Optimal (s,S) Booking Policies with Fixed Penalty

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Abstract: In this paper we investigate an inventory system with a fixed penalty cost that is incurred whenever there is a shortage. To our knowledge it is the first time a model with a fixed penalty cost for lost demand is investigated. Explicit solutions are derived in the case of uniform demands. In this paper the discussion is done in terms of applying an (s, S) policy into a hotel booking problem.

Key–Words: Dynamic Hotel Booking, Infinite Time Horizon, (s,S) policy.

1 Introduction

In the inventory control literature (s, S) policies have been used to specify ordering quantities when the demand is stochastic and there is a fixed set-up cost, c.f., Arrow (1951), Veinott (1966) and Song and Zipkin (1993). Iglehart (1963) derived an explicitly optimal (s, S) policy for the stationary infinite horizon version of the problem.

In this paper we use (s,S) policies in a hotel booking environment, to model an agent that books hotel rooms for its customers. In this environment we need to consider a fixed penalty cost for lost demand that is independent of the amount of the shortage. Often, agents use a booking policy of the following form. A fixed number of rooms S is booked at "booking epochs", to meet random customer demand at the agent's site. The booking process repeats instantaneously whenever the number of rooms at the disposal of the agent at the end of a time period falls below a critical level s. Let D_t, denote the demand customers generate for rooms in period t. We assume that D_t are i.i.d discrete random variables with probability mass function $f(x)$, $(x = 0, 1, 2, ...)$, and cumulative distribution function $F(x)$. Let the random variable X_t denote the number of rooms at the agent's disposal in beginning of period t. Under this booking policy X_t satisfies the following:

$$X_{t+1} = \begin{cases} X_t - D_t, & X_t - D_t > s \\ S, & otherwise \end{cases} \tag{1}$$

The relation between the transition probabilities for X_t and the probability mass function of the demand is the following:

- $P(X_{t+1} = x'|X_t = x) = f(x' - x)$, if $s < x' < S$, $x' \neq x$;
- $P(X_{t+1} = x|X_t = x) = f(0) + \sum_{d_t=x+1}^{+\infty} f(d_t)$;
- $P(X_{t+1} = S|X_t = x) = \sum_{d_t=x-s}^{x} f(d_t)$.

The agent typically has a profit generating operations policy that works as follows. The agent can get a gross profit of c units per room sold to a customer; if he can't satisfy the demand of the customer, then he incurs a fixed penalty fee p, (which can be explained as the cost paid by him to another agent for satisfying the demand of his customers). If the room still at the agent's disposal in the beginning the time period, i.e the room hasn't been occupied by a customer, the agent needs to pay a reservation fee h for each room at each time period.

The profit $u(x_t, d_t)$ in the time period t with available rooms x_t and demand d_t can be written as:

$$u(x_t, d_t) = \begin{cases} d_t c - x_t h, & \text{if } d_t \leq x_t \\ -p - x_t h, & otherwise. \end{cases} \tag{2}$$

2 Formulation

Using standard theory from Markovian processes we can calculate the stationary state distribution $\pi(i)$, $i = s + 1, \ldots, S$; where we say the system is in state i if $X_t = i$. Note that the number of states is $S - s$. Then we have that $\pi(i)$, is the unique solution to the system of equations (3),(4) and (5) below

If $s + 1 \leq i < S$,

$$\pi(i) = \sum_{j=1}^{S-i} f(j)\pi(i+j) + f(0)\pi(i) + \sum_{j=i+1}^{+\infty} f(j)\pi(i);$$

$$\pi(i)\sum_{j=1}^{i} f(j) = \sum_{j=1}^{S-i} f(j)\pi(i+j). \quad (3)$$

If $i = S$,

$$\pi(S) = \sum_{i=1}^{S}\sum_{j=i-s}^{i} f(j)\pi(i) + \sum_{j=S+1}^{+\infty} f(j)\pi(S) + f(0)\pi(S);$$

$$\pi(S)\sum_{j=1}^{S} f(j) = \sum_{i=1}^{S}\sum_{j=i-s}^{i} f(j)\pi(i). \quad (4)$$

And

$$\sum_{j=s+1}^{S} \pi(j) = 1 \quad (5)$$

Then we can get the expected profit g_i corresponding to state i as follows

$$g_i = c\sum_{j=0}^{i} jf(j) \; - \; p\sum_{j=i+1}^{+\infty} f(j) - ih. \quad (6)$$

Next we can got long run average profit for the whole system:

$$g = \sum_{j=s+1}^{S} g_j \pi(j). \quad (7)$$

3 Applications

Assume the demand follows the uniform distribution in $[0, m]$, $(m \geq S)$, then the probability mass function will be $f(x) = \frac{1}{m+1}$, $(x = 0, \cdots, m)$. We first obtain the solution of the stationary equations in the next theorem. We still assume $c > h$.

Theorem 1 *Under the suppositions made in this section, the stationary probability distribution will be*

$$\pi(i) = \begin{cases} \frac{s+1}{i(i+1)}, & s+1 \leq i < S; \\ \frac{s+1}{S}, & i = S. \end{cases} \quad (8)$$

Proof: Assume $\pi(S) = \theta$, then we have

$$\pi(S-1)\sum_{j=1}^{S-1} f(j) = \sum_{j=1}^{1} f(j)\pi(S-1+j);$$

$$\pi(S-1)\frac{S-1}{m+1} = \frac{1}{m+1}\pi(S);$$

$$\pi(S-1) = \frac{S}{(S-1)(S-1+1)}\theta.$$

Next we use mathematical induction to prove.

Assume $\forall i+1 \leq k < S$,

$$\pi(k) = \frac{S\theta}{k(k+1)}.$$

Then

$$\pi(i)\sum_{j=1}^{i} f(i) = \sum_{j=1}^{S-i} f(j)\pi(i+j);$$

$$\frac{i\pi(i)}{m+1} = \frac{1}{m+1}(\pi(S)+\pi(S-1)+\pi(S-2)+\cdots+\pi(i+1));$$

$$i\pi(i) = \theta(1 + \frac{S}{(S-1)S} + \frac{S}{(S-2)(S-1)} + \cdots + \frac{S}{(i+1)(i+2)});$$

$$i\pi(i) = \theta(1 + \frac{S}{S-1} - \frac{S}{S} + \frac{S}{S-2} - \frac{S}{S-1} + \cdots + \frac{S}{i+1} + \frac{S}{i+2});$$

$$i\pi(i) = \theta(\frac{S}{i+1});$$

$$\pi(i) = \frac{S\theta}{i(i+1)}.$$

That is to say, $\forall i \leq k < S$,$\pi(k) = \frac{S\theta}{k(k+1)}$. Then according to mathematic induction, we have

$$\forall s+1 \leq k < S, \pi(k) = \frac{S\theta}{k(k+1)}.$$

Hence,

$$\sum_{j=s+1}^{S} \pi(j) = \theta(\frac{S}{(s+1)(s+2)} + \frac{S}{(s+2)(s+3)} + \cdots + \frac{S}{(S-1)S} + 1);$$

$$\sum_{j=s+1}^{S} \pi(j) = \theta(\frac{S}{s+1} - \frac{S}{s+2} + \frac{S}{s+2} - \frac{S}{s+3} + \cdots + \frac{S}{S-1} - \frac{S}{S} + 1);$$

$$\sum_{j=s+1}^{S} \pi(j) = \frac{S\theta}{s+1}$$

Because of eq (4), $\sum_{j=s+1}^{S} \pi(j) = 1$, we have

$$\theta = \frac{s+1}{S}$$

Therefore,

- $\pi(i) = \frac{s+1}{i(i+1)}, s+1 \le i < S-s$;
- $\pi(S) = \frac{s+1}{S}$.

□

Remark 2 *Notice the expression of stationary probability distribution has no relation to the parameter of the uniform demand distribution.*

At the same time, under the uniform demand assumption and eq (6) we obtain:
$g_i = c\frac{i(i+1)}{2(m+1)} - p\frac{m-i}{m+1} - ih$
Then we can get the long run average profit for eq (7):

$$g = \sum_{i=s+1}^{S-1} (c\frac{i(i+1)}{2(m+1)} - p\frac{m-i}{m+1} - ih)\frac{s+1}{i(i+1)}$$

$$+(c\frac{S(S+1)}{2(m+1)} - p\frac{m-S}{m+1} - Sh)\frac{s+1}{S}.$$

$$g = \sum_{i=s+1}^{S-1} \left(c\frac{s+1}{2(m+1)} - p\frac{m(s+1)}{m+1}(\frac{1}{i(i+1)}) \right.$$

$$\left. +(\frac{p}{m+1} - h)\frac{s+1}{i+1} \right)$$

$$+(c\frac{S(S+1)}{2(m+1)} - p\frac{m-S}{m+1} - Sh)\frac{s+1}{S};$$

$$g = (s+1)\left(\frac{c(2S-s)}{2(m+1)} - \frac{pm}{m+1}(\frac{1}{s+1} - \frac{1}{S} + \frac{1}{S}) \right.$$

$$\left. +(\frac{p}{m+1} - h)(\sum_{i=s+1}^{S-1} \frac{1}{i+1} + 1) \right);$$

$$g = (s+1)\frac{c(2S-s)}{2(m+1)} - \frac{pm}{m+1}$$

$$+(s+1)(\frac{p}{m+1} - h)(\sum_{i=s+1}^{S-1} \frac{1}{i+1} + 1);$$

Here, g can be looked as a function about (s, S). Then we have

$$g(s,S) = (s+1)(\frac{p}{m+1} - h)(\sum_{i=s+1}^{S-1} \frac{1}{i+1} + 1) + (s+1)\frac{c(2S-s)}{2(m+1)} - \frac{pm}{m+1}. \quad (9)$$

Next we want to maximize with resect to s and S $g(s, S)$ and compute the following

$$\max_{0 \le s+1 \le S \le m} \{g(s,S)\}. \quad (10)$$

Theorem 3 *Under the assumptions made in this section, $(m-1, m)$ is the unique optimal solution for (10), if $\frac{p}{m+1} \ge h$.*

Proof: First for all $1 \le s+1 \le S \le m$, we have:

$$g(s,S) = (s+1)\frac{c(2S-s)}{2(m+1)} - \frac{pm}{m+1}$$

$$+(s+1)(\frac{p}{m+1} - h)(\sum_{i=s+1}^{S-1} \frac{1}{i+1} + 1);$$

$$g(s-1,S) = s\frac{c(2S-s+1)}{2(m+1)} - \frac{pm}{m+1}$$

$$+s(\frac{p}{m+1} - h)(\sum_{i=s}^{S-1} \frac{1}{i+1} + 1).$$

Then

$$g(s,S) - g(s-1,S) = \frac{c(2S-2s)}{2(m+1)}$$

$$+(\frac{p}{m+1} - h)(-\frac{s}{s+1} + \sum_{i=s+1}^{S-1} \frac{1}{i+1} + 1).$$

$$g(s,S) - g(s-1,S) = \frac{c(2S-2s)}{2(m+1)} + (\frac{p}{m+1} - h)\sum_{i=s}^{S-1} \frac{1}{i+1} \quad (11)$$

And according to the condition of the theorem,

$$\frac{c(2S-2s)}{2(m+1)} > 0,$$

$$(\frac{p}{m+1} - h)\sum_{i=s}^{S-1} \frac{1}{i+1} \ge 0.$$

Hence for all $1 \le s+1 \le S \le m$,

$$g(s,S) - g(s-1,S) > 0 \quad (12)$$

Second, for all $0 \le s+1 < S \le m$, we have:

$$g(s,S) = (s+1)\frac{c(2S-s)}{2(m+1)} - \frac{pm}{m+1}$$

$$+(s+1)(\frac{p}{m+1} - h)(\sum_{i=s+1}^{S-1} \frac{1}{i+1} + 1);$$

$$g(s,S-1) = (s+1)\frac{c(2S-s-2)}{2(m+1)} - \frac{pm}{m+1}$$

$$+(s+1)(\frac{p}{m+1} - h)(\sum_{i=s+1}^{S-2} \frac{1}{i+1} + 1);$$

Then

$$g(s,S)-g(s,S-1) = \frac{2(s+1)c}{2(m+1)}+(\frac{p}{m+1}-h)\frac{s+1}{S}. \quad (13)$$

And according to assumption of the theorem,

$$\frac{2(s+1)c}{2(m+1)} > 0,$$

$$(\frac{p}{m+1} - h)\frac{s+1}{S} \geq 0.$$

Hence for all $0 \leq s+1 < S \leq m$

$$g(s,S) - g(s,S-1) > 0 \quad (14)$$

Because of equation (12), (14), we have $\forall 0 \leq s+1 < S < m$,

$$g(s,S) < g(s,m) < g(m-1,m).$$

The proof is complete. □

Theorem 4 *Under the assumptions made in this section,*
a) if $\frac{c}{m+1} \geq h - \frac{p}{m+1} > 0$, *and* $0 \leq s+1 \leq S < m$, *then*

$$g(s,S) < g(s,m);$$

b) if $\frac{c}{m+1} \leq (h - \frac{p}{m+1})\frac{1}{m}$, *and* $0 \leq s+1 < S \leq m$, *then*

$$g(s,S) < g(s,s+1).$$

c) if $(h - \frac{p}{m+1})\frac{1}{m} < \frac{c}{m+1} < h - \frac{p}{m+1}$, *and* $0 \leq s+1 < S < m$, *then*

$$g(s,S) < max\{g(s,s+1), g(s,m)\}.$$

Proof: Because to all $0 \leq s+1 < S \leq m$, it follows from equation (13):

$$g(s,S)-g(s,S-1) = (s+1)\left(\frac{c}{m+1} - (h - \frac{p}{m+1})\frac{1}{S}\right).$$

If

$$\frac{c}{m+1} \geq h - \frac{p}{m+1},$$

then $\forall 2 \leq S \leq m$,

$$\frac{c}{m+1} \geq h - \frac{p}{m+1} > (h - \frac{p}{m+1})\frac{1}{S}.$$

Hence,

$$g(s,m) > g(s,m-1) > \cdots > g(s,s+2) \geq g(s,s+1),$$

Therefore, if $\frac{c}{m+1} \geq h - \frac{p}{m+1} > 0$, then $\forall 0 \leq s+1 \leq S < m$,

$$g(s,S) < g(s,m).$$

To show (b) note that if

$$\frac{c}{m+1} \leq (h - \frac{p}{m+1})\frac{1}{m},$$

then $\forall 0 \leq s+1 < S < m$,

$$\frac{c}{m+1} \leq (h - \frac{p}{m+1})\frac{1}{m} < (h - \frac{p}{m+1})\frac{1}{S};$$

$$(h - \frac{p}{m+1})\frac{1}{S} < (h - \frac{p}{m+1})\frac{1}{s+1}.$$

Hence,

$$g(s,s+1) > g(s,s+2) > \cdots > g(s,m-1) \geq g(s,m),$$

Therefore, if $\frac{c}{m+1} \leq (h - \frac{p}{m+1})\frac{1}{m}$, then $\forall 0 \leq s+1 < S \leq m$

$$g(s,S) < g(s,s+1).$$

To show (c) note that if $(h - \frac{p}{m+1})\frac{1}{m} < \frac{c}{m+1} < h - \frac{p}{m+1}$ and if we let $n^* = \lfloor (h(m+1)-p)c \rfloor$, then we have

$$n^* \leq (h(m+1)-p)c;$$

$$\frac{c}{m+1} \leq (h - \frac{p}{m+1})\frac{1}{n^*}.$$

And

$$n^*+1 \geq (h(m+1)-p)c$$

$$\frac{c}{m+1} \geq (h - \frac{p}{m+1})\frac{1}{n^*+1}.$$

Because $(h - \frac{p}{m+1})\frac{1}{m} < \frac{c}{m+1} < h - \frac{p}{m+1}$, $\forall 1 \leq n < n^*$,

$$\frac{c}{m+1} < (h - \frac{p}{m+1})\frac{1}{n};$$

$\forall n^*+1 < n \leq m$,

$$\frac{c}{m+1} > (h - \frac{p}{m+1})\frac{1}{n}.$$

Take $S^* = max\{n^*, s+1\}$, then we have

$$g(s,s+1) > g(s,s+2) > \cdots > g(s,S^*-1) \geq g(s,S^*);$$

$$g(s,m) > g(s,m-1) > \cdots > g(s,S^*+1) \geq g(s,S^*).$$

Hence, if $(h - \frac{p}{m+1})\frac{1}{m} < \frac{c}{m+1} < h - \frac{p}{m+1}$, then $\forall 0 \leq s+1 < S < m$,

$$g(s,S) < max\{g(s,s+1), g(s,m)\}.$$

The proof is complete. □

Lemma 5 *If $h - \frac{p}{m+1} > 0$, and $0 \leq s+1 \leq m$ then*

$$g(s, s+1) \leq g(m-1, m).$$

Proof: Because

$$g(s, s+1) = \frac{c(s+1)(s+2)}{2(m+1)} - \frac{pm}{m+1} + (s+1)(\frac{p}{m+1} - h);$$

$$g(s-1, s) = \frac{cs(s+1)}{2(m+1)} - \frac{pm}{m+1} + s(\frac{p}{m+1} - h),$$

then we have

$$g(s, s+1) - g(s-1, s) = \frac{c(s+1)}{m+1} - (h - \frac{p}{m+1}).$$

And we have $0 \leq s+1 \leq m$.

Hence we can find s^* which satisfies following:

- when $s \leq s^*$,

$$g(s, s+1) - g(s-1, s) \leq 0;$$

- when $s > s^*$,

$$g(s, s+1) - g(s-1, s) > 0.$$

Combining the above we have that either $s+1 = 0$ or $s+1 = m$ will make $g(s, s+1)$ achieve its biggest value. Let compare $s+1 = 0$ and $s+1 = m$:

When $s+1 = 0$, $g(-1, 0) = 0$;

When $s+1 = m$, $g(m-1, m) = \frac{cm(m-1)}{2(m+1)} - \frac{pm}{m+1} + m(\frac{p}{m+1} - h) = m(c-h) > 0$

That is to say, $g(m-1, m)$ make $g(s, s+1)$ attain its biggest value. □

Because of theorem 3, 4 and lemma 5, we can get following result:

Theorem 6 *Under the assumptions made in this section, we can find s^*, such that for all $\forall 0 \leq s+1 \leq S \leq m$ the following is true:*

$$g(s, S) \leq g(s^*, m).$$

4 Conclusion

All the results in the above analysis are valid for any inventory system with a fixed penalty cost that is incurred whenever there is a shortage. In this paper the discussion is done in terms of applying an (s, S) policy into a hotel booking problem. The contributions of this research are: it is the first model to analyze the (s, S) with fixed penalty cost for lost demand; second the paper describes a new hotel booking model; third, the case of unform demands is solved explicitly.

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