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Symmetry-Curvature Duality

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Abstract

Several studies have shown the importance of two very different descriptors for shape: symmetry structure and curvature extrema. The main theorem proved by this paper, i.e. the *Symmetry-Curvature Duality Theorem,* states that there is an important relationship between symmetry and curvature extrema: If we say that curvature extrema are of two opposite types, either maxima or minima, then the theorem states: Any segment of a smooth planar curve, bounded by two consecutive curvature extrema of the same type, has a unique symmetry axis, and the axis terminates at the curvature extremum of the opposite type. The theorem is initially proved using Brady's SLS as the symmetry analysis. However, the theorem is then generalized for any differential symmetry analysis. In order to prove the theorem, a number of results are established concerning the symmetry structure of Hoffman's and Richards' codons. All results are obtained first by observing that any codon is a string of two, three, or four spirals, and then by reducing the theory of codons to that of spirals. We show that the SLS of a codon is either (1) an SAT, which is a more restricted symmetry analysis that was introduced by Blum, or (2) an ESAT, which is a symmetry analysis that is introduced in the present paper and is dual to Blum's SAT.

Perceptual studies in both psychology and computer vision have revealed the importance of two very different structural characteristics for shape:

(1) *Symmetry*. Since the German gestalt school first showed that symmetry is a crucial organizing principle of shape (Wertheimer [24]; Goldmeier [8]), many studies have corroborated and extended their results. For example, Psotka [18] presented subjects with figures, such as the outline of a man, and asked the subjects to place a dot at the first place that came to mind. Pooling the results, he found that the dots were distributed along the local symmetry axes. Leyton [12, 13, 15, 16] gave subjects a set of twenty-two complex abstract and natural shapes; e.g. of animals, birds and plants, and asked them to place lines along the directions of greatest shape flexibility. The subjects consistently chose local symmetry axes. Richards & Kaufman [21] placed cutouts of figures against a TV screen exhibiting random "snow" and found that subjects saw

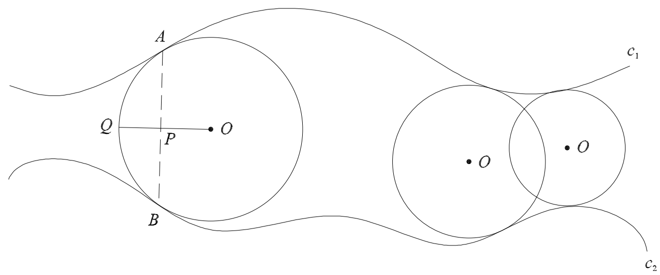


Figure 1: An illustration of the SAT construction for two curves c_1 and c_2. The SAT symmetry axis is the locus of the circle centers O.

flow patterns that accorded with the symmetry structure of the figures. Leyton [14, 15, 16] showed that human subjects successively prototypify objects by the successive introduction of global symmetries. Furthermore, Leyton [11] demonstrated that the orientation and form multistability phenomenon (e.g. Goldmeier [8]; Rock [22]) is due to the perceptual definition of shape as a symmetry structure. Again, several researchers have offered geometric theories of the encoding of shape in terms of what have been called generalized cones, cylinders, ribbons, etc. Despite the differences between these proposals, all of them crucially determine an axis of symmetry for each shape region encoded. One example is that of Blum [3], Blum & Nagel [4], called the Symmetric Axis Transform (SAT). Fig 1 shows the SAT for two curves c_1 and c_2. The SAT traces a trajectory of maximal inscribed discs between c_1 and c_2. The locus of disc centers, O, is then regarded as the symmetry axis (which might be curved). Other constructions, besides the SAT, include those by Binford [2], Brady [5], Brooks [7], and Leyton [17]. Various relationships between a number of these proposals have been elaborated by Rosenfeld [23].

(2) *Curvature extrema*. A structurally different type of shape description is curvature extrema. The pioneering paper by Attneave [1] showed that the information associated with a contour is not evenly distributed along it but is concentrated at points of curvature extrema. (For example, subjects, asked to represent a contour with a limited number of dots, tend to use the dots such that they correspond to points of maximal curvature on the contour.) Hoffman & Richards [10] and Richards & Hoffman [20] have examined curvature extrema with very different concerns. They have argued that a contour is perceptually partitioned at points of negative curvature extrema. They reasoned as follows: (1) Using the differential-topological notion of transversality, two smooth surfaces in 3-space tend to intersect at a contour of concave discontinuity of their tangent planes. (2) In particular, this can be said of two intersecting *parts* of a shape. (3) The contour of intersection can be considered to be a locus of greatest negative curvature. (4) The two-dimensional equivalent on a plane-curve is an extremum of negative curvature. (5) Thus the perceptual parts of a two-dimensional contour tend to be the segments bounded by points of greatest negative curvature. In support of this claim, Hoffman & Richards [10] present several compelling examples, including Rubin's classical face-vase illusion.

The above brief review clearly shows that two different descriptors, symmetry structure and curvature extrema, are fundamental to shape perception. In fact, it is important to notice just how different symmetry and curvature are, as structural characteristics. Firstly, symmetry is an algebraic concept; i.e. it is determined by a transitive group action. Furthermore, symmetry, as considered above, is reflectional symmetry, and is thus determined by a discrete group. In contrast, curvature is a continuity property determined by an asymmetric limiting process. Thus: Symmetry algebraically relates topologically separable points, i.e., points that can be isolated from each other in distinct neighborhoods of the contour. Curvature, on the other hand, is defined with respect to the converging non-separable neighborhood structure at a single point.

These considerations are reflected in the two different types of perceptual theories for which these two structural characteristics have been used. Theories of perceptual symmetry have concentrated on the perceptual binding together of quite separate parts of the contour. Theories involving curvature have concentrated on points of special significance, i.e. points of informational richness, or curve break-points.

Given these fundamental structural differences between symmetry and curvature, the theorem to be proved in this paper is somewhat surprising. Intuitively, it states: *The symmetry axes of a curve tend to terminate at points of curvature extrema.* In fact, the differential geometry is captured more by saying that the symmetry axis is *forced into* a point of maximal curvature.

It turns out that the theorem relates symmetry and curvature extrema even more intimately than this. If we let curvature extrema be of two opposite types, either maxima or minima[1], then the theorem states:

SYMMETRY-CURVATURE DUALITY THEOREM: *Any segment of a smooth planar curve, bounded by two consecutive curvature extrema of the same type, has a unique symmetry axis, and the axis terminates at the curvature extremum of the opposite type.*

One term in this theorem statement needs to be defined; it is the term *symmetry axis.* There are a number of alternative mathematical schemes for the description of symmetry. The scheme which we will use for most of the paper is one established by Brady [5] called SLS. However, we shall then generalize our argument to other symmetry schemes, quite easily, because we shall prove that, on segments of the type given in the theorem, a number of different symmetry schemes coincide. Again, amongst the curve segments themselves, it will become apparent that we only need to consider a subset of such segments. These are the segments that correspond to what Hoffman and Richards [20] call *codons*. Thus we shall first prove an apparently restricted statement of the above theorem in terms of Brady's SLS and Hoffman's and Richards' codons. An

[1] In fact, we will adopt the convention that curvature is a map from an entire curve to a single 1-dimensional continuum. That is, we will not assume two continua, one of positive curvature and the other of negative curvature. Note that the single continuum description arises from a specification of a single traversing direction to a curve

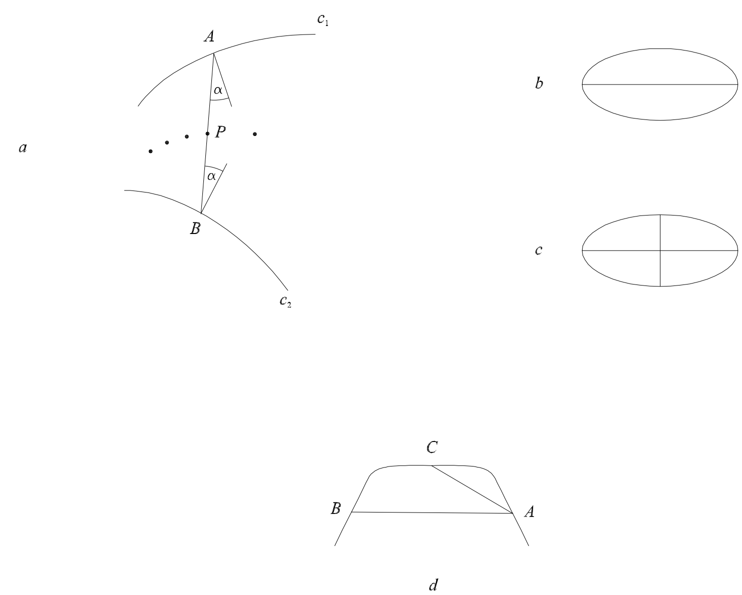


Figure 2: (a) The SLS construction for two curves c_1 and c_2. The SLS symmetry axis is the locus of chord points P. (b) The complete SAT for an ellipse. (c) The complete SLS for an ellipse. (d) A contour having two SLS cross sections at some point A.

extensive generalization will then be extracted from the proof of the restricted version. We begin now by briefly describing the SLS and codon analyses, respectively.

(1) *Brady's SLS.* Michael Brady [5] has provided an analysis of symmetry in shape that can be understood by considering Fig. 2a. The purpose is to find a line of symmetry for the two curved contours c_1 and c_2. In order to do this, points A and B are found on the contour such that the line AB subtends the *same* angle α with the normal at A and that at B. The cross-section AB is then divided by two; and the half-way point P is defined to be the symmetry point. The locus of dots in Fig. 2 represents the line of symmetry points established in this way. The curved symmetry axis is called the Smoothed Local Symmetry (SLS).

The importance of Brady's analysis is that it handles a wider variety of situations than some of the other analyses. For example, it is more exhaustive than the SAT (as defined by Blum). This is illustrated in Fig 2b which shows the SAT for a ellipse, in contrast to Fig 2c, which shows the ellipse's SLS. In this example, the SAT fails to produce the minor axis, which is captured by the SLS. The two figures illustrate also another generality: Every SAT induces an SLS (see Rosenfeld [23], for further discussion). This can be seen by looking back at Fig 1. The chord of an SAT disc subtends the same angle with the two radii at the tangent points A and B. These radii are also normal to the curves at A and B. Note that the SAT center, O, is not necessarily the SLS center, which is the point, P, that bisects the chord AB. This fact will be

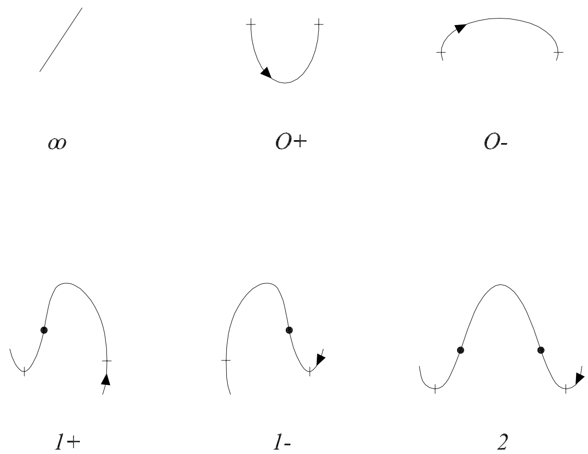


Figure 3: The codon classification. Slashes represent curvature minima, and dots represent inflection points.

crucial later when we prove that symmetry axes "are forced" to terminate at points of maximal curvature.

Observe finally that the exhaustiveness, of the SLS, results in the non-uniqueness of the latter. For example, as Brady & Asada [6] point out, a situation like Fig 2d yields two SLS cross-sections at A; i.e. the line-segments AB and AC.

(2) *Hoffman's and Richards' codon analysis.* As our brief review of some curvature literature indicated, Hoffman & Richards [10] and Richards & Hoffman [20] have argued that a smooth planar curve is perceptually partitioned into segments at the extrema of negative curvature. Any segment whose ends are two consecutive curvature minima is called a *codon*. Any curve can be segmented as a string of codons. In fact, Hoffman and Richards establish that there are only six possible types of codon; those given by the six possible orderings of curvature singularities as shown in Fig. 3. The first one given in Fig. 3 is trivial, and will be ignored throughout this article. On any codon shown, the direction of traversing the curve is in the direction of the arrow shown, and the curvature minima (indicated by slashes) are the points of greatest clockwise rotation of the arrow. The dots represent points of zero curvature. In this paper, we are going to make the harmless assumption that, except at the points of curvature extrema, the curvature of the codon changes strictly monotonically.

We now prove that there is a particularly close relationship between the above two very different kinds of structural descriptors; i.e. between the SLS and the codon-analysis:

SLS-CODON THEOREM: *The SLS of a codon is unique, and terminates at the point of maximal curvature on the codon.*

Table 1: A Redescription of the Codon Classification in Terms of Spirals.

Codon type	Spiral Sequence
0+	[+ve incr] · [+ve decr]
0-	[-ve incr] · [-ve decr]
1+	[+ve incr] · [+ve decr] · [-ve decr]
1-	[-ve incr] · [+ve incr] · [+ve decr]
2	[-ve incr] · [+ve incr] · [+ve decr] · [-ve decr]

Note: Each bracket-pair corresponds to a spiral. The symbols within each bracket give first, the curvature sign and second, the direction of change of curvature along the single 1-dimensional curvature continuum.

The proof of this theorem occupies most of the rest of the paper and reveals some surprisingly interesting facts about the relation between symmetry and curvature extrema. After the completion of the proof, we return to the Symmetry-Curvature Duality Theorem.

Proof. To prove this theorem, one observes first that a codon is itself built from a number of examples of only one type of primitive subpart. This can be seen as follows. To get to the subparts, we need first to break the codon at the point M of maximal curvature. One obtains two curves l_1 and l_2, which are respectively the curve before and the curve after M, in traversing the complete codon. We shall call the curves l_1 and l_2, the *limbs* of the codon. Now observe that each limb is a curve of monotonically changing curvature. A curve with monotonic curvature of *one sign,* is called a *spiral*. If a codon limb has no point of zero curvature, it must be a spiral. If the limb has a point of zero curvature then it consists of two spirals joined at the zero point. Thus each codon consists of a string of examples of only one type of subpart, a *spiral*. Codons 0+ and 0- consist of two spirals, codons 1+ and 1- of three spirals, and codon 2 of four spirals. If we take the direction of curve transversal into account, this leads to a redescription of the codon classification. Recall first (Footnote 1) that we will use the convention that curvature is given by a single one-dimensional continuum. Let us define a *positive increasing* spiral to be one which has positive curvature that increases as the spiral is traversed; and let us define a *negative increasing* spiral to be one with negative curvature where the curvature increases (moves in the *positive* direction along the curvature continuum) as the spiral is traversed. Correspondingly, we can define *positive decreasing* and *negative decreasing* spirals. Then the redescription of the codon classification is shown in Table 1.

We now prove a lemma that will be crucial to the proof of our theorem.

LEMMA 1: *An SLS cannot be constructed on a spiral.*[2]

Proof. Let the curve be parameterized by arc-length s. Let A and B be any pair of SLS-cross-section points. Define a Cartesian (x,y) plane at point A, such that the x-axis is along AB and the curve AB is in the negative half-plane of y, as shown in Fig 4. Define $\theta(s)$ to be the angle of the tangent at any point s to the x-axis. Now observe that

$$\begin{aligned} \int_{\theta(A)}^{\theta(B)} sin\theta \; d\theta \; &= \; cos\; \theta(A) \; - \; cos\; \theta(B) \\ &= \; cos(\frac{\pi}{2} - \alpha) \; - \; cos(\frac{\pi}{2} - \alpha) \\ &= \; 0 \end{aligned} \tag{1}$$

But observe:

$$\begin{aligned} \int_{\theta(A)}^{\theta(B)} sin\theta \; d\theta \; &= \; \int_{A}^{B} \frac{dy}{dx} \cdot \frac{d\theta}{ds} \cdot ds \\ &= \; \int_{A}^{B} \frac{dy}{dx} \cdot \kappa \cdot ds \\ &= \; y(B)\kappa(B) - y(A)\kappa(A) - \int_{A}^{B} y(s) \cdot d\kappa(s) \\ &= - \int_{A}^{B} y(s) \cdot d\kappa(s) \\ &\neq 0 \end{aligned} \tag{2}$$

But Eq (2) contradicts Eq (1). Therefore the SLS cannot be constructed[3]. ■

The lemma in particular shows that an SLS cannot be constructed on any spiral subpart of a codon. However, since a codon consists only of spirals, this means that if an SLS can be constructed, the SLS cross-section points A and B must each come from different spirals.

The first such case we consider is of two adjacent spirals which are linked by a point of zero curvature; i.e. by a dot as shown on codons 1+, 1-, and 2, in Fig. 3. Any such spiral-pair will be called a *bi-spiral.* It is easy to see that an SLS cannot be constructed on bi-spiral, because points A and B must come from opposite sides of the dot; and the cross-section must therefore cross over the contour.

[2]We will assume that normals cannot change sides on a curve.

[3]This proof is a variant of Ostrowski's proof of Vogt's theorem (see Guggenheimer [9])

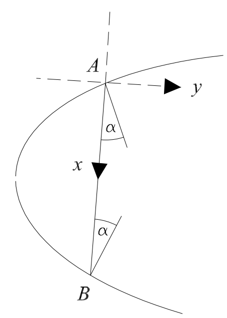


Figure 4: The figure required for Lemma 1.

Thus we are left with a crucial conclusion that no codon limb can have an SLS, because a codon limb consists either of a spiral or a bi-spiral. In other words, we have proved:

LEMMA 2: *The SLS cross-section points of a smooth planar curve must be separated by a curvature turning-point.* ■

In other words, the SLS cross-section points of a codon must lie on different limbs.

What is necessary now to prove is (1) that an SLS does in fact exist and that it does have the property of terminating at the maximal curvature point M; and (2) that there is no other possible SLS for a codon.

Existence

We first divide the five non-trivial codons into two classes: (1) the codons which have *positive* maximal curvature, and (2) the codons which have *negative* maximal curvature. The former will be called the *positive codons* and the latter the *negative codons.* Observe that four of the codons are positive; the only negative codon being 0-. Observe also, from Table 1, that the maximal curvature point M, for each codon, is flanked by two spirals of the same sign. These two spirals will be called the *s-region* of the codon.

We prove existence first for positive codons.

Recall that any SAT has an associated SLS. We shall call such an SLS, an *SAT-induced SLS.* We will now show that any positive codon has an SAT-induced SLS. To start the construction of an SAT, choose any point A on the codon (where $A \neq M$ and M is the point of maximal curvature). Without loss of generality, we will assume that $A \in l_1$. Put a circle at A, tangential to the codon at A and inside the codon (to the left of the traversal direction), and ensure that the circle is small enough not to touch or intersect any other point on the codon. Now simply increase the radius of the circle till it touches the codon at one other point. Two conditions are satisfied, which we prove in the two subsequent lemmas:

LEMMA 3: *A maximal inscribed circle cannot be tangential to a positive codon at more than two separate points.*

Proof. Suppose the maximal circle is tangential to the codon at three points A, B, and C. Suppose also that neither A, B, nor C is M (the point of maximal curvature). Then two of the three points, say A and C, must lie on the same limb, say l_1. However, the line AC is a chord of the circle. Furthermore, because angles $<OAC$ and $<OCA$ must be equal (where O is the center of the circle), the chord AC is an SLS cross-section; which contradicts Lemma 2, because A and C are on the same limb.

Now suppose that one of the three points, say C, is M. Then by Lemma 1, points A and B must lie on different limbs, say l_1 and l_2 respectively. Consider now the curve $l_1 \cup M$. This is a spiral or bi-spiral. But this means that points A and C lie on the same spiral or bi-spiral, which contradicts Lemma 1 and Lemma 2. ■

The next lemma requires an understanding of a codon's *evolute*. The evolute of a curve is the locus of its centers of curvature. The following are standard facts about evolutes (e.g. see Guggenheimer [9]): (1) The evolute of a convex curve is convex[4]; (2) the normal line, at any point on the curve, is tangent to the curve's evolute; and (3) the evolute at a point of maximal curvature is a cusp. Putting these facts together it is easy to show that the evolute of the s-region[5] of a codon is of the form shown in Fig 5; that is, two convex curves that form a cusp that is tangential to the normal at M and points towards M. Fig 5 puts the codon on a Cartesian frame with x-axis along the normal at M. Fig 5 also shows an arbitrary point A with center of curvature $E(A)$ on the evolute and normal that is tangent to the evolute at $E(A)$. It is now easy to prove:

LEMMA 4: *The center, O, of a maximal inscribed circle of a positive codon, must lie in between the two evolutes of the two codon limbs.*

Proof. Let the maximal circle be tangential to the evolute at A and B. Let us deal first with the s-region of the codon. By Lemma 1, the points A and B must lie on opposite spirals s_1 and s_2 respectively. But s_1 is separated from its evolute by the evolute of s_2, and vice versa. Therefore the normal lines must intersect in between the evolutes. However, the center of the inscribed circle is the intersection of normal lines; which means that the center is in between the evolutes of the two spirals.

When one now includes the non s-region, one merely adds evolutes that lie outside the codon. So the result remains. ■

[4] A curve AB is *convex* if the region bounded by the curve and the chord AB is a topologically convex set.

[5] We defined the s-region to be the two spirals flanking the maximal curvature point.

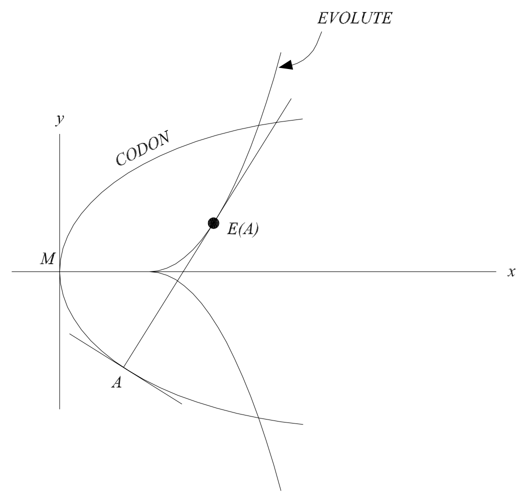


Figure 5: A positive codon with its evolute.

Observe now that Lemma 4 provides the following ordering constraint. Consider the normal line at A, and three points on that line: point A, the evolute point $E(A)$, and the point O which is the center of the maximal circle. Lemma 4 tells us that the order of these points along the normal must be A, O and $E(A)$. This order must be preserved under projection onto the x-axis in Fig 5.

Now observe that, as point A moves along the codon to M, point $E(A)$ moves along the evolute to the cusp point. The ordering constraint forces O to reach the cusp point before $E(A)$ does. However, the tangent at the cusp is the normal at M. Thus when O reaches the cusp point it is the center of a circle tangential to M. Observe however that at this point the circle cannot touch the codon also at another point on l_2, because $l_2 \cup M$ is a spiral or bi-spiral. Thus point B must have reached point M at the same time as point A. Which means that the SLS symmetry point, which lies on the chord AB, must have reached M.

This completes the proof that a positive codon has a unique SAT-induced SLS, and that this SLS terminates at the point of maximal curvature.

Observe that the above shows why, earlier, we had said intuitively that a symmetry axis is *forced into* the point of maximal curvature. We can now see that this refers to the fact that the SAT center O is forced in between the evolutes and is eventually pushed by the ordering constraints into the cusp point.

Let us now prove the existence of an SLS for negative codons. All negative codons are of the singularity form shown as codon type 0- in Fig. 3. Above, we proved the existence of an SLS for positive codons by showing that those codons have an SAT. However, we cannot apply the above proofs to negative codons because we find the following:

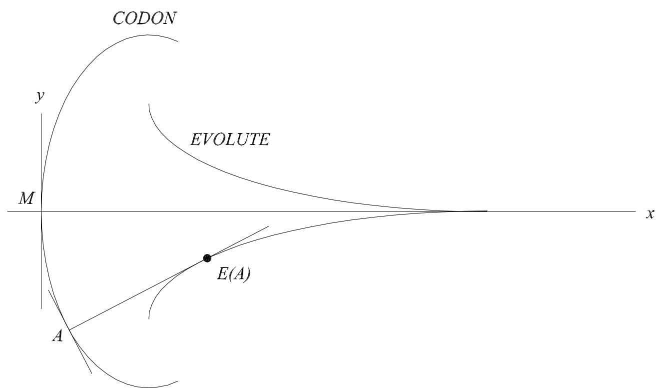


Figure 6: A negative codon with its evolute.

LEMMA 5: *An SAT cannot be constructed for a negative codon.*

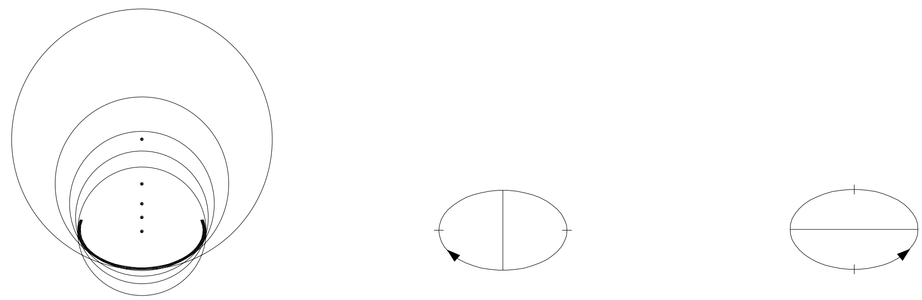
Material in the proof will be required later.

Proof. Fig 6 shows a negative codon with its evolute, such that the evolute cusp point is tangential to the x-axis of the Cartesian plane. In this case, the evolute of each spiral is on the *same* side of the x-axis as its spiral. Consider points A and B on different spirals. Their normals must intersect at some point O. However this is further along the normals from A and B than the centers of curvature $E(A)$ and $E(B)$. However, any inscribed circle that is tangential to the codon at two points must have a radius smaller or equal to the radii of curvature at both points. Thus an inscribed circle cannot exist, and an SAT is impossible. ■

It might appear at first that Lemma 5 precludes the use of any of the types of techniques used above in the existence proof for positive codons. However, the situation is rescued when one realizes that negative codons can have *exscribed* circles that will have the properties that we require.

We propose a new kind of symmetry analysis that is dual to the SAT. Instead of using circles that are (1) maximal and (2) inscribed, the new analysis uses circles that are (1) minimal and (2) exscribed. To distinguish these two types of analyses, let us call Blum's analysis, the **Inscribed SAT** (or just **ISAT**), and the new analysis, the **Exscribed SAT** (or just **ESAT**).

One of the things that our proof has so far shown is that positive codons yield an ISAT. We will now go on to show that negative codons yield an ESAT. To illustrate this, in advance, consider Fig. 7a. It shows the ESAT of a negative codon. The codon is the bold curve, and the circles are the minimal exscribed ones of the ESAT. The dotted line shows the locus of symmetry points. Now consider Fig 7b. It shows an ellipse; and an ellipse turns out to be the conjunction of two negative codons, bounded by the slashes



a *b* *c*

Figure 7: (a) A negative codon (thickened curve) with its ESAT. (b) An ellipse interpreted as a string of two negative codons. (c) An ellipse interpreted as a string of two positive codons.

shown (see Richards & Hoffman [20] for fuller discussion). However, looking at Fig 7a, it is easy to see that the ESAT of an ellipse creates the short axis.

Let us examine this situation a little more closely. Richards & Hoffman [20] show that the ellipse has in fact *two* interpretations depending on the direction of curve traversal[6] The interpretations are (1) the conjunction of two positive codons, as shown in Fig 7c; and (2) the conjunction of two negative codons, as shown in Fig 7b. We have seen that positive codons yield an SAT, which we now call an ISAT. This produces the *long* axis of an ellipse. We shall also see that negative codons yield an ESAT. This produces the *short* axis of an ellipse. Putting these together, we obtain the complete SLS of the ellipse; i.e. the two axes, as shown in Fig 2c.

Let us now return to the general SLS existence-proof for negative codons. It is easy to prove the duals of Lemmas 3 and 4 given earlier:

LEMMA 3′: *A minimal exscribed circle cannot be tangential to a negative codon at more than two separate points.*

LEMMA 4′: *The center, O, of a minimal exscribed circle of a negative codon must lie in between the two evolutes of the two codon limbs.*

Proofs. The proofs of these lemmas are obtained by writing out the proofs of Lemmas 3 and 4, and substituting the dual constructs; i.e. substituting "minimal" for "maximal", and "exscribed" for "inscribed", etc. ■

[6]As Richards and Hoffman [20] point out, the two directions also lead to two alternative figure-ground relationships because the interior of a figure is to the left of the transversal direction. Therefore, in Fig 7b, the ellipse is a hole, and in Fig 7c the ellipse is a solid shape.

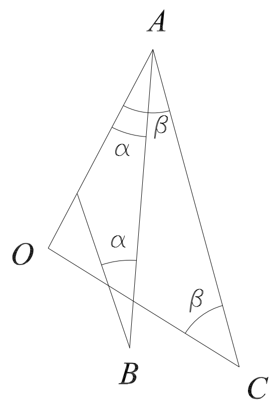


Figure 8: AB and AC are SLS chords – a situation that we prove does not exist.

Now observe that different ordering constraints emerge from Lemma 4′ than from Lemma 4. Going back to Fig 6, we see that, for negative codons, the order of points A, O, and $E(A)$ on the normal line at A must in fact be A, $E(A)$, O; which reverses the order of $E(A)$ and O for the positive codons. However, this new order is just what we need to prove our result for negative codons; as we can now see. As point A travels to point M, the point $E(A)$ travels to the evolute cusp-point. However, the new ordering constraint pushes O to the cusp point ahead of $E(A)$, ensuring that O reaches the cusp point. But the cusp point is the center of the circle of curvature at M. Thus, as before, the SLS center reaches M, and the existence result follows.

Uniqueness

Observe now that, although we have just proved that a codon does have an SLS – that which is induced either by the codon's unique ISAT, or unique ESAT - the codon might have another SLS, because there are many SLSs that are not induced by an ISAT or ESAT. It remains therefore to show that no such alternative SLS is possible for a codon.

We prove uniqueness first for positive codons. The situation we are going to describe is illustrated in Fig 8. Let A and B be cross-section points determined by the ISAT-induced SLS described above; i.e. there is a maximal inscribed circle tangential to the codon at A and B. If the SLS is not unique, then, without loss of generality, for some A, there is a point C on the same limb of B (say l_2) such that A and C are SLS cross-section points to the codon. Crucial to our analysis is the fact that the latter SLS also induces a circle (not a maximal inscribed one) that is tangential to the codon at points A and C. (This is easy to see: By the SLS construction, angle β, between the normal at A and AC, must be the same as the angle between the normal at C and AC. Furthermore, the inward normals must lie on the same side of AC, and thus must intersect at some point O. This means that triangle OAC is isoseles and OA is the same length as OC. Thus a circle S, with center at O, can be drawn tangentially to the codon at A and C.)

We shall call this new circle, the *SLS-induced circle.* All pairs of SLS-cross-section points, of any figure, have such a circle.

Observe now that, in the case we are considering here, the circle cannot be inscribed both at A and at C, because it would then be a maximal inscribed circle at A and would have to be tangential to B also; which would contradict Lemma 3.

This condition, that the SLS-induced circle cannot be inscribed both at A and at C, is crucial. For we shall now prove that this condition is impossible to comply with; i.e. that the SLS-induced circle has to be *inscribed* at both points. In order to do this we shall prove:

LEMMA 6: *A circle that is tangential to a positive codon at two distinct points must be inscribed.*

Although the proof of this lemma is simple, it is interesting, firstly because, once again, the theory of codons is reduced to a theory of spirals, and secondly because it makes use of a result about spirals that we have not already used. The result, known as *Kneser's Theorem* (e.g. Guggenheimer [9]), states that the circle of curvature of a spiral contains every smaller circle of curvature of that arc.

Proof. We shall divide all situations into three cases: (1) The two points where the circle touches the codon are both on the s-region; (2) one point is on the s-region, and the other is off the s-region; and (3) both points are off the s-region. The points will be labeled X_1 and X_2, and the circle will be labeled S.

Case (1): The tangent points are both on the s-region. Let us assume that the circle S is not inscribed at one of the two tangent points, say X_1. We show that the circle cannot be inscribed also at the other point. If the circle is not inscribed (but tangential) at X_1, then the radius of the circle must be greater than the radius of curvature at X_1. But this means that the circle S must contain the circle of curvature at X_1. However, by Kneser's theorem, this means that all of the spiral from X_1 to M must lie inside the circle S. However, it must then follow that all the spiral from X_2 to M must be contained in S. Therefore, S is not inscribed at X_2.

Now let O be the center of S. From the conclusion that S is not inscribed at X_2 as well as X_1, it follows that the radii of curvature of the codon at X_1 and X_2 are less than or equal to the radius of S. Thus: (1) the center $E(X_1)$ of curvature at X_1 must be on the line OX_1 between O and X_1; and (2) the center $E(X_2)$ of curvature at X_2 must be on the line OX_2 between O and X_2. However, this contradicts the ordering constraints that follow from Lemma 4, unless $O = E(X_1) = E(X_2)$, which would mean that X_1 and X_2 are not distinct, contradicting an assumption of the Lemma. Thus the circle has to be inscribed at both X_1 and X_2.

Case 2: One tangent point is on the s-region, and the other is off the s-region. If one tangent point, say X_1, is not on the s-region, the circle is inscribed at X_1 because the curvature at X_1 is not positive. Suppose now that the circle is not inscribed at X_2. Then, because X_2 is on the s-region, it must contain the spiral from X_2 to M (by Kneser's theorem). However, this means that the circle must contain the spiral from X_1 to M. But this contradicts the assumption that the codon curvature is not positive at X_1.

Case 3: The tangent points are both off the s-region. If both the tangent points are off the s-region, then the curvature at the points is negative and the circle must be inscribed at both points.
Thus, for the three possible cases the circle is inscribed at the two tangential points. ■

Now applying Lemma 6 to the SLS-induced circle that is tangent to the codon at A and C in Fig.8, we see that this latter circle must be inscribed, which means that it must be maximal, which means that it is the circle that is tangent also at B. But this contradicts Lemma 1, which forbids three tangential codon points on a maximal circle. Therefore C does not exist. Thus the ISAT-induced SLS is the unique SLS for a positive codon.

In order to prove uniqueness for negative codons, we prove the following Lemma:

LEMMA 6$'$: *A circle that is tangential to a negative codon at two distinct points must be exscribed.*

Proof. Because the negative codon equals its own s-region (see Table 1) we only have the dual of case 1 in Lemma 6. However, recall that the proof of case 1 was completely dependent on the ordering constraint A, O, $E(A)$, for normals on positive codons. Analogously, it is easy to show that the ordering constraint, A, $E(A)$, O, for normals of negative codons yields the dual result. ■

This completes the proof of the SLS-Codon Theorem. ■.

The SLS-Codon Theorem is only a particular example of the Symmetry-Curvature Duality Theorem, but its proof reveals how we can understand and prove the latter in its full generality, as follows:

First let us examine the nature of symmetry analyses for smooth curves. Consider again Fig 1 which shows two smooth planar curves c_1 and c_2. It seems reasonable to define local reflectional symmetry between points A and B (on curves c_1 and c_2 respectively) as the condition that the tangent vector at A is reflectionally symmetric to that at B about some line; i.e. mirror. This condition turns out to be equivalent to the requirement that there is a circle tangential to the curves at A and B. The line of reflection (i.e. mirror) contains a radius QO of the circle (see Fig 1).

Now, reflection in a mirror is not dependent upon the choice of the incidence point for the light-ray defining the reflection. In particular, the incidence point that yields

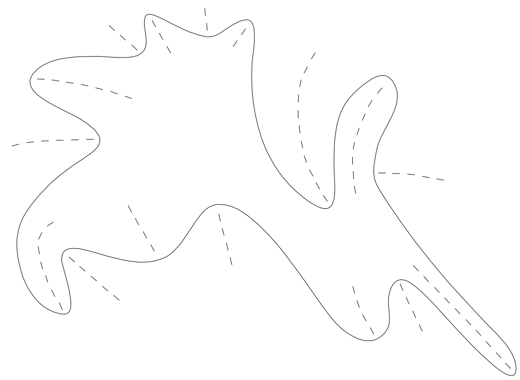


Figure 9: An illustration of the Symmetry-Curvature Duality Theorem. Note that three consecutive extrema comprise a codon when the curve is traversed in either one direction or the reverse.

symmetry at A and B, in Fig 1, can be any point along the extended line through Q and O. In the SAT of Blum [3], the incidence point is O; in the SLS of Brady [5], it is P; and in the Process-Inferring Symmetry Analysis (PISA) of Leyton [17], it is Q.

We therefore define a **differential symmetry axis** to be a differential trajectory of incidence points yielded by a trajectory of reflection lines (where each consecutive incidence point is selected from each consecutive reflection line). Note that the existence of a trajectory of reflection lines is equivalent to the existence of a trajectory of doubly-tangential circles.

We now claim that the Symmetry-Curvature Duality Theorem, in its full generality, applies to any **differential symmetry axis;** i.e. not just to the axis of the SLS but to the axes of the infinite variety of symmetry analyses (e.g. PISA, etc.) which have the above general type of construction.

The proof that the theorem applies to any differential symmetry analysis is given by a more abstract version of the proof of the SLS-Codon Theorem. More specifically, the proof falls into two halves as does the proof of the latter theorem. That is, one shows: (1) Any codon has a fully inscribed, or fully exscribed, trajectory of doubly-tangential circles that are no-where triply tangential. Furthermore, the trajectory terminates at the maximal curvature point. (2) The circle trajectory of any differential symmetry analysis, of a codon, must reduce to the codon's fully inscribed or fully exscribed system.[7] Since these two proposals were proved in the existence and uniqueness sections, respectively, of the preceding proof, the Symmetry-Curvature Duality Theorem follows. The theorem is illustrated in Fig 9. ■

[7]To ensure the the incidence points actually arrive at the extrema, and not just move towards the latter, the final circle might have to be contracted to a point.

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