(L(x, y))]\!] = \top)$
6. $\exists H, J$: $\mathcal{G}^M \approx_x H$ & $H \approx_y J$ — rearrange
   & $H(x) = \{[\![a]\!], [\![b]\!]\}$ & $J[\![\mathbf{3}(y)]\!] = \top$
   & $J[\![\delta_y(C(y))]\!] = \top$ & $J[\![\delta_x(L(x, y))]\!] = \top$
7. $\exists H, J$: $\mathcal{G}^M \approx_x H$ & $H \approx_y J$ — D3:$\boldsymbol{n}$, Fδ, D3:$B($
   & $H(x) = \{[\![a]\!], [\![b]\!]\}$ & $|J(y)| = 3$
   & $\forall K(J \approx_z K$ & $K(z) \subseteq K(y)$ & $|K(z)| = 1 \rightarrow K[\![C(z)]\!] = \top)$
   & $\forall K(J \approx_z K$ & $K(z) \subseteq K(x)$ & $|K(z)| = 1 \rightarrow K[\![L(z, y)]\!] = \top)$
8. $\exists H, J$: $\mathcal{G}^M \approx_x H$ & $H \approx_y J$ — D3:$B($, $\langle \mathsf{a} \rangle = \mathsf{a}$
   & $H(x) = \{[\![a]\!], [\![b]\!]\}$ & $|J(y)| = 3$
   & $\forall K(J \approx_z K$ & $K(z) \subseteq K(y)$ & $|K(z)| = 1 \rightarrow K(z) \in [\![C]\!])$
   & $\forall K(J \approx_z K$ & $K(z) \subseteq K(x)$ & $|K(z)| = 1 \rightarrow \langle K(z), K(y) \rangle \in [\![L]\!])$
9. $\exists H, J \exists \mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3$: $\mathcal{G}^M \approx_x H$ & $H \approx_y J$ — D2:$\approx_u$, $G(u)$
   & $H(x) = \{[\![a]\!], [\![b]\!]\}$ & $J(y) = \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}$ & $|\{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}| = 3$
   & $\forall K(J \approx_z K$ & $K(z) \subseteq \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}$ & $|K(z)| = 1 \rightarrow K(z) \in [\![C]\!])$
   & $\forall K(J \approx_z K$ & $K(z) \subseteq \{[\![a]\!], [\![b]\!]\}$ & $|K(z)| = 1 \rightarrow \langle K(z), \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\} \rangle \in [\![L]\!])$
10. $\exists H, J \exists \mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3$: $\mathcal{G}^M \approx_x H$ & $H \approx_y J$ — simplify
   & $H(x) = \{[\![a]\!], [\![b]\!]\}$ & $J(y) = \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}$ & $|\{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}| = 3$
   & $\forall \mathsf{d}(\mathsf{d} \in \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\} \rightarrow \{\mathsf{d}\} \in [\![C]\!])$
   & $\langle \{[\![a]\!]\}, \{\mathsf{c}_1, \mathsf{c}_2. \mathsf{c}_3\} \rangle \in [\![L]\!]$ & $\langle \{[\![b]\!]\}, \{\mathsf{c}_1, \mathsf{c}_2. \mathsf{c}_3\} \rangle \in [\![L]\!]$

   e.g. J: partial matrix, relevant variables & assignments only

x	y
a	c_1
a	c_2
a	c_3
b	c_1
b	c_2
b	c_3

11. $\exists \mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3$: — simplify
   $|\{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\}| = 3$ & $\{\{\mathsf{c}_1\}, \{\mathsf{c}_2\}, \{\mathsf{c}_3\}\} \subseteq [\![C]\!]$
   & $\langle \{[\![a]\!]\}, \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\} \rangle \in [\![L]\!]$ & $\langle \{[\![b]\!]\}, \{\mathsf{c}_1, \mathsf{c}_2, \mathsf{c}_3\} \rangle \in [\![L]\!]$

3. DISTRIBUTIVE VP WITH NARROW SCOPE OBJECT

(3) Adam and Beth (each) *lifted (a different stack of) three crates.* dist. VP, narrow obj

(3_{B}) $\exists x(x = a \oplus b \wedge \delta_x(\exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge L(x, y))))$

1. $\models_M \exists x(x = a \oplus b \wedge \delta_x(\exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge L(x, y))))$
2. $\mathcal{G}^M[\![\exists x(x = a \oplus b \wedge \delta_x(\exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge L(x, y))))]\!] = \top$ D4:|=
3. $\exists H\text{: } \mathcal{G}^M \approx_x H \;\&\; H[\![(x = a \oplus b \wedge \delta_x(\exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge L(x, y))))]\!] = \top$ D3:∃
4. $\exists H\text{: } \mathcal{G}^M \approx_x H \;\&\; H(x) = \{[\![a]\!]\} \cup \{[\![b]\!]\}$ D3:∧, F=, D3:⊕, *A*
   $\&\; H[\![\delta_x(\exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge L(x, y)))]\!] = \top$
5. $\exists H\text{: } \mathcal{G}^M \approx_x H \;\&\; H(x) = \{[\![a]\!], [\![b]\!]\}$ df. ∪, Fδ
   $\&\; \forall K(H \approx_z K \;\&\; K(z) \subseteq K(x) \;\&\; |K(z)| = 1$
   $\rightarrow K[\![\exists y(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge L(z, y))]\!] = \top)$
6. $\exists H\text{: } \mathcal{G}^M \approx_x H \;\&\; H(x) = \{[\![a]\!], [\![b]\!]\}$ D2:$\approx_u$, D3:∃
   $\&\; \forall K(H \approx_z K \;\&\; K(z) \subseteq \{[\![a]\!], [\![b]\!]\} \;\&\; |K(z)| = 1$
   $\rightarrow \exists J(K \approx_y J \;\&\; J[\![(\mathbf{3}(y) \wedge \delta_y(C(y)) \wedge L(z, y))]\!] = \top))$
7. $\exists H\text{: } \mathcal{G}^M \approx_x H \;\&\; H(x) = \{[\![a]\!], [\![b]\!]\}$ D3:∧, ***n***, *B*(
   $\&\; \forall K(H \approx_z K \;\&\; K(z) \subseteq \{[\![a]\!], [\![b]\!]\} \;\&\; |K(z)| = 1$
   $\rightarrow \exists J(K \approx_y J \;\&\; |J(y)| = 3 \;\&\; J[\![\delta_y(C(y))]\!] = \top \;\&\; \langle J(z), J(y)\rangle \in [\![L]\!]))$
8. $\exists H\text{: } \mathcal{G}^M \approx_x H \;\&\; H(x) = \{[\![a]\!], [\![b]\!]\}$ D2:$\approx_u$, $J(u)$
   $\&\; \exists J(\{h[x/[\![a]\!]]\text{: } h \in H\} \approx_y J \;\&\; |J(y)| = 3 \;\&\; J[\![\delta_y(C(y))]\!] = \top \;\&\; \langle\{[\![a]\!]\}, J(y)\rangle \in [\![L]\!])$
   $\&\; \exists J(\{h[x/[\![b]\!]]\text{: } h \in H\} \approx_y J \;\&\; |J(y)| = 3 \;\&\; J[\![\delta_y(C(y))]\!] = \top \;\&\; \langle\{[\![b]\!]\}, J(y)\rangle \in [\![L]\!])$
9. $\exists H\text{: } \mathcal{G}^M \approx_x H \;\&\; H(x) = \{[\![a]\!], [\![b]\!]\}$ D2:$\approx_u$, Fδ, D3:*B*(
   $\&\; \exists J \exists \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3(\{h[x/[\![a]\!]]\text{: } h \in H\} \approx_y J \;\&\; J(y) = \{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\} \;\&\; |\{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}| = 3$
   $\&\; \forall K(J \approx_z K \;\&\; K(z) \subseteq \{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\} \;\&\; |K(z)| = 1 \rightarrow K(z) \in [\![C]\!])$
   $\&\; \langle\{[\![a]\!]\}, \{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}\rangle \in [\![L]\!])$
   $\&\; \exists J \exists \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3(\{h[x/[\![b]\!]]\text{: } h \in H\} \approx_y J \;\&\; J(y) = \{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\} \;\&\; |\{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}| = 3$
   $\&\; \forall K(J \approx_z K \;\&\; K(z) \subseteq \{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\} \;\&\; |K(z)| = 1 \rightarrow K(z) \in [\![C]\!])$
   $\&\; \langle\{[\![b]\!]\}, \{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}\rangle \in [\![L]\!])$
10. $\exists H\text{: } \mathcal{G}^M \approx_x H \;\&\; H(x) = \{[\![a]\!], [\![b]\!]\}$ D2:$\approx_u$, $G(u)$
    $\&\; \exists J \exists \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3(\{h[x/[\![a]\!]]\text{: } h \in H\} \approx_y J \;\&\; J(y) = \{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\} \;\&\; |\{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}| = 3$
    $\&\; \forall \mathrm{d}(\mathrm{d} \in \{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\} \in [\![C]\!]) \;\&\; \langle\{[\![a]\!]\}, \{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}\rangle \in [\![L]\!])$
    $\&\; \exists J \exists \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3(\{h[x/[\![b]\!]]\text{: } h \in H\} \approx_y J \;\&\; J(y) = \{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\} \;\&\; |\{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}| = 3$
    $\&\; \forall \mathrm{d}(\mathrm{d} \in \{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\} \in [\![C]\!]) \;\&\; \langle\{[\![b]\!]\}, \{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}\rangle \in [\![L]\!])$

e.g. *J*: partial matrix, relevant variables & assignments only

x	y
a	c_1
a	c_2
a	c_3
b	c_1'
b	c_2'
b	c_3'

11. $\exists \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3(|\{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}| = 3 \;\&\; \{\{\mathrm{c}_1\}, \{\mathrm{c}_2\}, \{\mathrm{c}_3\}\} \subseteq [\![C]\!] \;\&\; \langle\{[\![a]\!]\}, \{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}\rangle \in [\![L]\!])$ smpl.
    $\&\; \exists \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3(|\{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}| = 3 \;\&\; \{\{\mathrm{c}_1\}, \{\mathrm{c}_2\}, \{\mathrm{c}_3\}\} \subseteq [\![C]\!] \;\&\; \langle\{[\![b]\!]\}, \{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}\rangle \in [\![L]\!])$

QUANTIFIERS IN PlL

1 QUANTIFIERS AS 2-PLACE PREDICATES

Extend PlL with:

DEFINITION 2 (PlL-models and assignments)

M. A PlL-*model* is a structure $M = \langle D, [\![\cdot]\!]\rangle$ such that:

⋮

iii. for any $X, Y \in \mathcal{P}(D)$:

$\langle X, Y\rangle \in [\![all]\!]$ iff $X \subseteq Y$

$\langle X, Y\rangle \in [\![sm]\!]$ iff $X \cap Y \neq \varnothing$

$\langle X, Y\rangle \in [\![most]\!]$ iff $|X \cap Y| > |X\backslash Y|$

$\langle X, Y\rangle \in [\![n!]\!]$ iff $|X \cap Y| = n$

ABBREVIATIONS 2 (proper part, maximization, induced quantifiers)

- $(A \subset B) := ((A \subseteq B) \wedge \neg(A = B))$ — for $A, B \in$ **Trm**
- $\mathbf{M}_u(\phi) := (\neg\exists v(u \subset v \wedge \phi[u/v]) \wedge \phi)$ — $\phi \in$ **For**, no v in ϕ
- $\mathbf{Q}u(\phi, \psi) := \exists u\exists v(\mathbf{M}_v(\phi[u/v]) \wedge \mathbf{M}_u(u \subseteq v \wedge \psi) \wedge Q(v, u))$ — $Q \in \{all, sm, most, n!\}$

FACTS

F⊂ $G[\![(A \subset B)]\!] = \top$, iff $G[\![A]\!] \subset G[\![B]\!]$

F**M** $G[\![\mathbf{M}_u(\phi)]\!] = \top$, iff $\neg(\exists H\!: G \approx_v H \;\&\; G(u) \subset H(v) \;\&\; H[\![\phi[u/v]]\!] = \top) \;\&\; G[\![\phi]\!] = \top$

F**Q** $G[\![\mathbf{Q}u(\phi, \psi)]\!] = \top$, iff $\exists H\exists K(G \approx_u H \;\&\; H \approx_v K \;\&\; K[\![\mathbf{M}_v(\phi[u/v])]\!]$
$\;\&\; K[\![\mathbf{M}_u(u \subseteq v \wedge \phi)]\!] \;\&\; \langle K(v), H(u)\rangle \in [\![Q]\!])$

Some paradigm examples:

(1) All the women gathered in the square. ↓MON↑, collective VP

(1′) $\mathbf{all}\, x(\delta_x(W(x)), G(x))$
$:= \exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(W(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge G(x)) \wedge all(x', x))$

(2) Every child is asleep. ↓MON↑, distributive VP

(2′) $\mathbf{all}\, x(\delta_x(C(x)), \delta_x(S(x)))$
$:= \exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(C(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge \delta_x(S(x))) \wedge all(x', x))$

(3) Most women gathered in the square. MON↑, collective VP

(3′) $\mathbf{most}\, x(\delta_x(W(x)), G(x))$
$:= \exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(W(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge G(x)) \wedge most(x', x))$

(4) None of the women gathered in the square. ↓MON↓, collective VP

(4′) $\neg\, \mathbf{sm}\, x(\delta_x(W(x)), G(x))$
$:= \exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(W(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge G(x)) \wedge sm(x', x))$

(5) (Last year) two scientists (*jointly*) wrote an important paper. collective VP

(5′) $\mathbf{2!}\, x(\delta_x(S(x)), \exists y(P(y) \wedge \mathbf{1}(y) \wedge W(x, y)))$
$:= \exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(S(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge \exists y(P(y) \wedge \mathbf{1}(y) \wedge W(x, y))) \wedge 2!(x', x))$

(6) (Last year) two scientists (*each*) wrote an important paper. distributiveVP

(6′) $\mathbf{2!}\, x(\delta_x(S(x)), \delta_x(\exists y(P(y) \wedge \mathbf{1}(y) \wedge W(x, y))))$
$:= \exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(S(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge \delta_x(\exists y(P(y) \wedge \mathbf{1}(y) \wedge W(x, y)))) \wedge 2!(x', x))$

2. Sample Truth Conditions and Models

(1) All the women gathered in the square. ↓MON↑, collective VP

(1′) $\models_M$ **all** $x(\delta_x(W(x)), G(x))$

iff $\models_M \mathbf{\exists} x \mathbf{\exists} x'(\mathbf{M}_{x'}(\delta_{x'}(W(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge G(x)) \wedge all(x', x))$

iff $\exists X \in \mathcal{P}(D^M)\backslash\{\emptyset\}$:

$\forall d(d \in X \rightarrow \{d\} \in [\![W]\!]) \;\&\; X \in [\![G]\!]$

$\& \neg\exists Y(X \subset Y \;\&\; \forall d(d \in Y \rightarrow \{d\} \in [\![W]\!]))$

Sample models:	Witness for $\exists X$	Truth values	
$M_0 = \langle\{a, b, c, d\}, [\![\cdot]\!]^{M0}\rangle$	none	predicted:	$\not\models_{M0}$ (1′)
$[\![W]\!]^{M0} = \{\{a\}, \{b\}, \{c\}\}$		MB intuition:	$\models_{M0}$ (1′)
$[\![G]\!]^{M0} = \{\{a, b, c, d\}\}$			**Oops!**
		Solution: constrain $[\![\cdot]\!]^M$ for collective predicates	
$M_1 = \langle\{a, b, c, d\}, [\![\cdot]\!]^{M1}\rangle$	$X = \{a, b, c\}$	predicted:	$\models_{M1}$ (1′)
$[\![W]\!]^{M1} = \{\{a\}, \{b\}, \{c\}\}$		MB intuition:	$\models_{M1}$ (1)
$[\![G]\!]^{M1} = \{X \mid X \subseteq \{a, b, c\} \;\&\; \lvert X\rvert \geq 2\}$			✓
$M_2 = \langle\{a, b, c, d\}, [\![\cdot]\!]^{M2}\rangle$	none	predicted:	$\not\models_{M2}$ (1′)
$[\![W]\!]^{M2} = \{\{a\}, \{b\}, \{c\}\}$		MB intuition:	$\not\models_{M2}$ (1)
$[\![G]\!]^{M2} = \{X \mid X \subseteq \{a, b, d\} \;\&\; \lvert X\rvert \geq 2\}$			✓
$M_3 = \langle\{a, b, c, d\}, [\![\cdot]\!]^{M3}\rangle$	$X = \{a, b, c\}$	predicted:	$\models_{M3}$ (1′)
$[\![W]\!]^{M3} = \{\{a\}, \{b\}, \{c\}\}$		MB intuition:	$\models_{M3}$ (1′)
$[\![G]\!]^{M3} = \{X \mid X \subseteq \{a, b, c, d\} \;\&\; \lvert X\rvert \geq 2\}$			✓

(2) Every child is asleep. ↓MON↑, distributive VP

(2′) $\models_M$ **all** $x(\delta_x(C(x)), \delta_x(S(x)))$

iff $\models_M \mathbf{\exists} x \mathbf{\exists} x'(\mathbf{M}_{x'}(\delta_{x'}(C(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge \delta_x(S(x))) \wedge all(x', x))$

iff $\exists X \in \mathcal{P}(D^M)\backslash\{\emptyset\}$:

$\forall d(d \in X \rightarrow \{d\} \in [\![C]\!] \;\&\; d \in [\![S]\!])$

$\& \neg\exists Y(X \subset Y \;\&\; \forall d(d \in Y \rightarrow \{d\} \in [\![C]\!]))$

Sample models:	Witness for $\exists X$	Truth values	
$M_1 = \langle\{a, b, c, d\}, [\![\cdot]\!]^{M1}\rangle$	$X = \{a, b, c\}$	predicted:	$\models_{M1}$ (2′)
$[\![C]\!]^{M1} = \{\{a\}, \{b\}, \{c\}\}$		MB intuition:	$\models_{M1}$ (2)
$[\![S]\!]^{M1} = \{\{a\}, \{b\}, \{c\}\}$			✓
$M_2 = \langle\{a, b, c, d\}, [\![\cdot]\!]^{M2}\rangle$	none	predicted:	$\not\models_{M2}$ (2′)
$[\![C]\!]^{M2} = \{\{a\}, \{b\}, \{c\}\}$		MB intuition:	$\not\models_{M2}$ (2′)
$[\![S]\!]^{M2} = \{\{a\}, \{b\}, \{d\}\}$			✓
$M_3 = \langle\{a, b, c, d\}, [\![\cdot]\!]^{M3}\rangle$	$X = \{a, b, c\}$	predicted:	$\models_{M3}$ (2′)
$[\![C]\!]^{M3} = \{\{a\}, \{b\}, \{c\}\}$		MB intuition:	$\models_{M2}$ (2′)
$[\![S]\!]^{M3} = \{\{a\}, \{b\}, \{c\}, \{d\}\}$			✓

(3) Most (of the) women gathered in the square. MON↑, collective VP

(3′) $\models_M$ **most** $x(\delta_x(W(x)), G(x))$

iff $\models_M \exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(W(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge G(x)) \wedge most(x', x))$

iff $\exists X, X' \in \mathcal{P}(D^M)\backslash\{\varnothing\}$:

$\forall d(d \in X' \to \{d\} \in [\![W]\!]$

$\&\ X \subseteq X' \ \&\ X \in [\![G]\!] \ \&\ |X' \cap X| > |X' \backslash X|$

$\&\ \neg\exists Y(X' \subset Y \subseteq D^M \ \&\ \forall d: d \in Y \to \{d\} \in [\![W]\!]^M)$

$\&\ \neg\exists Y(X \subset Y \subseteq X' \ \&\ Y \in [\![G]\!]^M)$

Sample models:	Witnesses for X', X	Truth values	
$M_1 = \langle\{a, b, c, d\}, [\![\cdot]\!]^{M1}\rangle$	$X' := \{a, b, c\}$	predicted:	$\models_{M1}$ (3′)
$[\![W]\!]^{M1} = \{\{a\}, \{b\}, \{c\}\}$	$X := \{a, b, c\}$	MB intuition:	$\models_{M1}$ (3)
$[\![G]\!]^{M1} = \{X \mid X \subseteq \{a, b, c\}\ \&\ \lvert X \rvert \geq 2\}$			✓
$M_2 = \langle\{a, b, c, d\}, [\![\cdot]\!]^{M2}\rangle$	$X' := \{a, b, c\}$	predicted:	$\models_{M2}$ (3′)
$[\![W]\!]^{M2} = \{\{a\}, \{b\}, \{c\}\}$	$X := \{a, b\}$	MB intuition:	$\models_{M2}$ (3)
$[\![G]\!]^{M2} = \{X \mid X \subseteq \{a, b, d\}\ \&\ \lvert X \rvert \geq 2\}$			✓
$M_3 = \langle\{a, b, c, d, e\}, [\![\cdot]\!]^{M3}\rangle$	$X' := \{a, b, c\}$	predicted:	$\not\models_{M3}$ (3′)
$[\![W]\!]^{M3} = \{\{a\}, \{b\}, \{c\}\}$	none for X	MB intuition:	$\not\models_{M3}$ (3)
$[\![G]\!]^{M3} = \{X \mid X \subseteq \{a, d, e\}\ \&\ \lvert X \rvert \geq 2\}$			✓

(4) None of the women gathered in the square. ↓MON↓, collective VP

(4′) $\models_M \neg$ **sm** $x(\delta_x(Wx), Gx)$

iff $\models_M \neg\exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(W(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge G(x)) \wedge sm(x', x))$

iff $\neg\exists X, X' \in \mathcal{P}(D^M)\backslash\{\varnothing\}$:

$\forall d(d \in X' \to \{d\} \in [\![W]\!]^M) \ \&\ \varnothing \subset X \subseteq X' \ \&\ X \in [\![G]\!]^M$

Sample models:	Counterexamples for X', X	Truth values	
$M_1 = \langle\{a, b, c, d, e\}, [\![\cdot]\!]^{M1}\rangle$	$X' := \{a, b, c\}$	predicted:	$\models_{M1}$ (4′)
$[\![W]\!]^{M1} = \{\{a\}, \{b\}, \{c\}\}$	none for X	MB intuition:	$\models_{M1}$ (4)
$[\![G]\!]^{M1} = \{X \mid X \subseteq \{d, e\}: \lvert X \rvert \geq 2\}$			✓
$M_2 = \langle\{a, b, c, d, e\}, [\![\cdot]\!]^{M2}\rangle$	$X' := \{a, b, c\}$	predicted:	$\models_{M2}$ (4′)
$[\![W]\!]^{M2} = \{\{a\}, \{b\}, \{c\}\}$	none for X	MB intuition:	$\not\models_{M2}$ (4)
$[\![G]\!]^{M2} = \{X \mid X \subseteq \{a, d, e\}\ \&\ \lvert X \rvert \geq 2\}$		***remark***: possibly a different reading of (4), with *gather = come to the gathering*	
$M_3 = \langle\{a, b, c, d, e\}, [\![\cdot]\!]^{M3}\rangle$	none for X'	predicted:	$\models_{M3}$ (4′)
$[\![W]\!]^{M3} = \varnothing$	hence none for X	MB intuition:	$\neg(\models_{M3}$ (4))
$[\![G]\!]^{M3} = \{X \mid X \subseteq \{a, d, e\}: \lvert X \rvert \geq 2\}$		***problem***: (4) *presupposes* a plural set of women	

(4″) $\models_M \exists x'(\mathbf{M}_{x'}(\delta_{x'}(W(x'))) \wedge \neg\exists x(\mathbf{M}_x(x \subseteq x' \wedge G(x)) \wedge sm(x', x)))$

iff $\exists X' \in \mathcal{P}(D^M)\backslash\{\varnothing\}$:

$\forall d(d \in X' \to \{d\} \in [\![W]\!]^M) \ \&\ \neg\exists X(\varnothing \subset X \subseteq X' \ \&\ X \in [\![G]\!]^M)$

3. DETAILED DERIVATIONS

(1) All the women gathered in the square. $\downarrow$MON$\uparrow$, collective VP

0. $\models_M$ **all** $x(\delta_x(W(x)), G(x))$
1. $\models_M \exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(W(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge G(x)) \wedge all(x', x))$ A2.**Q**
2. $\mathcal{G}^M[\![\exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(W(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge G(x)) \wedge all(x', x))]\!] = \top$ D4:$\models$
3. $\exists H, K$: $\mathcal{G}^M \approx_x H$ & $H \approx_{x'} K$ F**Q**
   & $K[\![\mathbf{M}_{x'}(\delta_{x'}(W(x')))]\!] = \top$ & $K[\![\mathbf{M}_x(x \subseteq x' \wedge G(x))]\!] = \top$
   & $\langle K(x'), K(x)\rangle \in [\![all]\!]$
4. $\exists H, K$: $\mathcal{G}^M \approx_x H$ & $H \approx_{x'} K$ F**M**, D2.$[\![all]\!]$
   & $\neg\exists J(K \approx_y J$ & $K(x') \subset J(y)$ & $J[\![\delta_y(W(y))]\!] = \top)$
   & $K[\![\delta_{x'}(W(x'))]\!] = \top$
   & $\neg\exists J(K \approx_y J$ & $K(x) \subset J(y)$ & $J[\![y \subseteq x' \wedge G(y)]\!] = \top)$
   & $K[\![x \subseteq x' \wedge G(x)]\!] = \top$ & $K(x') \subseteq K(x)$
5. $\exists H, K$: $\mathcal{G}^M \approx_x H$ & $H \approx_{x'} K$ D3.$\wedge$, $\subseteq$, $B($, A, D2.$\approx$
   & $\neg\exists J(K \approx_y J$ & $K(x') \subset J(y)$ & $J[\![\delta_y(W(y))]\!] = \top)$
   & $K[\![\delta_{x'}(W(x'))]\!] = \top$
   & $\neg\exists J(K \approx_y J$ & $K(x) \subset J(y)$ & $J(y) \subseteq K(x')$ & $J(y) \in [\![G]\!])$
   & $K(x) \subseteq K(x')$ & $K(x) \in [\![G]\!]$ & $K(x') \subseteq K(x)$
6. $\exists H, K$: $\mathcal{G}^M \approx_x H$ & $H \approx_{x'} K$ rearr., simplify
   & $K[\![\delta_{x'}(W(x'))]\!] = \top$
   & $K(x) = K(x')$ & $K(x) \in [\![G]\!]$
   & $\neg\exists J(K \approx_y J$ & $K(x') \subset J(y)$ & $J[\![\delta_y(W(y))]\!] = \top)$
7. $\exists H, K$: $\mathcal{G}^M \approx_x H$ & $H \approx_{x'} K$ Fδ, D2.$\approx_u$, D3.$B($
   & $\forall I(K \approx_y I$ & $I(y) \subseteq K(x')$ & $|I(y)| = 1 \rightarrow I(y) \in [\![W]\!])$
   & $K(x) = K(x')$ & $K(x) \in [\![G]\!]$
   & $\neg\exists J(K \approx_y J$ & $K(x') \subset J(y)$ & $\forall I'(J \approx_z I'$ & $I'(z) \subseteq J(y)$ & $|I'(z)| = 1 \rightarrow I'(y) \in [\![W]\!]))$
8. $\exists H, K$: $\mathcal{G}^M \approx_x H$ & $H \approx_{x'} K$ elim. $\forall I$
   & $\forall d(d \in K(x') \rightarrow \{d\} \in [\![W]\!])$
   & $K(x) = K(x')$ & $K(x) \in [\![G]\!]$
   & $\neg\exists J(K \approx_y J$ & $K(x') \subset J(y)$ & $\forall d(d \in J(y) \rightarrow \{d\} \in [\![W]\!]))$
9. $\exists H, K$: $\mathcal{G}^M \approx_x H$ & $H \approx_{x'} K$ elim. $\exists J$
   & $\forall d(d \in K(x') \rightarrow \{d\} \in [\![W]\!])$ & $K(x) = K(x')$ & $K(x) \in [\![G]\!]$
   & $\neg\exists Y(K(x') \subset Y$ & $\forall d(d \in Y \rightarrow \{d\} \in [\![W]\!]))$
10. $\exists X, X' \in \mathcal{P}(D^M)\setminus\{\varnothing\}$: elim. $\exists H$, $\exists K$
    $\forall d(d \in X' \rightarrow \{d\} \in [\![W]\!])$ & $X = X'$ & $X \in [\![G]\!]$
    & $\neg\exists Y(X' \subset Y$ & $\forall d(d \in Y \rightarrow \{d\} \in [\![W]\!]))$
11. $\exists X \in \mathcal{P}(D^M)\setminus\{\varnothing\}$: elim. $\exists X'$
    $\forall d(d \in X \rightarrow \{d\} \in [\![W]\!])$ & $X \in [\![G]\!]$
    & $\neg\exists Y(X \subset Y$ & $\forall d(d \in Y \rightarrow \{d\} \in [\![W]\!]))$

(2) Every child is asleep. ↓MON↑, distributive VP

0. $\models_M$ **all** $x(\delta_x(C(x)), \delta_x(S(x)))$
1. $\models_M \exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(C(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge \delta_x(S(x))) \wedge all(x', x))$
2. $\mathcal{G}^M[\![\exists x \exists x'(\mathbf{M}_{x'}(\delta_{x'}(C(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge \delta_x(S(x))) \wedge all(x', x))]\!] = \top$ D4:$\models$
3. $\exists H, K$: $\mathcal{G}^M \approx_x H \;\&\; H \approx_{x'} K$ F**Q**
   $\&\; K[\![\mathbf{M}_{x'}(\delta_{x'}(C(x')))]\!] = \top \;\&\; K[\![\mathbf{M}_x(x \subseteq x' \wedge \delta_x(S(x)))]\!] = \top$
   $\&\; \langle K(x'), K(x)\rangle \in [\![all]\!]$
4. $\exists H, K$: $\mathcal{G}^M \approx_x H \;\&\; H \approx_{x'} K$ F**M**, D2.$[\![all]\!]$
   $\&\; \neg\exists J(K \approx_y J \;\&\; K(x') \subset J(y) \;\&\; J[\![\delta_y(C(y))]\!] = \top)$
   $\&\; K[\![\delta_{x'}(C(x'))]\!] = \top$
   $\&\; \neg\exists J(K \approx_y J \;\&\; K(x) \subset J(y) \;\&\; J[\![y \subseteq x' \wedge \delta_x(S(x))]\!] = \top)$
   $\&\; K[\![x \subseteq x' \wedge \delta_x(S(x)))]\!] = \top \;\&\; K(x') \subseteq K(x)$
5. $\exists H, K$: $\mathcal{G}^M \approx_x H \;\&\; H \approx_{x'} K$ D3.$\wedge$, $\subseteq$, $B($, A
   $\&\; \neg\exists J(K \approx_y J \;\&\; K(x') \subset J(y) \;\&\; J[\![\delta_y(C(y))]\!] = \top)$
   $\&\; K[\![\delta_{x'}(C(x'))]\!] = \top$
   $\&\; \neg\exists J(K \approx_y J \;\&\; K(x) \subset J(y) \;\&\; J(y) \subseteq J(x') \;\&\; K[\![\delta_{x'}(S(x'))]\!] = \top)$
   $\&\; K(x) \subseteq K(x') \;\&\; K[\![\delta_{x'}(S(x'))]\!] = \top \;\&\; K(x') \subseteq K(x)$
6. $\exists H, K$: $\mathcal{G}^M \approx_x H \;\&\; H \approx_{x'} K$ rearrange, simplify
   $\&\; K[\![\delta_{x'}(C(x'))]\!] = \top$
   $\&\; K(x) = K(x') \;\&\; K[\![\delta_{x'}(S(x'))]\!] = \top$
   $\&\; \neg\exists J(K \approx_y J \;\&\; K(x') \subset J(y) \;\&\; J[\![\delta_y(C(y))]\!] = \top)$
7. $\exists H, K$: $\mathcal{G}^M \approx_x H \;\&\; H \approx_{x'} K$ Fδ, D3.$B($,D2.$\approx_u$
   $\&\; \forall I(K \approx_y I \;\&\; I(y) \subseteq K(x') \;\&\; |I(y)| = 1 \rightarrow I(y) \in [\![C]\!])$
   $\&\; H(x) = K(x') \;\&\; \forall I(K \approx_y I \;\&\; I(y) \subseteq H(x) \;\&\; |I(y)| = 1 \rightarrow I(y) \in [\![S]\!])$
   $\&\; \neg\exists J(K \approx_y J \;\&\; K(x') \subset J(y)$
   $\quad \&\; \forall I(J \approx_z I \;\&\; I(z) \subseteq J(y) \;\&\; |I(z)| = 1 \rightarrow I(z) \in [\![C]\!]))$
8. $\exists H, K$: $\mathcal{G}^M \approx_x H \;\&\; H \approx_{x'} K$ elim. $\forall I$
   $\&\; \forall d(d \in K(x') \rightarrow \{d\} \in [\![C]\!])$
   $\&\; H(x) = K(x') \;\&\; \forall d(d \in H(x) \rightarrow \{d\} \in [\![S]\!])$
   $\&\; \neg\exists J(K \approx_y J \;\&\; K(x') \subset J(y) \;\&\; \forall d(d \in J(y) \rightarrow \{d\} \in [\![C]\!]))$
9. $\exists H, K$: $\mathcal{G}^M \approx_x H \;\&\; H \approx_{x'} K$ elim. $\exists J$, simplify
   $\&\; H(x) = K(x') \;\&\; \forall d(d \in K(x') \rightarrow \{d\} \in [\![C]\!] \;\&\; \{d\} \in [\![S]\!])$
   $\&\; \neg\exists Y(K(x') \subset Y \;\&\; \forall d(d \in Y \rightarrow \{d\} \in [\![C]\!]))$
10. $\exists X, X' \in \mathcal{P}(D^M)\backslash\{\varnothing\}$: elim. $\exists H$, $\exists K$
   $X = X' \;\&\; \forall d(d \in X' \rightarrow \{d\} \in [\![C]\!] \;\&\; \{d\} \in [\![S]\!])$
   $\&\; \neg\exists Y(X' \subset Y \;\&\; \forall d(d \in Y \rightarrow \{d\} \in [\![C]\!]))$
11. $\exists X \in \mathcal{P}(D^M)\backslash\{\varnothing\}$: elim. $\exists X'$
   $\forall d(d \in X \rightarrow \{d\} \in [\![C]\!] \;\&\; \{d\} \in [\![S]\!])$
   $\&\; \neg\exists Y(X \subset Y \;\&\; \forall d(d \in Y \rightarrow \{d\} \in [\![C]\!]))$

(3) Most (of the) women gathered in the square. ↓MON↑, collective VP

0. $\models_M$ **most** $x(\delta_x(W(x)), G(x))$
1. $\models_M \exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(W(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge G(x)) \wedge most(x', x))$ A2.**Q**
2. $\mathcal{G}^M[\![\exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(W(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge G(x)) \wedge most(x', x))]\!] = \top$ D4:$\models$
3. $\exists H, K$: $\mathcal{G}^M \approx_x H$ & $H \approx_{x'} K$ F**Q**
   & $K[\![\mathbf{M}_{x'}(\delta_{x'}(W(x')))]\!] = \top$ & $K[\![\mathbf{M}_x(x \subseteq x' \wedge G(x))]\!] = \top$
   & $\langle K(x'), H(x)\rangle \in [\![most]\!]$
4. $\exists H, K$: $\mathcal{G}^M \approx_x H$ & $H \approx_{x'} K$ F**M**, rearrange
   & $K[\![\delta_{x'}(W(x'))]\!] = \top$
   & $K[\![x \subseteq x' \wedge G(x)]\!] = \top$ & $\langle K(x'), H(x)\rangle \in [\![most]\!]$
   & $\neg\exists J(K \approx_y J$ & $K(x') \subset J(y)$ & $J[\![\delta_y(W(y))]\!] = \top)$
   & $\neg\exists J(K \approx_y J$ & $K(x) \subset J(y)$ & $J[\![y \subseteq x' \wedge G(y)]\!] = \top)$
5. $\exists H, K$: $\mathcal{G}^M \approx_x H$ & $H \approx_{x'} K$ D3.$\wedge$, $\subseteq$, $B($, A, D2.$\approx$
   & $K[\![\delta_{x'}(W(x'))]\!] = \top$
   & $H(x) \subseteq K(x')$ & $H(x) \in [\![G]\!]$ & $\langle K(x'), H(x)\rangle \in [\![most]\!]$
   & $\neg\exists J(K \approx_y J$ & $K(x') \subset J(y)$ & $J[\![\delta_y(W(y))]\!] = \top)$
   & $\neg\exists J(K \approx_y J$ & $H(x) \subset J(y)$ & $J(y) \subseteq K(x')$ & $J(y) \in [\![G]\!])$
6. $\exists H, K$: $\mathcal{G}^M \approx_x H$ & $H \approx_{x'} K$ Fδ, D3.$B($, A
   & $\forall I(K \approx_z I$ & $I(z) \subseteq K(x')$ & $|I(z)| = 1 \rightarrow I(z) \in [\![W]\!])$
   & $H(x) \subseteq K(x')$ & $K(x) \in [\![G]\!]$ & $\langle K(x'), H(x)\rangle \in [\![most]\!]$
   & $\neg\exists J(K \approx_y J$ & $K(x') \subset J(y)$ & $\forall I(J \approx_z I$ & $I(z) \subseteq J(y)$ & $|I(z)| = 1 \rightarrow I(z) \in [\![W]\!]))$
   & $\neg\exists J(K \approx_y J$ & $H(x) \subset J(y)$ & $J(y) \subseteq K(x')$ & $J(y) \in [\![G]\!])$
7. $\exists H, K$: $\mathcal{G}^M \approx_x H$ & $H \approx_{x'} K$ elim. $\forall d$
   & $\forall d(d \in K(x') \rightarrow \{d\} \in [\![W]\!])$
   & $H(x) \subseteq K(x')$ & $H(x) \in [\![G]\!]$ & $\langle K(x'), H(x)\rangle \in [\![most]\!]$
   & $\neg\exists J(K \approx_y J$ & $K(x') \subset J(y)$ & $\forall d(d \in J(y) \rightarrow \{d\} \in [\![W]\!]))$
   & $\neg\exists J(K \approx_y J$ & $H(x) \subset J(y)$ & $J(y) \subseteq K(x')$ & $J(y) \in [\![G]\!])$
8. $\exists H, K$: $\mathcal{G}^M \approx_x H$ & $H \approx_{x'} K$ elim. $\exists J$, simplify
   & $\forall d(d \in K(x') \rightarrow \{d\} \in [\![W]\!])$
   & $H(x) \subseteq K(x')$ & $H(x) \in [\![G]\!]$ & $\langle K(x'), H(x)\rangle \in [\![most]\!]$
   & $\neg\exists Y(K(x') \subset Y$ & $\forall d(d \in Y \rightarrow \{d\} \in [\![W]\!]))$
   & $\neg\exists Y(H(x) \subset Y \subseteq K(x')$ & $Y \in [\![G]\!])$
9. $\exists X, X' \in \mathcal{P}(D^M)\backslash\{\varnothing\}$: elim. $\exists H, K$
   & $\forall d(d \in X' \rightarrow \{d\} \in [\![W]\!])$
   & $X \subseteq X'$ & $X \in [\![G]\!]$ & $\langle X', X\rangle \in [\![most]\!]$
   & $\neg\exists Y(X \subset Y$ & $\forall d(d \in Y \rightarrow \{d\} \in [\![W]\!]))$
   & $\neg\exists Y(X \subset Y \subseteq X'$ & $Y \in [\![G]\!])$
10. $\exists X, X' \in \mathcal{P}(D^M)\backslash\{\varnothing\}$: D2.$[\![most]\!]$
   & $\forall d(d \in X' \rightarrow \{d\} \in [\![W]\!])$
   & $X \subseteq X'$ & $X \in [\![G]\!]$ & $|X' \cap X| > |X' \backslash X|$
   & $\neg\exists Y(X \subset Y$ & $\forall d(d \in Y \rightarrow \{d\} \in [\![W]\!]))$
   & $\neg\exists Y(X \subset Y \subseteq X'$ & $Y \in [\![G]\!])$

(4) None of the women gathered in the square. ↓MON↑, collective VP

0. $\models_M\ \neg\mathbf{sm}\ x(\delta_x(W(x)), G(x))$
1. $\models_M \neg\exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(W(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge G(x)) \wedge sm(x', x))$ A2.**Q**
2. $\mathcal{G}^M[\![\exists x\exists x'(\mathbf{M}_{x'}(\delta_{x'}(W(x'))) \wedge \mathbf{M}_x(x \subseteq x' \wedge G(x)) \wedge sm(x', x))]\!] \neq \top$ D4:|=, D3.¬
3. $\neg\exists H, K$: $\mathcal{G}^M \approx_x H \,\&\, H \approx_{x'} K$ F**Q**
   $\&\ K[\![\mathbf{M}_{x'}(\delta_{x'}(W(x')))]\!] = \top \,\&\, K[\![\mathbf{M}_x(x \subseteq x' \wedge G(x))]\!] = \top$
   $\&\ \langle K(x'), H(x)\rangle \in [\![sm]\!]$
4. $\neg\exists H, K$: $\mathcal{G}^M \approx_x H \,\&\, H \approx_{x'} K$ F**M**, rearrange
   $\&\ K[\![\delta_{x'}(W(x'))]\!] = \top$
   $\&\ K[\![x \subseteq x' \wedge G(x)]\!] = \top \,\&\, \langle K(x'), H(x)\rangle \in [\![sm]\!]$
   $\&\ \neg\exists J(K \approx_y J \,\&\, K(x') \subset J(y) \,\&\, J[\![\delta_y(W(y))]\!] = \top)$
   $\&\ \neg\exists J(K \approx_y J \,\&\, K(x) \subset J(y) \,\&\, J[\![y \subseteq x' \wedge G(y)]\!] = \top)$
5. $\neg\exists H, K$: $\mathcal{G}^M \approx_x H \,\&\, H \approx_{x'} K$ D3.∧, ⊆, *B*(, *A*, D2.≈
   $\&\ K[\![\delta_{x'}(W(x'))]\!] = \top$
   $\&\ H(x) \subseteq K(x') \,\&\, H(x) \in [\![G]\!] \,\&\, \langle K(x'), H(x)\rangle \in [\![sm]\!]$
   $\&\ \neg\exists J(K \approx_y J \,\&\, K(x') \subset J(y) \,\&\, J[\![\delta_y(W(y))]\!] = \top)$
   $\&\ \neg\exists J(K \approx_y J \,\&\, H(x) \subset J(y) \,\&\, J(y) \subseteq K(x') \,\&\, J(y) \in [\![G]\!])$
6. $\neg\exists H, K$: $\mathcal{G}^M \approx_x H \,\&\, H \approx_{x'} K$ Fδ, D3.*B*(, *A*
   $\&\ \forall I(K \approx_z I \,\&\, I(z) \subseteq K(x') \,\&\, |I(z)| = 1 \rightarrow I(z) \in [\![W]\!])$
   $\&\ H(x) \subseteq K(x') \,\&\, K(x) \in [\![G]\!] \,\&\, \langle K(x'), H(x)\rangle \in [\![sm]\!]$
   $\&\ \neg\exists J(K \approx_y J \,\&\, K(x') \subset J(y) \,\&\, \forall I(J \approx_z I \,\&\, I(z) \subseteq J(y) \,\&\, |I(z)| = 1 \rightarrow I(z) \in [\![W]\!]))$
   $\&\ \neg\exists J(K \approx_y J \,\&\, H(x) \subset J(y) \,\&\, J(y) \subseteq K(x') \,\&\, J(y) \in [\![G]\!])$
7. $\neg\exists H, K$: $\mathcal{G}^M \approx_x H \,\&\, H \approx_{x'} K$ elim. $\forall d$, simplify
   $\&\ \forall d(d \in K(x') \rightarrow \{d\} \in [\![W]\!])$
   $\&\ H(x) \subseteq K(x') \,\&\, H(x) \in [\![G]\!] \,\&\, \langle K(x'), H(x)\rangle \in [\![sm]\!]$
   $\&\ \neg\exists J(K \approx_y J \,\&\, K(x') \subset J(y) \,\&\, \forall d(d \in J(y) \rightarrow \{d\} \in [\![W]\!]))$
   $\&\ \neg\exists J(K \approx_y J \,\&\, H(x) \subset J(y) \subseteq K(x') \,\&\, J(y) \in [\![G]\!])$
8. $\neg\exists H, K$: $\mathcal{G}^M \approx_x H \,\&\, H \approx_{x'} K$ elim. $\exists J$
   $\&\ \forall d(d \in K(x') \rightarrow \{d\} \in [\![W]\!])$
   $\&\ H(x) \subseteq K(x') \,\&\, H(x) \in [\![G]\!] \,\&\, \langle K(x'), H(x)\rangle \in [\![sm]\!]$
   $\&\ \neg\exists Y(K(x') \subset Y \,\&\, \forall d(d \in Y \rightarrow \{d\} \in [\![W]\!]))$
   $\&\ \neg\exists Y(H(x) \subset Y \subseteq K(x') \,\&\, Y \in [\![G]\!])$
9. $\neg\exists X, X' \in \mathcal{P}(D^M)\backslash\{\varnothing\}$: elim. $\exists H, K$, D2.$[\![sm]\!]$
   $\&\ \forall d(d \in X' \rightarrow \{d\} \in [\![W]\!])$
   $\&\ X \subseteq X' \,\&\, X \in [\![G]\!] \,\&\, X' \cap X \neq \varnothing$
   $\&\ \neg\exists Y(X \subset Y \,\&\, \forall d(d \in Y \rightarrow \{d\} \in [\![W]\!]))$
   $\&\ \neg\exists Y(X \subset Y \subseteq X' \,\&\, Y \in [\![G]\!])$
10. $\neg\exists X, X' \in \mathcal{P}(D^M)\backslash\{\varnothing\}$: simplify
   $\&\ \forall d(d \in X' \rightarrow \{d\} \in [\![W]\!])$
   $\&\ \varnothing \subset X \subseteq X' \,\&\, X \in [\![G]\!]$