Production-Inventory Systems with Lost-sales and Compound Poisson Demands

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Abstract

This paper considers a continuous-review, single-product production-inventory system with a constant replenishment rate, compound Poisson demands and lost-sales. Two objective functions that represent metrics of operational costs are considered: (1) the sum of the expected discounted inventory holding costs and lost-sales penalties, both over an infinite time horizon, given an initial inventory level; and (2) the long-run time average of the same costs. The goal is to minimize these cost metrics with respect to the replenishment rate. It is, however, not possible to obtain closed form expressions for the aforementioned cost functions directly in terms of positive replenishment rate (PRR). To overcome this difficulty, we construct a bijection from the PRR space to the space of positive roots of Lundberg’s fundamental equation, to be referred to as the Lundberg Positive Root (LPR) space. This transformation allows us to derive closed form expressions for the aforementioned cost metrics with respect to the LPR variable, in lieu of the PRR variable. We then proceed to solve the optimization problem in the LPR space, and finally recover the optimal replenishment rate from the optimal LPR variable via the inverse bijection. For the special cases of constant or loss-proportional penalty and exponentially distributed demand sizes, we obtain simpler explicit formulas for the optimal replenishment rate.

Keywords and Phrases: Compound Poisson arrivals, integro-differential equation, Laplace transform, Lundberg’s fundamental equation, lost-sales, production-inventory system, constant replenishment rate.
1. Introduction

Production-inventory systems with constant production rates are implemented by a variety of manufacturing firms. Examples can be found in: (1) glass manufacturing, where glass furnaces often produce at constant rates (Federal Register, 2009); (2) sugar mills, where raw sugar is produced utilizing a constant production rate (Grunow et al. 2007); (3) the electronic computer industry, where displays are manufactured at constant production rates (Display Development News, 2000); and (4) the pharmaceutical industry, where cell-free proteins and other products are generally produced at constant production rates (Membrane & Separation Technology News, 1997). Additional examples can be found in the carpet manufacturing industry, where the yarning and dyeing processes operate at constant rates over long periods of time. These constant rates are selected by the manufacturer at the production planning stage by taking into account the anticipated demands and its cost structures. At the manufacturing stage, it produces carpet rolls continuously, and specifically, at full capacity for carpet dyeing.

Production-inventory systems with constant production rates are typically deployed when there are high setup times and high setup costs, where frequent modification (e.g., interruption or rate change) of the production line is financially or operationally prohibitive. Thus, for both financial and operational reasons, it is critical to establish the proper production process early in the planning process. The importance of the production rate is self-evident: an overly high production rate results in high holding costs due to excess inventory, while a low production rate results in high penalty costs due to frequent stockouts and subsequent lost-sales. Thus, it is reasonable to expect that there exists an optimal production rate that balances these two costs. Furthermore, manufactures often employ “full capacity” in production. For example, the refinery industry has Operable Capacity Utilization Rate at 92% or even higher. Consequently, the production capacity level corresponding to the production rate has a critical impact on the firm’s cost structure, its inventory policies and its service levels, as well as its management and staff support requirements [cf. Jacobs and Chase (2013)]. This study sheds light on the optimal production capacity of a firm from a long-term cost minimization perspective.

We study a continuous-review single-product production-inventory system with a constant production/replenishment rate and compound Poisson demands, subject to lost-sales. In the sequel, we will use the terms production and replenishment interchangeably. Unsatisfied demand may be partially fulfilled from on-hand inventory (if any) and all excess demand (shortage) is lost; such excess demand

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1 According to a recent publication of the Independent Statistics & Analysis, U.S. Energy Information Administration.
will be referred to as the *lost-sales size*. The system incurs two types of costs: a holding cost and a lost-sales cost. The holding cost is incurred as a function of the inventory on hand, and assessed at a constant rate per unit on-hand inventory per unit time. The lost-sales cost is a penalty imposed at each loss occurrence, and is assumed to be a function of the lost-sales size. The goal of this paper is to derive the optimal replenishment rates that minimize two objective functions that represent metrics of operational costs: (1) the sum of expected discounted inventory holding costs and lost-sales penalties over an infinite time horizon, given an initial inventory level; and (2) the long-run time average of the same costs.

The main objective of this paper is twofold: (1) to provide closed form expressions for the respective objective functions of the conditional expected discounted costs and of the time-average costs; and (2) to minimize the aforementioned objective functions with respect to the replenishment rate. To this end, we first derive an integro-differential equation for the conditional expected discounted cost function until the first lost-sale occurrence. However, a closed form formula for that cost function is not available. To overcome this difficulty, we observe that the original optimization problem in terms of the replenishment rate parameter can be reformulated and solved in a tractable form in terms of another variable, and then the requisite optimal replenishment rate can be recovered. More specifically, let the original space of all positive replenishment rates be referred to as the *PRR space*, and define a related space consisting of all positive roots of the so-called *Lundberg’s fundamental equation* (see Gerber and Shiu (1998) and Eq. (4.7)), to be referred to as the *Lundberg positive roots (LPR)* space. The two spaces, PRR and LPR, will be shown to be related by a bijection (i.e., a one-one and onto mapping); see Eq. (4.9). Indeed, the cost function over the PRR space does not have a closed form expression, while the same cost function over the LPR space does, thereby facilitating its optimization. Finally, having obtained the optimal solution in the LPR space, we shall provide an algorithm to compute the requisite optimal replenishment rate in the PRR space via the inverse bijection [cf. Figure 3]. We further obtain explicit solutions for the special cases in which the lost-sales penalty function is either: (1) a constant penalty for each lost-sales occurrence, or (2) a loss-proportional penalty. Finally, a numerical study is performed to illustrate the results and demonstrate additional properties of the system.

The methodology employed in this paper gives rise to interesting connections between inventory management and queueing and insurance risk models. In particular, this study is connected to some important aspects of G/M/1 queues in equilibrium, such as the joint distribution of the busy period and the idle period [cf. Perry *et al.* (2005), Adan *et al.* (2005) and Perry (2011)].
In summary, the main analytical contributions of this paper are: (1) a closed form formula for the expected discounted cost function for any initial inventory level, general demand size distributions and general penalty functions; (2) a characterization of the optimal constant replenishment rate that minimizes the expected discounted cost function for general demand size distributions and general penalty functions; (3) closed form expressions for the optimal replenishment rate and the attendant costs for the case of exponential demand size, for both constant penalty and loss-proportional penalty functions; (4) a closed form formula for the long-run time-average cost function for general demand size distributions and general penalty functions; this cost function can also be optimized using the same approach employed for the expected discounted cost function.

The remainder of this paper is organized as follows. Section 2 reviews related literature. Section 3 formulates the production-inventory model under study. Section 4 derives a closed form expression for the expected discounted cost function and Section 5 treats its optimization. Section 6 presents a set of numerical studies. Section 7 examines the long-run time-average cost function and its optimization. Section 8 presents ideas on extensions of the model to incorporate variable production cost and service level constraints. Finally, Section 9 concludes this paper.

2. Literature Review

This section first reviews the literature on continuous-review production-inventory systems, and then compares the production-inventory model with related queueing and insurance risk models.

Most papers on continuous-review inventory systems assume that orders are placed and replenished in batches or lot sizes. One of the well-known ordering policies is the continuous-review \((s,S)\) policy; see Scarf (1960) for a seminal work. In contrast, our study considers a production-inventory system where inventory is replenished continuously at a constant rate, and the goal is to find the optimal replenishment rate. Constant production or replenishment rates are common in continuous-review production-inventory systems. For example, Doshi et al. (1978) consider a production-inventory control model of finite capacity that switches between two possible production rates based on two critical stock-levels. The main result of that paper is a formula for the long-run time-average cost as a function of two critical levels of the production rate. De Kok et al. (1984) deal with a production-inventory model subject to a service level constraint, where excess demand is backlogged and the production rate can be dynamically switched between two possible rates. The authors derive a useful approximation for the switch-over level. For the same model, De Kok (1985) considers the corresponding lost-sales case and provides an approximation for the switch-over level. Gavish and Graves (1980) consider a production-inventory system, where the
demand process is Poisson and demand size is constant. The authors assume that excess demand is backlogged and the production facility may be set up or shut down. They treat their system as an $M/D/1$ queue and minimize the expected cost per unit time. Graves and Keilson (1981) extend the model of Gavish and Graves (1980) by considering a compound Poisson demand process. The problem is analyzed as a constrained Markov process, using the compensation method, and a closed-form expression is derived for the expected system cost as a function of the policy parameters. For a similar setting, Graves (1982) derives the steady-state distribution of the inventory level using queueing theory. More recently, Perry et al. (2005) study a production-inventory system with a fixed and constant replenishment rate under an $M/G$ (i.e., a compound Poisson) demand process and two “clearing policies” (sporadic and continuous) to avoid high inventory levels. The paper derives explicit results for the associated expected discounted cost functions under both types of clearing policies. We note that while the literature above assumes the replenishment rate to be exogenous and fixed, our paper treats this parameter as a decision variable.

The underlying inventory process studied in this paper can also be ascribed a variety of interpretations, drawn from the contexts of queueing and insurance risk systems. In what follows, we provide a literature review on such connections; interested readers are also referred to Prabhu (1997) for a general treatment of such models under the theme of stochastic models.

The similarity between queueing and inventory models is well recognized in the literature, and a number of papers treat one model from the perspective of the other. From a queueing vantage point, the inventory level can be interpreted as the attained waiting time in a $G/M/1$ queue, provided idle periods are removed; see Adan et al (2005), Prabhu (1965) and references therein. An inventory analysis generally includes an explicit cost structure and a solution for optimal policies, while researchers in queueing theory have been more interested in the underlying probabilistic structure. However, some papers address inventory problems using queueing theory; two cases in point are Graves (1982) and Perry et al. (2005). Cost optimization has also been considered in queueing models. Such research has been directed towards finding optimal operating policies for a queuing system subject to a given cost/reward structure. Such optimization problems have been considered by Bell (1971), Heyman (1968), Lee and Srinivasan (1989) and Sobel (1969).

In the context of classical insurance risk models, the inventory level can be interpreted as a surplus (or capital, or risk reserve) level of an insurance firm, under a constant rate of premium inflows and compound Poisson claim arrivals; see Asmussen (2000), Gerber and Shiu (1998). Risk theory in general,
and ruin probability in particular, are traditionally considered essential topics in the insurance literature. Since the seminal paper by Lundberg (1932), many studies have addressed this topic; cf. Gerber and Shiu (1997, 1998) and Rolski et al. (1999). Two typical questions of interest in classical ruin theory are (a) the deficit at ruin; and (b) the time to ruin. To address those two questions, Gerber and Shiu (1998) have introduced a comprehensive penalty function, the so-called *Gerber-Shiu penalty function*, as a function of surplus immediately prior to ruin and the deficit at ruin; this function has been widely discussed in the recent insurance literature. Additional extensions based on the Gerber-Shiu penalty function include barrier or threshold strategies; see Boxma et al. (2011), Lin et al. (2003), Lin and Pavlova (2006), and references therein. Recently, Boxma et al. (2011) and Löpker and Perry (2010) have further studied insurance risk models (time to ruin, ruin probability, and the total dividend) using methods and results from queueing theory. In most of these studies, it is noted that the inventory process can be interpreted as the content process of a queuing or an insurance risk model. In contrast, the present study differs from the above in terms of its objective function and its conditions for system stability; in particular, our paper treats cost computation and optimization while the insurance literature is primarily interested in dividends and risk (e.g., time to ruin and ruin probability), and the queuing literature mainly focuses on quantities such as service levels and workload in the system. Queueing theory also puts emphasis on stability conditions: a stable queue requires the traffic intensity to be strictly less than one; cf. Asmussen (2003) and Prabhu (1997). Stability conditions for an insurance risk model ensure that the average claim is less than the premium rate (i.e., a *positive* security-loading), such that the probability of ultimate ruin is less than one; cf. Eq. (2.5) in Gerber and Shiu (1998). In our production-inventory context, the condition that the average demand is greater than the replenishment rate (i.e., a *negative* security-loading) is necessary for the time-average cost optimization, whereas no such restriction is required for the expected discounted cost analysis.

In this paper, it is not possible to directly solve the integro-differential equation in Eq. (4.4). However, it is possible to solve equations that involve Laplace transforms [cf. Widder (1959)], and then invert the transformed functions to obtain the requisite functions. We note that the problem of inverting Laplace transforms is often difficult, so most studies focus on numerical approximations, e.g., Cohen (2007) and Shortle et al. (2004).

In addition to the contributions of analytical results listed in Section 1, the main methodology contributions of this paper are as follows: (1) we treat the original problem in terms of the LPR variable by taking advantage of *Lundberg’s fundamental* equation and a bijection between positive production rates (PRR variables) and Lundberg positive roots (LPR variables); and (2) we optimize this cost function
in the LPR space and then invert the optimal LPR variable to obtain the requisite optimal replenishment rate in the PRR space using the inverse bijection. To the best of our knowledge, no study in the inventory literature exploits such an optimization technique.

3. Model Formulation

We will use the following notational conventions and terminology. Let \( \mathbb{R} \) denote the set of real numbers and \( x^+ = \max\{x, 0\} \), for any \( x \in \mathbb{R} \). For a random variable \( X \), its probability density function (pdf) is denoted by \( f_x(x) \), its cumulative distribution function (cdf) by \( F_x(x) \) and its complementary cdf by \( \bar{F}_x(x) \). For two real functions \( f(x) \) and \( g(x) \) on \([0, \infty)\), their convolution function is given by

\[
(f * g)(u) = \int_0^u f(u - x) g(x) \, dx.
\]

The Laplace transform of a function \( f(x) \) is defined by

\[
\mathcal{L}[f](z) = \bar{f}(z) = \int_0^\infty e^{-zx} f(x) \, dx, \quad z \geq 0.
\]

For any non-negative random variable \( X \), we shall make repeated use of the following relation

\[
\bar{F}_X(z) = \int_0^\infty e^{-zx} \bar{F}_x(x) \, dx = \frac{1}{z} \left[ 1 + \int_0^\infty e^{-zx} d\bar{F}_x(x) \right] = \frac{1}{z} \left[ 1 - \bar{f}_X(z) \right], \quad (3.1)
\]

where the second equality follows from integration by parts. Throughout this paper, we will tacitly assume the existence of a basic probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \( \Omega \) is the sample space, \( \mathcal{F} \) is a \( \sigma \)-field of events, and \( \mathbb{P} \) is a probability measure on \( \mathcal{F} \). Finally, we assume continuously compounded discounting at rate, \( r > 0 \).

3.1 Inventory Process

We consider a continuous-review inventory system, subject to lost sales. The demand arrival stream constitutes a compound Poisson process with rate \( \lambda \) and arrival times \( \{A_i : i \geq 0\} \), where \( A_0 = 0 \) by convention. Thus, the corresponding sequence of inter-arrival times, \( \{T_i : i \geq 1\} \), where \( T_i = A_i - A_{i-1} \), is exponentially distributed and the sequence is \textit{identically independently distributed} (iid). The corresponding demand sizes form an iid sequence \( \{D_i : i \geq 1\} \) with a common pdf \( f_D(x) \) and common mean demand, \( \mathbb{E}[D] < \infty \), where the demand of size \( D_i \) arrives at time \( A_i \). Replenishment occurs at a
constant (deterministic) rate, \( \rho \geq 0 \). Let \( \{I(t) : t \geq 0\} \) denote the right-continuous inventory process, given by

\[
I(t) = I(0) + \rho t - \sum_{i=1}^{N_A(t)} [D_i - L(A_i)],
\]

where \( N_A(t) \) is the number of demands arriving over \((0, t]\) and

\[
L(A_i) = [D_i - I(A_i^-)]^+, \; i = 1, 2, \ldots
\]

is the lost-sales size at time \( A_i \). Let \( \{\tau_i : i \geq 0\} \) be the sequence of loss occurrence times, given by

\[
\tau_i = \inf \{A_j > \tau_{i-1} : L(A_j) > 0\},
\]

where \( \tau_0 = 0 \) by convention. Let \( \{J_k : k \geq 0\} \) be the sequence of random arrival indexes at which a loss occurs, namely, \( \tau_k = A_{J_k} \). Figure 1 illustrates a sample path of the inventory process with lost-sales over an infinite time horizon.

![Figure 1. A sample path of the inventory level process, \( \{I(t)\} \)](image)

We note that the inventory process \( \{I(t)\} \) of Eq. (3.2) is stable under the condition \( \rho < \lambda \mathbb{E}[D] \); cf. Proposition 1.1 in Asmussen (2000). In contrast, it is typically assumed \( \rho > \lambda \mathbb{E}[D] \) in queueing theory and classical risk insurance studies. In particular, queueing systems generally assume that the service rate is greater than the arrival rate [cf. Adan et al. (2005) and Asmussen (2003)]; otherwise the queue length explodes. Classical risk insurance analysis typically assumes that the premium rate is greater than the average claim to ensure a positive drift; cf. Gerber and Shiu (1997, 1998). In our model the stability condition \( \rho < \lambda \mathbb{E}[D] \) is only required when studying the time-average cost; it is not imposed for the
expected discounted cost, since in this case the objective cost function is always bounded due to discounting even if the inventory process is unstable.

### 3.2 Cost Functions
Recall that the production-inventory system under study incurs costs in the form of holding costs and lost-sales penalties. Specifically, a holding cost is incurred at rate $h$ per unit inventory per unit time while there is inventory on hand, and a penalty $w(x)$ is incurred whenever a customer's demand cannot be fully satisfied from on-hand inventory and there is a shortage of size $x$. The penalty function $w(x)$ is assumed to be non-decreasing in the lost-sales size, $x$, where $w(0) = 0$. Thus, the total discounted cost up until time $t$ is given by

$$C_\rho(t) = h\int_0^t e^{-rz}I(z) \, dz + \sum_{i=1}^{N(t)} e^{-rA_i}w(L(A_i)),$$

which is dependent on the initial inventory level $I(0) = u \geq 0$. Of particular interest is the conditional expected discounted cost function up until and including the first lost-sale occurrence, given by

$$c_\rho(u) = \mathbb{E}[C_\rho(t_1) \mid I(0) = u].$$

Furthermore, the conditional expected discounted cost function over the interval $(0, t]$ is given by

$$\Phi_\rho(t \mid u) = \mathbb{E}[C_\rho(t) \mid I(0) = u].$$

It is easy to show that the function $\Phi_\rho(t \mid u)$ is increasing and uniformly bounded in $t$, for any given $u$. Hence, it follows that the conditional expected total discounted cost function,

$$\Phi_\rho(u) = \lim_{t \to \infty} \Phi_\rho(t \mid u),$$

is well defined. In order to optimize $\Phi_\rho(u)$ with respect to $\rho$, we next derive the expected discounted cost function in Section 4, and then treat its optimization in Section 5. All proofs omitted from these sections are provided in the appendices.

### 4. Computation of the Expected Discounted Cost Function
To derive a closed form formula for the cost function $\Phi_\rho(u)$ of Eq. (3.8), we first establish, in the following theorem, that the expected discounted cost for an arbitrary initial inventory level can be decomposed into two terms: the discounted cost up until the first lost-sale occurrence and the expected discounted cost thereafter.
Theorem 1

Given any initial inventory $u \geq 0$, $\Phi_{\rho}(u)$ and $c_{\rho}(u)$ satisfy the following equation,

$$\Phi_{\rho}(u) = c_{\rho}(u) + d_{\rho}(u) \Phi_{\rho}(0), \quad (4.1)$$

where

$$d_{\rho}(u) = \mathbb{E}[e^{-rT_{I}} | I(0) = u]. \quad (4.2)$$

Proof. Follows readily from the strong Markov property of the process $\{I(t) : t \geq 0\}$. $\square$

In particular, setting $u = 0$ in Eq. (4.1), we obtain

$$\Phi_{\rho}(0) = \frac{c_{\rho}(0)}{1 - d_{\rho}(0)}. \quad (4.3)$$

The following two subsections study the component functions $c_{\rho}(u)$ and $d_{\rho}(u)$ of $\Phi_{\rho}(u)$.

4.1 The Cost Function $c_{\rho}(u)$

In this subsection we derive an integro-differential equation for $c_{\rho}(u)$ in Lemma 1 from which we will later obtain closed form expressions for $c_{\rho}(0)$ and $\tilde{c}_{\rho}(z)$ in Proposition 1.

Lemma 1

The function $c_{\rho}(u)$ defined by Eq. (3.6) is continuous, differentiable in $u \geq 0$ and satisfies

$$\rho \frac{\partial}{\partial u} c_{\rho}(u) - (\lambda + r) c_{\rho}(u) + \lambda \langle f_{D} * c_{\rho} \rangle (u) = -g(u), \quad (4.4)$$

where

$$g(u) = hu + \lambda \int_{u}^{\infty} f_{D}(x) w(x - u) dx, \quad u \geq 0. \quad (4.5)$$

To solve Eq. (4.4) for $c_{\rho}(u)$, we introduce the auxiliary function $\psi(z)$, given by

$$\psi(z) = \lambda \tilde{f}_{D}(z) + \rho z - \lambda - r. \quad (4.6)$$

where by convention, $\psi(z) = \infty$ if $\tilde{f}_{D}(z)$ does not exist. It is of interest to study the roots of the equation $\psi(z) = 0$, that is, the roots of the equation
\[ r - \rho z + \lambda [1 - \tilde{f}_D(z)] = 0. \] \hspace{1cm} (4.7)

Eq. (4.7) is well known in the context of insurance models, where it is referred to as *Lundberg’s fundamental equation*; cf. Gerber and Shiu (1998). An important property of the roots of that equation is as follows: for any \( r > 0 \), the equation \( \psi(z) = 0 \) has two distinct real roots, \( \xi \) and \( \theta \), where \( \xi > 0 \) and \( \theta < 0 \) (ibid.). Figure 2 depicts the structure of the function \( \psi(z) \) and its two roots.

![Figure 2. Illustration of the structure of the function \( \psi(z) \) and its two roots](image)

We note that either the negative root, \( \theta \), or the positive one, \( \xi \), can be employed later to derive the cost function via their one-one and onto relationships with \( \rho \). However, in the sequel, we shall employ \( \xi \) (rather than \( \theta \)) as a decision variable in deriving the cost functions and optimal solutions. There are two reasons for this preference. First, for \( \tilde{f}_D(\theta) \) to exist, the negative root \( \theta \) is constrained to be larger than a certain constant (determined by the demand distribution function), but such constant is generally difficult to identify. In contrast, the positive root \( \xi \) always guarantees the existence of \( \tilde{f}_D(\xi) \). Second, the time-average cost function, to be studied in Section 7, can be derived from the discounted cost function by taking the limit as the discount rate \( r \) tends to zero. In this case, \( \theta \) tends to zero, while \( \xi \) remains positive, which can also facilitate the study of the time-average cost case.

Next, setting \( z = \xi \) in Eq. (4.7), it follows that the Lundberg positive-root, \( \xi \), satisfies

\[ \lambda \tilde{f}_D(\xi) + \rho \xi - \lambda - r = 0. \] \hspace{1cm} (4.8)

Eq. (4.8) motivates the following Lemma which provides the basis for our solution methodology.
Lemma 2
(a) There is a bijection between $\rho$ and $\xi$, implicitly given by the equation
$$\rho = \frac{r}{\xi} + \lambda \tilde{F}_D(\xi).$$ (4.9)

(b) The function $\xi(\rho)$, implicitly defined by Eq. (4.9), is strictly decreasing in $\rho$ and satisfies:
\begin{align*}
(1) \quad & \lim_{\rho \to 0^+} \xi(\rho) = \infty \quad \text{and} \quad \lim_{\rho \to 0^+} \rho \xi(\rho) = \lambda + r; \\
(2) \quad & \lim_{\rho \to \infty} \xi(\rho) = 0 \quad \text{and} \quad \lim_{\rho \to \infty} \rho \xi(\rho) = r. \quad \square
\end{align*}

The bijection between $\rho$ and $\xi$, given by Eq. (4.9), allows us to derive a closed form formula for the attendant cost functions in terms of the LPR variable $\xi$ in lieu of the PRR variable, $\rho$. Furthermore, the optimization of the cost functions can be performed with respect to $\xi$, and the corresponding optimal $\xi^*$ can be used to recover the optimal $\rho^* = \rho(\xi^*)$ via the bijection function given by Eq. (4.9). Figure 3 depicts the idea of the solution methodology, which we dub the Bijection Solution Methodology.

We next establish expressions for $c_{\rho}(u)$ by solving Eq. (4.4). To this end, we take the Laplace transform with respect to $u$ on both sides of Eq. (4.4), which yields
$$\rho \left[ z \tilde{c}_{\rho}(z) - c_{\rho}(0) \right] - (\lambda + r) \tilde{c}_{\rho}(z) + \lambda \tilde{f}_D(z) \tilde{c}_{\rho}(z) = -\tilde{g}(z), \quad z > 0. \quad (4.10)$$
Rearranging and simplifying Eq. (4.10), we obtain
\[ \psi(z) \cdot \tilde{c}_\rho(z) - \rho c_\rho(0) = -\tilde{g}(z), \quad z > 0, \]  

where \( \psi(z) \) is given by Eq. (4.6). The following result provides closed form formulas for \( c_\rho(0) \) and \( \tilde{c}_\rho(z) \) in terms of the LPR variable \( \xi \).

**Proposition 1**

For \( \rho > 0 \),

\[ c_\rho(0) = \frac{1}{\rho} \tilde{g}(\xi); \quad (4.12) \]

\[ \tilde{c}_\rho(z) = \frac{\tilde{g}(\xi) - \tilde{g}(z)}{\psi(z)}, \quad z \neq \xi. \quad (4.13) \]

\( \Box \)

Next, substituting Eq. (4.9) into Eq. (4.12) yields another expression for \( c_\rho(0) \) in terms of the LPR variable \( \xi \), given by

\[ c_{\rho(\xi)}(0) = \frac{\xi \tilde{g}(\xi)}{r + \lambda \xi \tilde{F}_D(\xi)}. \quad (4.14) \]

The expression above allows us to optimize \( c_{\rho(\xi)}(0) \) with respect to \( \xi \) rather than \( \rho \), where the latter is very difficult or even impossible. The optimal \( \rho^* \) can then be recovered from the optimal \( \xi^* \) via the bijection of Eq. (4.9). The minimization of \( \Phi_\rho(u) \) with respect to \( \rho \) can be performed in a similar manner.

We mention that for the limiting case of \( \rho = 0 \), it can be readily shown by Eq. (3.6) that

\[ c_0(0) = \frac{\lambda}{\lambda + r} \mathbb{E}[w(D)]. \quad (4.15) \]

Alternatively, the above result can be obtained by taking limits on both sides of Eq. (4.12), resulting in

\[ \lim_{\rho \to 0} c_\rho(0) = \lim_{\rho \to 0} \frac{\xi \tilde{g}(\xi)}{\rho \xi} = \frac{\lim_{\xi \to \infty} \xi \tilde{g}(\xi)}{\lambda + r} = \frac{g(0)}{\lambda + r}, \]

where the second equality holds by Lemma 2, part (b) and the third holds by a property of the Laplace transform. The above equation can now be rewritten as Eq. (4.15) by Eq. (4.5).
We note that if there is no holding cost (i.e., $h = 0$), then $c_\rho(0)$ represents the expected discounted value of the deficit at ruin in a classical insurance model. Gerber and Shiu (1998) have given a representation analogous to Eq. (4.12) for this case. If we further specify $w(x)$ to be an exponential function, then Eq. (4.13) can be interpreted in a queueing context as the joint Laplace transform of the busy period and the idle period; cf. Prabhu (1997), Asmussen (2003) and Adan et al. (2005).

4.2 The Function $d_\rho(u)$

In this subsection, we derive a closed form formula for $d_\rho(0)$ and provide an explicit expression for $\tilde{d}_\rho(z)$. Note that by Eqs. (3.5) and (3.6), $c_\rho(u)$ can be written as

$$c_\rho(u) = \mathbb{E}\left[h \int_0^{\tau_1} e^{-rz} I(z) \, dz + e^{-r\tau_1} w(L(\tau_1)) \mid I(0) = u\right].$$

The above equation implies that $d_\rho(u)$, given by Eq. (4.2), is a special case of $c_\rho(u)$ when $h = 0$ and $w(x) = 1$. The results for $d_\rho(u)$ contained in the next proposition can be obtained from their counterparts for $c_\rho(u)$.

**Proposition 2**

For $\rho > 0$,

$$d_\rho(0) = \frac{\lambda}{\rho} \tilde{F}_D(\xi) = 1 - \frac{r}{\rho \xi}, \quad (4.16)$$

$$\tilde{d}_\rho(z) = \frac{r}{\psi(z)} \left[ \frac{1}{z} - \frac{1}{\xi} \right] + \frac{1}{z}, \quad z \neq \xi. \quad (4.17)$$

Note that the definition of $d_\rho(u)$ given by Eq. (4.2) implies its continuity in $\rho$ and

$$d_0(0) = \mathbb{E}[e^{-rA_1}] = \frac{\lambda}{\lambda + r},$$

by virtue of Eq. (4.2), where $\tau_1 = A_1$ when $\rho = 0$. Alternatively, this can be verified by substituting $\lim_{\rho \to 0} \rho \xi(\rho) = \lambda + r$ (cf. Lemma 2) into Eq. (4.16). Note also that $\lim_{\rho \to \infty} d_\rho(0) \to 0$ in view of Eq. (4.2), since $\tau_1 \to \infty$ while $\rho \to \infty$. This can be alternatively verified using the fact that $\lim_{\rho \to \infty} \rho \xi(\rho) = r$ (cf. Lemma 2) and Eq. (4.16).
4.3 The Function $\Phi_\rho(u)$

It appears that it is not possible to derive a closed form expression for $\Phi_\rho(u)$ as a function of $\rho$. However, the Bijection Solution Methodology allows us to derive a closed form expression for $\Phi_\rho(u) = \Phi_\rho(\xi)(u)$ as function of $\xi$. The main results in this subsection are presented in Theorem 2 and Theorem 3. To keep the notation simple, we will use $\rho$ and $\xi$ interchangeably, exploiting the bijection between them. In this fashion, $\Phi_\rho(u)$ and $\Phi_\xi(u)$ denote the same function but given in terms of $\rho$ and $\xi$, respectively. Similar notational conventions will be adopted in the sequel for other quantities, e.g., $\bar{e}_\rho$ and $\bar{e}_\xi$ for the time-average cost in Section 7, as well as $v_\rho$ and $v_\xi$ for the production cost in Section 8.

**Theorem 2**

For a zero initial inventory level,

$$\Phi_\xi(0) = \frac{\rho \xi}{r} c_\rho(0) = \frac{\xi}{r} \tilde{g}(\xi);$$  \hspace{1cm} (4.18)

while for an arbitrary initial inventory level $u \geq 0$,

$$\Phi_\xi(u) = c_\xi(u) + \xi \tilde{g}(\xi) d_\xi(u), \quad u \geq 0;$$ \hspace{1cm} (4.19)

$$\tilde{\Phi}_\xi(z) = \xi \tilde{g}(\xi) \left[ \frac{1}{rz} + \frac{1}{z \psi(z)} \right] \tilde{g}(z), \quad z \neq \xi.$$ \hspace{1cm} (4.20)

We next obtain a renewal-type representation of $\Phi_\xi(u)$ by inverting Eq. (4.20).

**Corollary 1**

For any initial inventory $u \geq 0$, $\Phi_\xi(u)$ satisfies the equation,

$$\Phi_\xi(u) = \Phi_\xi(0) + \left\{ G_\xi * \eta_\xi \right\}(u), \quad u \geq 0,$$ \hspace{1cm} (4.21)

where $\Phi_\xi(0)$ is given by Eq. (4.18), $G_\xi(x)$ is given by

$$G_\xi(x) = \xi \tilde{g}(\xi) - g(x).$$ \hspace{1cm} (4.22)

and $\eta_\xi(u)$ is the inverse Laplace transform of $\frac{1}{\psi(z)}$ at $u \geq 0$. \hspace{1cm} $\square$
In view of Corollary 1, \( \Phi_\xi(u) \) can be obtained by computing the convolution of \( \eta_\xi(u) \) and \( G_\xi(x) \). To derive a closed form expression for \( \Phi_\xi(u) \), we introduce the function,

\[
V_\rho(z) = \frac{(z - \xi)(z - \theta)}{\psi(z)}. \tag{4.23}
\]

We define \( V_\rho(\xi) \) and \( V_\rho(\theta) \) to be the limits of \( V_\rho(z) \) as \( z \) tends to \( \xi \) and \( \theta \), respectively. Note that by the L’Hôpital rule, \( V_\rho(\xi) \) and \( V_\rho(\theta) \) can be further simplified as

\[
V_\rho(\xi) = \frac{\xi - \theta}{\psi'(\xi)}, \tag{4.24}
\]

\[
V_\rho(\theta) = \frac{\theta - \xi}{\psi'(\theta)}. \tag{4.25}
\]

where the derivatives \( \psi'(\xi) \) and \( \psi'(\theta) \) can be obtained from Eq. (4.6).

The following theorem provides an explicit formula for \( \Phi_\xi(u) \) and is a key result of the paper.

**Theorem 3**

For any initial inventory level \( u \geq 0 \),

\[
\Phi_\xi(u) = \frac{\xi \bar{g}(\xi)}{\rho} + \frac{V_\rho(\xi)}{\xi - \theta} \left[ e^{\xi u} \int_u^\infty g(x) e^{-\xi x} \, dx - \bar{g}(\xi) \right] + \frac{V_\rho(\theta)}{\xi - \theta} \left[ e^{\theta u} \int_0^u g(x) e^{-\theta x} \, dx + \frac{\xi \bar{g}(\xi)}{\theta} \left( e^{\theta u} - 1 \right) \right], \tag{4.26}
\]

where \( V_\rho(\xi) \) and \( V_\rho(\theta) \) are given by Eqs. (4.24) and (4.25), respectively. \( \square \)

Theorem 3 shows that the expected discounted cost \( \Phi_\xi(u) \) depends on the initial inventory level, \( u \), in a complicated way. We further observe that Eq. (4.26) reduces to Eq. (4.18) when the initial inventory level \( u \) is zero.

In the following two subsections, we investigate two special cases of the penalty function: constant lost-sales penalty and loss-proportional penalty.

### 4.3.1 Constant Lost-Sales Penalty

In this case we have \( \mathbf{w}(x) = K_0 \), for \( x > 0 \), where \( K_0 > 0 \) is a constant. Accordingly, Eq. (4.5) becomes
\[ g(u) = h u + \lambda K_0 \tilde{F}_D(u), \quad u \geq 0, \]  

and the corresponding Laplace transform is given by

\[ \tilde{g}(z) = \frac{h}{z^2} + \lambda K_0 \tilde{F}_D(z). \]  

Next, setting \( z = \xi \) and substituting \( \tilde{F}_D(\xi) \) from Eq. (4.9) into Eq. (4.28), we have

\[ \tilde{g}(\xi) = \frac{h}{\xi^2} + K_0 \left( \rho - \frac{r}{\xi} \right). \]  

Now substituting Eq. (4.29) into Eq. (4.18) yields

\[ \Phi_\xi(0) = \frac{h}{r} + K_0 \left( \frac{\rho \xi}{r} - 1 \right). \]  

Finally, substituting Eqs. (4.27) and (4.29) into Eq. (4.26) yields

\[ \Phi_\xi(u) = \Phi_\xi(0) + \frac{V_{\rho(\xi)}(\xi)}{\xi - \theta} \phi_1^\xi(u, \xi) + \frac{V_{\rho(\theta)}(\theta)}{\xi - \theta} \phi_2^\xi(u, \theta). \]  

where \( \Phi_\xi(0) \) is given by Eq. (4.30) and

\[ \phi_1^\xi(u, \xi) = \frac{h}{\xi} u + \lambda K_0 e^{\xi u} \int_u^\infty \tilde{F}_D(x) e^{-\xi x} dx - \frac{K_0}{\xi} (\rho \xi - r); \]

\[ \phi_2^\xi(u, \theta) = -\frac{h}{\theta} u + \lambda K_0 e^{\theta u} \int_0^u \tilde{F}_D(x) e^{-\theta x} dx + \frac{1}{\theta} \left( r \Phi_\rho(0) - \frac{h}{\theta} \right) (e^{\theta u} - 1). \]

### 4.3.2 Loss-Proportional Penalty

In this case, we have \( w(x) = K_1 x \), for \( x \geq 0 \), where \( K_1 > 0 \) is a constant. Accordingly, Eq. (4.5) becomes

\[ g(u) = h u + \lambda K_1 \int_u^\infty x f_D(x) dx, \quad u \geq 0, \]  

and the corresponding Laplace transform is given by

\[ \tilde{g}(z) = \frac{h}{z^2} + \lambda K_1 \left( \frac{\mu_D}{z} - \frac{1 - \tilde{f}_D(z)}{z} \right), \]  

where \( \mu_D = \mathbb{E}[D] \). Next, setting \( z = \xi \) in Eq. (4.33) and using \( \tilde{f}_D(\xi) \) as given by Eq. (4.8), we have

\[ \tilde{g}(\xi) = \frac{h}{\xi^2} + \lambda K_1 \left[ \frac{\mu_D}{\xi} - \frac{\rho}{\lambda \xi} + \frac{r}{\lambda \xi^2} \right]. \]
Now substituting Eq. (4.34) into Eq. (4.18) yields

$$\Phi_\xi(0) = \frac{h}{r\xi} + K_1 \left[ \frac{\lambda \mu - \rho + 1}{r} \xi \right].$$

(4.35)

Finally, substituting Eqs. (4.32) and (4.34) into Eq. (4.26) yields

$$\Phi_\xi(u) = \Phi_\xi(0) + \frac{V_\rho^\prime(\xi)}{\xi - \theta} \phi_1^p(u, \xi) + \frac{V_\rho^{\prime}(\theta)}{\xi - \theta} \phi_2^p(u, \theta),$$

(4.36)

where $\Phi_\xi(0)$ is given by Eq. (4.35) and

$$\phi_1^p(u, \xi) = \frac{h}{\xi} u + \lambda K_1 e^{\xi u} \int_x^\infty z f_D(z) e^{-\xi z} dz + 1 \left( \frac{h}{\xi} - r \Phi_\rho(0) \right);$$

$$\phi_2^p(u, \theta) = -\frac{h}{\theta} u + \lambda K_1 e^{\theta u} \int_0^u z f_D(z) e^{-\theta z} dz + 1 \left( r \Phi_\rho(0) - \frac{h}{\theta} \right)(e^{\theta u} - 1).$$

### 4.4 Computation of $\Phi_\xi(u)$ for Exponential Demand-Size Distributions

In this subsection, we derive the function $\Phi_\xi(u)$, subject to each penalty function, for the case of exponentially distributed demand sizes with rate $\beta > 0$. Thus,

$$f_D(x) = \beta e^{-\beta x}, \quad x \geq 0$$

(4.37)

and

$$\tilde{f}_D(z) = \frac{\beta}{\beta + z}, \quad z \geq 0.$$  

(4.38)

Substituting Eq. (4.38) into Eq. (4.6) yields

$$\psi(z) = \frac{\lambda \beta}{\beta + z} + \rho z - \lambda - r = \frac{(z - \theta)(z - \xi)}{V_\rho(z)},$$

(4.39)

where

$$V_\rho(z) = \frac{z + \beta}{\rho}.$$  

(4.40)

Hence, the two real roots of the equation $\psi(z) = 0$ are given by

$$\xi = \frac{\lambda + r - \rho \beta + \sqrt{(\lambda + r - \rho \beta)^2 + 4 r \rho \beta}}{2 \rho}\geq 0,$$

(4.41)

$$\theta = \frac{\lambda + r - \rho \beta - \sqrt{(\lambda + r - \rho \beta)^2 + 4 r \rho \beta}}{2 \rho}\leq 0.$$  

(4.42)
4.4.1 Constant Lost-Sales Penalty

Recall that in this case, \( w(x) = K_0 \), \( x > 0 \), so Eq. (4.31) can be written as

\[
\Phi_p(u) = a_0 + a_1 u + a_2 e^{\theta u},
\]

where

\[
a_0 = \frac{h}{r} \left( \frac{1}{\xi} + \frac{1}{\beta + \theta} \right),
\]

\[
a_1 = \frac{h}{r},
\]

\[
a_2 = \frac{\lambda K_0 \xi}{r(\beta + \xi)} - \frac{h}{r} \left( \frac{1}{\beta + \theta} \right)
\]

In Eq. (4.43), the initial inventory level, \( u \), appears in both a linear term and an exponential term. Since \( \theta < 0 \), it follows that when \( u \) is relatively small, the exponential term dominates the linear term, while for a relatively large \( u \), the opposite is true. A numerical study of \( \Phi_p(u) \) with exponential demand distribution is presented in Section 6. Finally, for the special case with \( u = 0 \), we have

\[
\Phi_p(0) = a_0 + a_2 = \frac{h}{r} \left( \frac{1}{\xi} + \frac{\lambda K_0 \xi}{\beta + \xi} \right),
\]

and a closed form expression for the optimal \( \rho^* \) is provided in Table 1.

4.4.2 Loss-Proportional Penalty

Recall that in this case, \( w(x) = K_1 \), \( x > 0 \), so Eq. (4.36) can be written as

\[
\Phi_p(u) = a_0 + a_4 e^{-\beta u} + a_5 e^{\theta u},
\]

where \( a_0 \) and \( a_4 \) are given by Eqs. (4.44) and (4.45) respectively, and

\[
a_4 = \frac{\lambda K_1}{\rho} \left[ \frac{1}{(\xi - \theta)(\beta + \theta)} - \frac{1}{\theta(\beta + \xi)} \right] - \frac{h}{r} \left( \frac{1}{\beta + \theta} \right);
\]

\[
a_5 = -\frac{\lambda K_1}{\rho(\beta + \xi)(\beta + \theta)}.
\]

In Eq. (4.47) the initial inventory level, \( u \), appears in a linear term and two distinct exponential terms, each with a negative exponent. It follows that when \( u \) is relatively small, the exponential terms dominate the linear term, while for a relatively large \( u \), the opposite is true. Finally, for the special case \( u = 0 \), we have
and a closed form expression for the optimal $\rho^*$ is provided in Table 2.

5. Optimization of the Replenishment Rate

In this section, we optimize the expected discounted cost function $\Phi_\rho(u)$ with respect to the replenishment rate, $\rho$, via an optimization of $\Phi_\xi(u)$ with respect to $\xi$. We first provide a general structural result in subsection 5.1 for an optimal replenishment rate, $\rho^*$ (admitting the possibility of multiple optimal replenishment rates), and then describe computational simplifications in subsection 5.2 for some selected demand-size distributions.

5.1 Optimal Replenishment Rate

Observe that the cost function $\Phi_\xi(u)$, given by Eq. (4.26), is expressed in terms of the two roots, $\theta$ and $\xi$. In the sequel, we shall express $\Phi_\xi(u)$ in terms of $\xi$ alone by expressing $\theta$ in terms of $\xi$. To this end, we set $z = 0$ in Eq. (4.23), and deduce the relation as follows by the fact that $\psi(0) = -r$ in light of Eq. (4.6),

$$\theta = -r V_\rho(0) / \xi.$$  

Substituting Eq. (5.1) into Eq. (4.26) then yields

$$\Phi_\xi(u) = \frac{\xi \bar{g}(\xi)}{\rho} + \frac{\xi V_\xi(\xi)}{\xi^2 + r V_\xi(0)} \left[ e^{\xi u} \int_{u}^{\infty} g(x) e^{-\xi x} \, dx - \bar{g}(\xi) \right]$$

$$+ \frac{\xi V_\xi \left(-r V_\xi(0) / \xi\right)}{\xi^2 + r V_\xi(0)} \left[ e^{-r V_\xi(0) u / \xi} \int_{0}^{u} g(x) e^{r V_\xi(0) x / \xi} \, dx + \frac{\xi^2 \bar{g}(\xi)}{r V_\xi(0)} \left( e^{-r V_\xi(0) u / \xi} - 1 \right) \right].$$  

5.1.1 The boundedness of $\Phi_\xi(u)$ guarantees the existence of a global minimizing point, $\xi^* = \arg\min_{\xi > 0} \{\Phi_\xi(u)\}$. However, the function $\Phi_\xi(u)$ is not convex in general. In fact, it is challenging to prove the uniqueness of the global minimizer, and this still remains an open problem.

In light of Theorem 3, a minimizer, $\xi^*$, can be computed in several ways. A straightforward, but relatively time-consuming method, is global search. However, when $\Phi_\xi(u)$ is convex, the availability of
the derivative $\frac{\partial}{\partial \xi} \Phi_\xi(u)$ allows us to apply the relatively fast Newton’s Method. The above discussion can be summarized as follows.

**Corollary 2**

Given $I(0) = u$, the optimal replenishment rates for $\Phi_\rho(u)$ are given by

$$\rho^* = \frac{r + \lambda \left[1 - \bar{f}_D(\xi^*)\right]}{\xi^*},$$

where $\xi^* = \arg\min_{\xi > 0} \{\Phi_\xi(u)\}$ and $\Phi_\xi(u)$ is given by Eq. (5.2).

**5.2 Optimal Replenishment Rate under Delayed Replenishment**

Suppose the system operates under delayed replenishment, that is, replenishment starts only after the first lost-sale occurrence. For example, suppose the system has an initial setup period during which replenishment is unavailable (e.g., a production facility which requires a setup time to gear up for production). Accordingly, minimizing the corresponding expected discounted cost, $\hat{\Phi}_\rho(u)$, over an infinite time horizon can be written as

$$\hat{\Phi}_\rho(u) = c_0(u) + d_0(u) \Phi_\rho(0).$$

From Eq. (5.4), it is readily seen that minimizing $\hat{\Phi}_\rho(u)$ with respect to $\rho$ is equivalent to minimizing $\Phi_\rho(0)$ with respect to $\rho$, since only the second term is a function of $\rho$. In the following two subsections, we treat the optimization of $\Phi_\rho(0)$ for the special cases of constant lost-sales penalty and loss-proportional penalty.

**5.2.1 Constant Lost-Sales Penalty**

Recall that in this case, $w(x) = K_0$, $x > 0$, where $K_0 > 0$ is a constant, and $\Phi_\xi(0)$ is given by Eq. (4.30). In view of Eq. (4.9), Eq. (4.30) can be rewritten as

$$\Phi_\xi(0) = \frac{h + \lambda \xi K_0 \left[1 - \bar{f}_D(\xi)\right]}{r \xi}.$$  

By Eq.(5.5), the optimal $\xi^*$ is given by

$$\xi^* = \arg\min_{\xi > 0} \left\{\frac{h}{\xi} - \lambda K_0 \bar{f}_D(\xi)\right\}. $$
Table 1 exhibits the optimal $\xi^*$, $\rho^*$ and $\Phi_{p^*}(0)$ with closed-form formulas, when available, for selected demand distributions; detailed derivations are given in Appendix B.

**Table 1. Optimal expected discounted costs subject to constant penalty under various demand distributions**

<table>
<thead>
<tr>
<th>Demand Distribution</th>
<th>$\xi^*$</th>
<th>$\rho^*$</th>
<th>$\Phi_{\rho^*}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D = d$, $d &gt; 0$</td>
<td>$\arg\min_{\xi &gt; 0} \left{ \frac{h}{\xi} - \lambda K_0 e^{-\xi d} \right}$</td>
<td>$\frac{r + \lambda \left( 1 - e^{-\xi^* d} \right)}{\xi^*}$</td>
<td>$\frac{h + K_0 \xi^* (\rho^* \xi - r)}{r \xi}$</td>
</tr>
<tr>
<td>$D \sim \text{Exp}(\beta)$, $\beta &gt; 0$</td>
<td>$\frac{\beta \sqrt{h}}{\sqrt{\lambda} \beta K_0 - \sqrt{h}}$, if $\beta \lambda K_0 &gt; h$</td>
<td>$\frac{\sqrt{\lambda} K_0 - \sqrt{h}}{\beta} + \frac{\lambda}{\sqrt{h} \sqrt{\lambda} \beta K_0}$, if $\beta \lambda K_0 &gt; h$</td>
<td>$\frac{2 \sqrt{\lambda} \beta K_0 - h}{\beta}$, if $\beta \lambda K_0 &gt; h$</td>
</tr>
<tr>
<td>$D \sim \text{U}(a, b)$, $0 \leq a &lt; b$</td>
<td>$\arg\min_{\xi &gt; 0} \left{ \frac{h}{\xi} - \lambda K_0 \frac{e^{-\xi a} - e^{-\xi b}}{(b-a) \xi} \right}$</td>
<td>$\frac{r + \lambda \left( 1 - e^{-\xi^* a} - e^{-\xi^* b} \right)}{(b-a) \xi^*}$</td>
<td>$\frac{h + K_0 \xi^* (\rho^* \xi - r)}{r \xi}$</td>
</tr>
<tr>
<td>$D \sim \Gamma(\alpha, \beta)$, $\alpha, \beta &gt; 0$</td>
<td>$\arg\min_{\xi &gt; 0} \left{ \frac{h}{\xi} - \lambda K_0 (1 + \xi / \beta)^{-\alpha} \right}$</td>
<td>$\frac{r + \lambda \left( 1 - 1 + \xi^* / \beta \right)^{-\alpha}}{\xi^*}$</td>
<td>$\frac{h + K_0 \xi^* (\rho^* \xi - r)}{r \xi}$</td>
</tr>
</tbody>
</table>

In the table above and elsewhere, the $\arg\min$ operation corresponds to a search for the optimal $\xi^*$, whenever a closed form formula for it is either unavailable or not readily available. In particular, for an exponential demand distribution, the optimal solution is available in closed form, and the condition $\beta \lambda K_0 > h$ ensures a positive optimal replenishment rate; otherwise, it is optimal to have zero replenishment and bear the repeated penalty costs (a degenerate case).

### 5.2.2 Loss-Proportional Penalty

Recall that in this case, $\omega(x) = K_1 x$, for $x > 0$, where $K_1 > 0$ is constant, and $\Phi_{\xi}(0)$ is given by Eq. (4.35). In view of Eq. (4.9), Eq. (4.35) can be rewritten as

$$\Phi_{\xi}(0) = \frac{1}{r} \left[ \frac{h}{\xi} - \lambda K_1 \frac{1 - \tilde{f}_D(\xi)}{\xi} \right] + \frac{\lambda K_1 \mu_D}{r},$$

where $\mu_D = \mathbb{E}[D]$. Consequently, by Eq. (5.7), the optimal $\xi^*$ is given by

$$\xi^* = \arg\min_{\xi > 0} \left\{ \frac{h}{\xi} - \lambda K_1 \frac{1 - \tilde{f}_D(\xi)}{\xi} \right\}.$$
Table 2 exhibits the optimal $\xi^*$, $\rho^*$ and $\Phi_{P^*}(0)$ with closed-form formulas, when available, for selected demand distributions; detailed derivations are given in Appendix B.

Table 2. Optimal expected discounted costs subject to loss-proportional penalty under various demand distributions

<table>
<thead>
<tr>
<th>Distribution</th>
<th>$\xi^*$</th>
<th>$\rho^*$</th>
<th>$\Phi_{P^*}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D = d$</td>
<td>$d &gt; 0$</td>
<td>$\arg\min_{\xi &gt; 0} \left{ h - \lambda K_1 \frac{1 - e^{-\xi d}}{\xi} \right}$</td>
<td>$r + \lambda \frac{1 - e^{-\xi d}}{\xi}$</td>
</tr>
<tr>
<td>$D \sim \text{Exp}(\beta)$</td>
<td>$\beta &gt; 0$</td>
<td>$\begin{cases} \frac{\beta h}{\lambda K_1 - \sqrt{h}} &amp; \text{if } \lambda K_1 &gt; h \ \frac{\lambda K_1 - \sqrt{h}}{\beta} &amp; \text{otherwise} \end{cases}$</td>
<td>$\begin{cases} \frac{\lambda K_1 - \sqrt{h}}{\beta} + \frac{\lambda}{\sqrt{h}} &amp; \text{if } \lambda K_1 &gt; h \ 0 &amp; \text{otherwise} \end{cases}$</td>
</tr>
<tr>
<td>$D \sim U(a, b)$</td>
<td>$0 \leq a &lt; b$</td>
<td>$\arg\min_{\xi &gt; 0} \left{ h - \lambda K_1 \frac{1 - e^{-\alpha \xi a} - e^{-\beta \xi b}}{\xi} \right}$</td>
<td>$\begin{cases} \frac{\lambda}{\xi} \left[ \frac{1 - e^{-a \xi}}{\xi} - \frac{1 - e^{-b \xi}}{\xi} \right] &amp; \text{if } \lambda K_1 &gt; h \ 0 &amp; \text{otherwise} \end{cases}$</td>
</tr>
<tr>
<td>$D \sim \Gamma(\alpha, \beta)$</td>
<td>$\alpha, \beta &gt; 0$</td>
<td>$\arg\min_{\xi &gt; 0} \left{ h - \lambda K_1 \frac{1 - (1 + \xi / \beta)\alpha}{\xi} \right}$</td>
<td>$\begin{cases} \frac{\lambda}{\xi} \left[ 1 - 1 + \xi / \beta \right]^{-\alpha} &amp; \text{if } \lambda K_1 &gt; h \ 0 &amp; \text{otherwise} \end{cases}$</td>
</tr>
</tbody>
</table>

Again, for an exponential demand distribution, the optimal solution is available in closed form, and the condition $\lambda K_1 > h$ ensures a positive optimal replenishment rate; otherwise, it is optimal to have zero replenishment and bear the repeated penalty costs (a degenerate case).

6. Numerical Study

This section contains two numerical studies of production-inventory systems with selected demand-size distributions, subject to constant lost-sales penalty. Both studies were conducted with the following common parameters: $\lambda = 1$, $h = 1$, $K_0 = 100$, and $r = 0.1$. Recall that only the exponential demand-size distribution gives rise to a closed-form optimal solution; in all other cases, optimal solutions were obtained by a simple search.

6.1 Optimal Numerical Solutions for Zero Initial Inventories

In this study we compute and compare the numerical values of $\Phi_{P^*}(0)$ for increasing mean demand sizes, and under the following demand-size distributions: constant, exponential, uniform and Gamma. Table 3
displays the optimal \( \rho^* \) and \( \xi^* \) as functions of the mean demand, \( \mathbb{E}[D] = 1/\beta \), for the four aforementioned demand-size distributions.

Table 3. Optimal \( \Phi_{\rho^*}(0) \) for selected demand-size distributions

<table>
<thead>
<tr>
<th>( \lambda \mathbb{E}[\eta] )</th>
<th>( D = 1/\beta )</th>
<th>( D \sim \text{Exp} \beta )</th>
<th>( D \sim U(0,2/\beta) )</th>
<th>( D \sim \Gamma(4,1/(4\beta)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho^* )</td>
<td>( \Phi_{\rho^*}(0) )</td>
<td>( \rho^* )</td>
<td>( \Phi_{\rho^*}(0) )</td>
<td>( \rho^* )</td>
</tr>
<tr>
<td>0.05</td>
<td>0.27</td>
<td>44.47</td>
<td>0.27</td>
<td>44.22</td>
</tr>
<tr>
<td>0.30</td>
<td>0.82</td>
<td>108.03</td>
<td>0.80</td>
<td>106.54</td>
</tr>
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<td>221.40</td>
<td>2.16</td>
<td>215.04</td>
</tr>
<tr>
<td>3.30</td>
<td>4.63</td>
<td>346.27</td>
<td>4.20</td>
<td>330.32</td>
</tr>
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<td>5.30</td>
<td>6.66</td>
<td>432.79</td>
<td>5.87</td>
<td>407.43</td>
</tr>
<tr>
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<td>7.64</td>
<td>468.98</td>
<td>6.61</td>
<td>439.00</td>
</tr>
<tr>
<td>7.30</td>
<td>8.59</td>
<td>501.97</td>
<td>7.28</td>
<td>467.37</td>
</tr>
<tr>
<td>8.30</td>
<td>9.47</td>
<td>532.35</td>
<td>7.94</td>
<td>493.19</td>
</tr>
<tr>
<td>9.30</td>
<td>10.36</td>
<td>560.62</td>
<td>8.60</td>
<td>516.92</td>
</tr>
<tr>
<td>10.00</td>
<td>10.95</td>
<td>579.30</td>
<td>9.02</td>
<td>532.46</td>
</tr>
<tr>
<td>15.00</td>
<td>14.86</td>
<td>693.46</td>
<td>11.58</td>
<td>624.60</td>
</tr>
<tr>
<td>20.00</td>
<td>18.37</td>
<td>784.52</td>
<td>13.61</td>
<td>694.43</td>
</tr>
<tr>
<td>25.00</td>
<td>21.23</td>
<td>860.50</td>
<td>15.00</td>
<td>750.00</td>
</tr>
<tr>
<td>30.00</td>
<td>23.87</td>
<td>925.43</td>
<td>16.14</td>
<td>795.45</td>
</tr>
</tbody>
</table>

From Table 3, it can be seen that the respective \( \rho^* \) and the corresponding \( \Phi_{\rho^*}(0) \) increase in this order of distributions: exponential, uniform, Gamma and constant. Note that as the average demand increases, \( \rho^* \) and \( \Phi_{\rho^*}(0) \) increase as expected. Furthermore, for each selected demand-size distribution, we observe that \( \rho^* > \lambda \mathbb{E}[D] \) for \( \mathbb{E}[D] < 7 \) (case 1), whereas \( \rho^* < \lambda \mathbb{E}[D] \) for \( \mathbb{E}[D] > 15 \) (case 2). One possible explanation for these observations can be derived by examining the optimal production attendant to a demand rate, noting that discounting implies that the objective function is driven by the behavior of the system in an initial interval (starting at 0). Thus, in case 1, the optimal production rate would be driven above the demand rate, because otherwise, the inventory level would stay low, thereby incurring excessive penalty costs. Conversely, in case 2, the optimal production rate would be driven below the demand rate, because otherwise, the inventory level would stay high, thereby incurring excessive holding costs.

The above observation can be explained analytically for the case of exponential demand, \( D \sim \text{Exp}(\beta) \), with the aid of the explicit solution given in Table 1. In particular, assuming that \( \beta \lambda K_0 > h \) holds, the
optimal production rate is given in closed form by
\[ \rho^* = \frac{\sqrt{\lambda \beta K_0} - \sqrt{h}}{\beta} \left( \frac{r}{\sqrt{h}} + \frac{\lambda}{\sqrt{\lambda \beta K_0}} \right), \]
whence the difference \( \rho^* - \lambda \mathbb{E}[D] \) is given by
\[ \rho^* - \frac{\lambda}{\beta} = \sqrt{\frac{\lambda}{\beta}} \left( \frac{r\sqrt{K_0}}{\sqrt{h}} - \frac{\sqrt{h}}{\beta \sqrt{K_0}} \right) = \frac{r}{\beta}. \] (6.1)

Thus, for sufficiently large \( \beta \), i.e., sufficiently small \( \mathbb{E}[D] \), the right-hand side of Eq. (6.1) becomes positive, implying \( \rho^* > \lambda \mathbb{E}[D] \). Conversely, for sufficiently small but positive \( \beta \), i.e., sufficiently large \( \mathbb{E}[D] \), the right-hand side of Eq. (6.1) becomes negative, implying \( \rho^* < \lambda \mathbb{E}[D] \). Furthermore, by Eq. (6.1), the cut-off point for \( \rho^* = \lambda \mathbb{E}[D] \) is identified by \( \lambda = r^2 \beta \left( r \beta \sqrt{\frac{K_0}{h}} - \sqrt{\frac{h}{K_0}} \right)^{-2}. \)

In this numerical study with the selected parameters and \( \lambda = 1 \), it shows that the cut-off mean demand is \( \mathbb{E}[D] = 13.7 \). That is, \( \rho^* > \lambda \mathbb{E}[D] \) for \( \mathbb{E}[D] < 13.7 \), whereas \( \rho^* < \lambda \mathbb{E}[D] \) for \( \mathbb{E}[D] > 13.7 \), which explains our observations.

In the next numerical study, we use the same parameters as before, but fix \( \mathbb{E}[D] = 10 \) and vary the value of the coefficient of variation \( c_v \) (ratio of standard deviation to mean) of the random demand. For each selected value of \( c_v \), we chose the parameters of Uniform and Gamma distributions for \( D \) so as to keep the corresponding values of \( c_v \) the same. Table 4 displays several such parameter values and the corresponding \( \rho^* \), \( \xi^* \) and \( \Phi_{\rho^*}(0) \) for selected \( c_v \) ranging between \( 1/\sqrt{3} \) to \( 1/4 \).

<table>
<thead>
<tr>
<th>( c_v )</th>
<th>( D \sim U(a, b) )</th>
<th>( D \sim \Gamma(\alpha, \beta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/\sqrt{3}</td>
<td>0.041</td>
<td>10.165</td>
</tr>
<tr>
<td>1/2</td>
<td>0.040</td>
<td>10.363</td>
</tr>
<tr>
<td>1/3</td>
<td>0.039</td>
<td>10.676</td>
</tr>
<tr>
<td>1/4</td>
<td>0.039</td>
<td>10.783</td>
</tr>
</tbody>
</table>
From Table 4, it can be seen that the respective $\rho^*$ and the corresponding $\Phi_{\rho^*}(0)$ increase in $c_v$. For each case, it is shown $\rho^* > \lambda \mathbb{E}[D] = 10$. Note that although the variation in $\xi^*$, $\rho^*$ and $\Phi_{\rho^*}(0)$ is not significant compared with the change in $c_v$, it reveals to what extent the optimal rates depend on more than the first two moments of the demand distribution. Furthermore, observe that when the demand distribution is $\Gamma(\alpha, \beta)$, we have larger $\rho^*$ and $\Phi_{\rho^*}(0)$ than their counterparts for demand distribution $U(a, b)$. This phenomenon can be explained by the longer tail of the $\Gamma(\alpha, \beta)$ distribution [cf. De Kok (1987)].

6.2 Optimal Numerical Solutions for Arbitrary Initial Inventory Levels

In this study we compute and compare the numerical values of $\xi^*$, $\rho^*$ and $\Phi_{\rho^*}(u)$ for selected demand-size distributions (constant, exponential and uniform) with increasing initial inventory levels and for low and high average demands. Table 5 and Table 6 display $\rho^*$, $\xi^*$ and $\Phi_{\rho^*}(u)$ for sample low and high demands as functions of the initial inventory level, $I(0) = u$.

<table>
<thead>
<tr>
<th>$I(0) = u$</th>
<th>$D = 1/\beta$</th>
<th>$D \sim \text{Exp}(\beta)$</th>
<th>$D \sim U(0, 2/\beta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi^*$</td>
<td>$\rho^*$</td>
<td>$\Phi_{\rho^*}(u)$</td>
<td>$\xi^*$</td>
</tr>
<tr>
<td>0</td>
<td>0.076</td>
<td>3.169</td>
<td>272.590</td>
</tr>
<tr>
<td>5</td>
<td>0.130</td>
<td>2.530</td>
<td>157.450</td>
</tr>
<tr>
<td>10</td>
<td>0.194</td>
<td>2.173</td>
<td>150.260</td>
</tr>
<tr>
<td>15</td>
<td>0.301</td>
<td>1.835</td>
<td>175.720</td>
</tr>
<tr>
<td>20</td>
<td>0.372</td>
<td>1.679</td>
<td>197.900</td>
</tr>
<tr>
<td>25</td>
<td>0.513</td>
<td>1.445</td>
<td>229.660</td>
</tr>
<tr>
<td>30</td>
<td>0.547</td>
<td>1.399</td>
<td>262.250</td>
</tr>
<tr>
<td>35</td>
<td>0.717</td>
<td>1.202</td>
<td>295.100</td>
</tr>
<tr>
<td>40</td>
<td>1.160</td>
<td>0.864</td>
<td>330.040</td>
</tr>
<tr>
<td>45</td>
<td>1.714</td>
<td>0.623</td>
<td>364.120</td>
</tr>
<tr>
<td>50</td>
<td>7.598</td>
<td>0.145</td>
<td>392.380</td>
</tr>
</tbody>
</table>
Table 6. Optimal quantities for selected demand-size distributions under a high demand with \( \lambda \mathbb{E}[D] = 1/\beta = 20 \)

<table>
<thead>
<tr>
<th>( I(0) = u )</th>
<th>( D = 1/\beta )</th>
<th>( D \sim \text{Exp}(\beta) )</th>
<th>( D \sim U(0, 2/\beta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \xi^* )</td>
<td>( \rho^* )</td>
<td>( \Phi_{\rho^*}(u) )</td>
</tr>
<tr>
<td>0</td>
<td>0.030</td>
<td>18.299</td>
<td>784.500</td>
</tr>
<tr>
<td>5</td>
<td>0.031</td>
<td>18.251</td>
<td>774.650</td>
</tr>
<tr>
<td>10</td>
<td>0.032</td>
<td>17.943</td>
<td>759.150</td>
</tr>
<tr>
<td>15</td>
<td>0.032</td>
<td>17.943</td>
<td>736.390</td>
</tr>
<tr>
<td>20</td>
<td>0.032</td>
<td>17.943</td>
<td>705.290</td>
</tr>
<tr>
<td>25</td>
<td>0.035</td>
<td>17.324</td>
<td>705.480</td>
</tr>
<tr>
<td>30</td>
<td>0.036</td>
<td>16.994</td>
<td>707.410</td>
</tr>
<tr>
<td>35</td>
<td>0.036</td>
<td>16.994</td>
<td>707.410</td>
</tr>
<tr>
<td>40</td>
<td>0.040</td>
<td>16.303</td>
<td>706.100</td>
</tr>
<tr>
<td>45</td>
<td>0.040</td>
<td>16.303</td>
<td>709.320</td>
</tr>
<tr>
<td>50</td>
<td>0.042</td>
<td>15.929</td>
<td>715.630</td>
</tr>
</tbody>
</table>

Table 5 and Table 6 above reveal similar behavior patterns of \( \rho^* \) and \( \Phi_{\rho^*}(u) \), as functions of \( I(0) = u \).

For each demand-size distribution in each table, \( \rho^* \) decreases as \( I(0) = u \) increases, while the corresponding \( \Phi_{\rho^*}(u) \) first decreases and then increases in \( u \). Also, for any given initial inventory level, \( \Phi_{\rho^*}(u) \) increases as the average demand, \( \lambda \mathbb{E}[D] \), increases. Moreover, for each demand-size distribution, the optimal initial inventory level \( u^* = \arg\min_{u \geq 0} \{ \Phi_{\rho^*}(u) \} \) increases in the average demand.

For example, \( u^* = 10 \) in Table 5 and \( u^* \in [25, 35] \) in Table 6 are cases in point. In other words, a larger demand size is more beneficial when the initial inventory level is high. This is intuitive since higher demand is more likely to deplete the inventory quickly, which reduces the holding cost incurred due to a high initial inventory level. We also observe that in each of these tables, \( \rho^* \) decreases in the demand-size distribution in this order: constant, uniform and exponential; this, however, does not generally hold for \( \Phi_{\rho^*}(u) \).

7. Time-Average Cost and Optimization

The long-run time-average (undiscounted) cost can be treated similarly to its discounted counterpart. In this case, we need to assume the stability condition, \( \rho < \lambda \mathbb{E}[D] \) (or equivalently \( \mathbb{E}[\tau_1] < \infty \));
otherwise the long-run time-average cost is infinite. In the sequel, we derive the time-average cost directly from the results for the discounted cost by taking limits as \( r \downarrow 0 \) and using the renewal reward theorem (cf. Ross (1996)).

For \( r = 0 \), the *Lundberg’s fundamental equation* of Eq. (4.7) becomes

\[
\lambda \hat{f}_D(z) + \rho z - \lambda = 0.
\]  
(7.1)

Under the stability condition \( \rho < \lambda \mathbb{E}[D] \), it follows that Eq. (7.1) has two real roots: \( \theta_0 = 0 \) and \( \xi_0 > 0 \). Next, by Eqs. (3.1) and (7.1), one has

\[
\rho = \lambda \tilde{F}_D(\xi_0),
\]  
(7.2)

which implies that \( \rho \) and \( \xi_0 \) are connected by a bijection.

In view of Eq. (7.2), the stability condition \( \rho < \lambda \mathbb{E}[D] \) can be written as \( \tilde{F}_D(\xi_0) < \mathbb{E}[D] \). Since \( \tilde{F}_D(z) \) is monotonically decreasing in \( z \) and \( \tilde{F}_D(z) = \mathbb{E}[D] \) at \( z = 0 \), the stability condition \( \rho < \lambda \mathbb{E}[D] \) in the PRR space, can be equivalently expressed as \( \xi_0 > 0 \) in the LPR space.

Under the stability condition \( \xi_0 > 0 \) in the LPR space (i.e., \( \rho < \lambda \mathbb{E}[D] \) in the PRR space), the inventory process over time intervals of the form \( (\tau_i, \tau_{i+1}] \) is a renewal process, and the corresponding cost process can be regarded as a renewal reward process, with finite expectations of inter-renewal times and cycle rewards. Consequently, by Theorem 3.6.1 in Ross (1996), the long-run time-average cost is independent of the initial inventory level, and can be represented by

\[
\bar{c}_\rho = \frac{c_\rho(0)}{\mathbb{E}[\tau_1 \mid I(0) = 0]},
\]  
(7.3)

where \( c_\rho(0) \) is given by Eq. (3.6) with \( r = 0 \) and \( u = 0 \). Next, we use \( \xi_0 \) as the decision variable to derive \( \bar{c}_\rho \) in closed form and analyze its optimal solution. Following our notational fashion, we let \( \bar{c}_\rho \) and \( \bar{c}_{\xi_0} \) denote the time-average cost function of Eq. (7.3) in terms of \( \rho \) and \( \xi_0 \), respectively. To derive the time average cost, we use the fact that \( d_\rho(u) \), defined by Eq. (4.2), can be interpreted as the moment generating function of \( \tau_1 \) at \( -r \); cf. Karr (1993). Consequently, by Eq. (4.16), the expected time to the first shortage conditioned on the initial inventory level can be written as

\[
\mathbb{E}[\tau_1 \mid I(0) = 0] = \frac{1}{\rho \xi_0} = \frac{1}{\lambda - \lambda \hat{f}_D(\xi_0)}.
\]  
(7.4)
Note also that Eq. (7.4) can be interpreted as the expected value of the time to ruin in the classical insurance model [cf. Gerber and Shiu (1998)], conditioned on a zero initial surplus level. The following theorem provides a closed form expression for the time-average cost.

**Theorem 4**
Under the stability condition $\rho < \lambda \mathbb{E}[D]$, the time-average cost is given by

$$
\overline{C}_{\xi_0} = \xi_0 \overline{g}(\xi_0),
$$

where $\xi_0 > 0$. $\square$

We mention that De Kok (1987) studies a corresponding production-inventory system, but with two switchable production rates, and provides an approximation for the time-average of inventory holding and switching cost (cf. Eq. (2.10) therein). Actually, our production-inventory model can be treated as the aforementioned model, provided the two production rates as the equal and there is no switching cost. In this case, the approximated carrying cost in De Kok (1987) is exactly equivalent to the time-average holding cost in Eq. (7.5). However, the approximation proposed by De Kok (1987) only accounts for the holding cost but ignores the lost-sale penalty component.

In view of Theorem 4, minimizing $\overline{C}_\rho$ with respect to $\rho$ is equivalent to minimizing $\overline{C}_{\xi_0}$ with respect to the positive variable $\xi_0$. To this end, we first optimize $\overline{C}_{\xi_0} = \xi_0 \overline{g}(\xi_0)$ in the LPR space to find the optimal $\xi_0^*$, and then compute the corresponding optimal $\rho^*$ in the PRR space. The following corollary provides a general structural result for the optimal replenishment rate, $\rho^*$.

**Corollary 3**
The optimal replenishment rate for the time-average cost $\overline{C}_\rho$ under the stability condition $\rho < \lambda \mathbb{E}[D]$ is given by

$$
\rho^* = \lambda \overline{F}_D(\xi_0^*),
$$

where

$$
\xi_0^* = \arg\min_{\xi_0 > 0} \{ \xi_0 \overline{g}(\xi_0) \}.
$$

$\square$
8. Further Extensions

The research presented in this paper can be extended in several directions. First, the methodology can be extended to include in the objective function a variable production cost modeled as a nonnegative and increasing function $a(\rho)$ of the replenishment rate. In this case, the expected discounted production cost is

$$v_\rho = \int_0^\infty a(\rho) e^{-rt} dt = \frac{a(\rho)}{r}. \quad (8.1)$$

By Eq. (4.9), we can rewrite $v_\rho$ in Eq. (8.1) as

$$v_\xi = \frac{a\left(\frac{r}{\xi} + \lambda \tilde{F}_D(\xi) \right)}{r}, \quad (8.2)$$

where $v_\rho$ and $v_\xi$ denote the same cost function, but of $\rho$ and $\xi$, respectively. Finally, we can express the total expected discounted cost function as $\Phi_\xi(u) + v_\xi$, where $\Phi_\xi(u)$ is given by Eq. (5.2) and $v_\xi$ by Eq. (8.2). This closed form of the objective function allows one to compute the optimal $\xi^*$ directly, from which the optimal replenishment rate $\rho^*$ can be recovered via Eq. (5.3).

For the case of time-average cost, adding the production cost $a(\rho)$ to Eq. (7.5) yields the total cost function representation

$$a(\rho) + \xi_0 \tilde{g}(\xi_0) = a\left(\lambda \tilde{F}_D(\xi_0) + \xi_0 \tilde{g}(\xi_0)\right). \quad (8.3)$$

by virtue of Eq. (7.2). The above closed form expression allows one to compute the optimal $\xi_0^*$ directly. The requisite optimal replenishment rate $\rho^*$ can then be obtained from Eq. (7.6).

Second, we point out that the results of this paper can be applied to cost optimization (discounted or time-average) subject to a given service-level constraint, e.g., a fill rate $\pi$, defined as the percentage of demand arrivals that are immediately satisfied in full from inventory on hand. Let the lost-sales rate be denoted by $\bar{\pi} = 1 - \pi$. Then, $\bar{\pi} = \lim_{t \to \infty} N_B(t) / N_A(t)$, where $N_A(t)$ and $N_B(t)$ denote the number of demand arrivals and lost-sale occurrences, respectively, in the interval $(0, t]$. The lost-sales rate, $\bar{\pi}$, can be alternatively represented as [cf. Ross (1996), Theorem 3.4.4]
\[ \bar{\pi} = \frac{\mathbb{E}[T_1]}{\mathbb{E}[T_1 \mid I(0) = 0]} . \]  

(8.4)

Substituting \( \mathbb{E}[T_1] = 1/\lambda \) and Eq. (7.4) into Eq. (8.4) yields

\[ \bar{\pi} = \frac{\beta \xi_0}{\lambda} = 1 - \tilde{f}_D(\xi_0) . \]  

(8.5)

Consequently, we have the following representation for the fill rate

\[ \pi = \tilde{f}_D(\xi_0) . \]  

(8.6)

For optimization problems with objective functions of expected discounted cost or long-run time-average cost, constrained by a given minimal fill rate, \( 0 < \pi' \leq 1 \), one can apply Eq. (8.6) to compute the critical value \( \xi' \) such that \( \tilde{f}_D(\xi') = \pi' \). It follows that the cost optimization problem (e.g., the time average cost studied in Section 7) with a constrained fill rate, \( \pi' \), can be solved by a search in the LPR space, restricted to the interval \( 0 < \xi_0 \leq \xi' \), in lieu of the original search space, \( \xi_0 > 0 \).

9. Conclusions and Future Research

This paper investigated a continuous-review single-product production-inventory system with a constant replenishment rate, compound Poisson demands and lost-sales. Two objective functions that represent metrics of operational costs were investigated: (1) the sum of the expected discounted inventory holding costs and the lost-sales penalties, over an infinite time horizon, given an initial inventory level; and (2) the long-run time-average of the same costs. A bijection between the PRR space and LPR space was established to facilitate optimization. For any initial inventory level, a closed form expression was derived for the expected discounted cost, given an initial inventory level, in terms of an LPR variable. The resultant cost function was then readily optimized in the LPR space, and the requisite optimal value of the replenishment rate was recovered via the aforementioned bijection. In addition, the time-average cost was also derived in closed form under a stability condition, and an optimization methodology similar to the one used for the expected discounted cost, was applied to optimize the requisite time-average cost.

Additional work in this area may include the following. First, for the general model (with general cost functions and general demand distributions), one might admit multiple optimal replenishment rates, though it is likely that a single optimal replenishment rate is unique under fairly general conditions. The type of conditions necessary to ensure uniqueness is a future research topic. Second, one might introduce inventory capacity constraints (e.g., base stock level), such that replenishment is suspended or shut down when the inventory level reaches or is at capacity. Third, it is of interest to investigate similar production-inventory systems with discrete replenishment, that is, where replenishment orders are triggered by
demand arrivals that drop the inventory level below some prescribed base stock level. Finally, regarding the discrete-time version of these problems, we note that the integro-differential equation obtained in Lemma 1 is no longer valid, since its derivation is based on time continuity. Therefore, a different approach which utilizes Markov chain and/or renewal theory might be employed to treat the corresponding discrete-time models.

Acknowledgments
We are indebted to the Area Editor, the Associate Editor and three anonymous referees for many constructive comments and useful suggestions.

References
Appendix A

A.1 Proof of Lemma 1

For any given initial inventory level \( u \geq 0 \), consider a time interval \((0, s)\), \( s > 0 \). We have the following two disjoint events and the corresponding conditional expected discounted cost functions on those events:

1. On the event \( \{ A_1 > s \} \), the corresponding conditional expected discounted cost is

\[
E[C_\rho(\tau_1)1_{\{A_1 > s\}} \mid I(0) = u] = \int_s^\infty \lambda e^{-\lambda t} \left[ h \int_0^s (u + \rho z) e^{-\rho z} dz + c_\rho(u + \rho s) e^{-\rho s} \right] dt
\]

\[
= e^{-\lambda s} \left[ h \int_0^s (u + \rho z) e^{-\rho z} dz + c_\rho(u + \rho s) e^{-\rho s} \right].
\]

Here, the first term in the integral above is the discounted holding cost over \((0, s)\), the second is the discounted residual cost over \((s, \tau_1)\), and we use the relation \( \{ A_1 > s \} \subset \{ s \leq \tau_1 \} \).

2. On the event \( \{ A_1 \leq s \} \), the corresponding conditional expected discounted cost can be expressed as

\[
E[C_\rho(\tau_1)1_{\{A_1 \leq s\}} \mid I(0) = u] = \int_0^s \lambda e^{-\lambda t} M(u, t) dt,
\]

where \( M(u, t) = E[C_\rho(\tau_1) \mid A_1 = t, I(0) = u] \) is given by

\[
M(u, t) = h \int_0^t (u + \rho z) e^{-\rho z} dz + e^{-\rho t} \int_0^{u+\rho t} f_D(x) c_\rho(u + \rho t - x) dx
\]

\[
+ e^{-\rho t} \int_{u+\rho t}^\infty f_D(x) w(x - (u + \rho t)) dx.
\]

Next, adding Eqs. (A.1) and (A.2) yields

\[
c_\rho(u) = h e^{-\lambda s} \int_0^s (u + \rho z) e^{-\rho z} dz + h \int_0^s e^{-(\lambda + \rho) t} \int_0^t (u + \rho z) e^{-\rho z} dz dt
\]

\[
+ c_\rho(u + \rho s) e^{-(\lambda + \rho) s} + \int_0^s \int_0^{u+\rho t} f_D(x) c_\rho(u + \rho t - x) dx dt
\]

\[
+ \int_0^s \int_{u+\rho t}^\infty f_D(x) w(x - (u + \rho t)) dx dt.
\]

- 33 -
In the following, we shall prove the continuity and differentiability of $c_p(u)$ for $u \geq 0$. To prove continuity, let $s \downarrow 0$ in Eq. (A.4), and observe that $c_p(u) = \lim_{s \downarrow 0} c_p(u + \rho s)$ since all the integrals on the right hand side vanish.

We proceed to prove differentiability by definition. By Eq. (A.4), one has

$$c_p(u + \rho s) - c_p(u) = \frac{\rho s}{\rho s}
- \frac{1}{\rho s} h e^{-\lambda s} \int_0^s (u + \rho z) e^{-r z} dz + h \int_0^s \lambda e^{-\lambda t} \int_0^t (u + \rho z) e^{-r z} dz dt
\]

$$+ \frac{1}{\rho s} c_p(u + \rho s) \left[1 - e^{-(\lambda + r) s}\right] - \frac{1}{\rho s} \int_0^s \lambda e^{-(\lambda + r) t} \int_0^{u + \rho t} f_d(x) c_p(u + \rho t - x) dx dt
\]

$$- \frac{1}{\rho s} \int_0^s \lambda e^{-(\lambda + r) t} \int_{u + \rho t}^\infty f_d(x) w(x - (u + \rho t)) dx dt$$

The equation above shows that $c_p(u)$ is right differentiable by taking the limit $\rho s \downarrow 0$ (equivalently, as $s \downarrow 0$) and the right derivative is

$$\frac{\partial^+}{\partial u} c_p(u) = - \frac{h u}{\rho} + \frac{\lambda + r}{\rho} c_p(u) - \frac{\lambda}{\rho} \int_0^u f_d(x) c_p(u - x) dx dt$$

$$- \frac{\lambda}{\rho} \int_u^\infty f_d(x) w(x - u) dx dt \quad (A.5)$$

For $0 < s < u / \rho$, replacing $u$ with $u - \rho s$ in Eq. (A.4) yields

$$c_p(u - \rho s) = h e^{-\lambda s} \int_0^u (u - \rho s + \rho z) e^{-r z} dz + h \int_0^u \lambda e^{-\lambda t} \int_0^t (u - \rho s + \rho z) e^{-r z} dz dt
\]

$$+ c_p(u) e^{-(\lambda + r) s} + \int_0^u \lambda e^{-(\lambda + r) t} \int_0^{u - \rho s + \rho t} f_d(x) c_p(u - \rho s + \rho t - x) dx dt
\]

$$+ \int_0^u \lambda e^{-(\lambda + r) t} \int_{u - \rho s + \rho t}^\infty f_d(x) w(x - (u - \rho s + \rho t)) dx dt$$

From the equation above, one further has
\[ \frac{c_\rho(u - \rho s) - c_\rho(u)}{-\rho s} = \]
\[ = -\frac{h}{\rho s} e^{-\lambda s} \int_0^s (u - \rho s + \rho z) e^{-r z} dz - \frac{h}{\rho s} \int_0^s \lambda e^{-\lambda t} \int_0^t (u - \rho s + \rho z) e^{-r z} dz dt \]
\[ + \frac{1}{\rho s} c_\rho(u) \left[ 1 - e^{-\lambda t} \right] - \frac{1}{\rho s} \int_0^s \lambda e^{-\lambda t} \int_0^{u - \rho s + \rho t} f_D(x) c_\rho(u - \rho s + \rho t - x) dx dt \]
\[ - \frac{1}{\rho s} \int_0^s \lambda e^{-\lambda t} \int_{u - \rho s + \rho t}^\infty f_D(x) w(x - (u - \rho s + \rho t)) dx dt. \]

In a similar vein, the equation above implies that \( c_\rho(u) \) is left differentiable by taking the limit \( \rho s \downarrow 0 \) (equivalently, as \( s \downarrow 0 \)), which yields the same expression as that of Eq. (A.5). Thus, \( c_\rho(u) \) is differentiable for \( u \geq 0 \) (its derivative at \( u = 0 \) is given by the right derivative).

Finally, we conclude that Eq. (A.5) holds for both the left and right derivatives for \( u > 0 \), and the remainder of the proof of Eq. (4.4) uses Eq. (4.5) and simple rearrangement of terms. \( \square \)

### A.2 Proof of Lemma 2

To prove part (a), note that by Eq. (4.8),
\[ \rho = \frac{r}{\xi} + \frac{\lambda}{\xi} \left[ 1 - \bar{f}_D(\xi) \right]. \tag{A.6} \]
Eq. (4.9) now follows from the above equation with the aid of Eq. (3.1). Furthermore, Eq. (4.9) defines implicitly a mapping \( \rho \rightarrow \xi \) from the PRR space to the LPR space. This mapping is one-one, since by Eq. (4.9), \( \xi(\rho_1) = \xi(\rho_2) \) implies \( \rho_1 = \rho_2 \); it is onto, since every \( \xi > 0 \) inverse maps to some \( \rho > 0 \), again by Eq. (4.9). This mapping is, therefore, a bijection.

To prove part (b), we differentiate Eq. (4.9) with respect to \( \rho \), yielding
\[ 1 = -\xi'(\rho) \left[ \frac{r}{\xi'(\rho)} + \lambda \int_0^\infty x e^{-x \xi(\rho)} \bar{F}_D(x) dx \right]. \]
The equation above implies that \( \xi'(\rho) < 0 \) since each term in the brackets on the right-hand side above is strictly positive for all \( \rho \geq 0 \), which in turn implies the requisite result.
Next, we prove \( \lim_{\rho \to 0} \xi(\rho) = \infty \) in part (1) of (b) by contradiction. Suppose \( \lim_{\rho \to 0} \xi(\rho) = \bar{\xi} < \infty \) exists.

Then, the following positive limit exists,

\[
L_0 = \lim_{\rho \to 0} \left[ \frac{r}{\xi(\rho)} + \lambda \tilde{F}_D(\xi(\rho)) \right] = \frac{r}{\bar{\xi}} + \lambda \tilde{F}_D(\bar{\xi}) > 0.
\]

However, taking the limit on both sides of Eq. (4.9) as \( \rho \downarrow 0 \) yields \( L_0 = 0 \), which contradicts the above, thereby establishing the requisite result.

To prove \( \lim_{\rho \to 0} \rho \xi(\rho) = \lambda + r \) in part (1) of (b), note that Eq. (4.8) can be rewritten as

\[
\rho \xi(\rho) = \lambda + r - \lambda \tilde{f}_D(\xi(\rho)).
\]  
(\text{A.7})

Since \( \lim_{\rho \to 0} \xi(\rho) = \infty \) (just proven before) implies \( \lim_{\rho \to 0} \tilde{f}_D(\xi(\rho)) = 0 \), the requisite result now follows by taking the limits as \( \rho \downarrow 0 \) on both sides of Eq. (A.7).

Next, we prove \( \lim_{\rho \to \infty} \xi(\rho) = 0 \) in part (2) of (b) by contradiction. Suppose \( \lim_{\rho \to \infty} \xi(\rho) = \bar{\xi} > 0 \) exists.

Then, the following finite limit exists,

\[
L_\infty = \lim_{\rho \to \infty} \left[ \frac{r}{\xi(\rho)} + \lambda \tilde{F}_D(\xi(\rho)) \right] = \frac{r}{\bar{\xi}} + \lambda \tilde{F}_D(\bar{\xi}) < \infty.
\]

However, sending \( \rho \to \infty \) on both sides of Eq. (4.9) yields \( L_\infty = \infty \), which contradicts the above, thereby establishing the requisite result.

Finally, to prove \( \lim_{\rho \to \infty} \rho \xi(\rho) = r \) in part (2) of (b) we again use Eq. (A.7). Since \( \lim_{\rho \to \infty} \xi(\rho) = 0 \) (just proven before) implies \( \lim_{\rho \to \infty} \tilde{f}_D(\xi(\rho)) = 1 \), the requisite result now follows by taking the limits as \( \rho \downarrow 0 \) on both sides of Eq. (A.7). □
A.3 Proof of Proposition 1

For \( \rho > 0 \), Eq. (4.12) follows by setting \( z = \xi \) in Eq. (4.11) and noting that its first term now vanishes by virtue of the equation \( \psi(\xi) = 0 \). Eq. (4.13) follows immediately by substituting Eq. (4.12) into Eq. (4.11) and dividing the resultant equation by \( \psi(z) \neq 0 \).

A.4 Proof of Proposition 2

To prove Eq. (4.16), consider the special case where \( h = 0 \) and \( w(x) = 1 \) for \( x > 0 \). In this case, Eq. (3.6) implies

\[
c_{\rho}(u) = d_{\rho}(u),
\]

(A.8)

and Eq. (4.5) becomes

\[
g(u) = \lambda \int_{u}^{\infty} f_D(x) dx = \lambda \tilde{F}_D(u),
\]

(A.9)

while in view of Eqs. (A.8) and (A.9), Eq. (4.12) becomes

\[
d_{\rho}(0) = \frac{\lambda}{\rho} \tilde{F}_D(\xi).
\]

(A.10)

By virtue of Eqs. (3.1) and (4.6), we further have

\[
\tilde{F}_D(z) = \frac{\rho z - r - \psi(z)}{\lambda z}.
\]

(A.11)

Consequently, setting \( z = \xi \) above, noting that \( \psi(\xi) = 0 \), and substituting the resultant \( \tilde{F}_D(\xi) \) into Eq. (A.10) yields Eq. (4.16).

To prove Eq. (4.17), we take the Laplace transform of Eq. (A.9) and substitute it into Eq. (4.13) to obtain

\[
\tilde{a}_{\rho}(z) = \frac{\lambda}{\psi(z)} \left[ \tilde{F}_D(\xi) - \tilde{F}_D(z) \right], \quad z \neq \xi.
\]

(A.12)

Finally, Eq. (4.17) follows by substituting Eq. (A.11) into Eq. (A.12).

A.5 Proof of Theorem 2

Setting \( u = 0 \) in Eq. (4.1) and rearranging yield
\[ \Phi_{\rho}(\xi)(0) = \frac{c_{\rho}(\xi)(0)}{1-d_{\rho}(\xi)(0)}. \]  

(A.13)

Eq. (4.18) now follows by substituting Eqs. (4.12) and (4.16) into Eq. (A.13). Finally, Eq. (4.19) follows readily by substituting Eq. (4.18) into Eq. (4.1), while Eq. (4.20) obtains by taking Laplace transforms of both sides of Eq. (4.19) and substituting \( \tilde{c}_{\rho}(z) \) from Eq. (4.13) and \( \tilde{d}_{\rho}(z) \) from Eq. (4.17).

\[ \Box \]

A.6 Proof of Corollary 1

Eq. (4.20) can be rewritten as

\[ \tilde{x}_{\xi}(z) = \frac{\xi \tilde{g}(\xi)}{r} \times \frac{1}{z} \psi(z) \left[ \frac{\xi \tilde{g}(\xi)}{z} - \tilde{g}(z) \right]. \]

Eq. (4.21) now follows by inverting the equation above, noting that \( \frac{\xi \tilde{g}(\xi)}{r} = \Phi_{\xi}(0) \) by Eq. (4.18) and \( \frac{\xi \tilde{g}(\xi)}{z} - \tilde{g}(z) = \tilde{g}_{\xi}(z) \) by Eq. (4.22).

\[ \Box \]

A.7 Proof of Theorem 3

To prove Theorem 3, we first show that the inverse Laplace transform of \( \frac{1}{\psi(z)} \) is given by

\[ \eta_{\xi}(u) = L^{-1} \left[ \frac{1}{\psi(z)} \right] (u) = \frac{V_{\rho}(\xi)}{\xi - \theta} e^{\xi u} + \frac{V_{\rho}(\theta)}{\theta - \xi} e^{\theta u}. \]  

(A.14)

where the values of \( V_{\rho}(\xi) \) and \( V_{\rho}(\theta) \) are the limits of \( V_{\rho}(z) \) at \( \xi \) and \( \theta \), given by Eq. (4.24) and Eq. (4.25), respectively. This inverse Laplace transform is obtained by a standard application of the Residue Theorem [Churchill (1971)] and contour integration, and by taking advantage of the fact that Lundberg’s fundamental equation \( \psi(z) = 0 \) has two distinct roots, \( \xi \) and \( \theta \). Accordingly, by Eq. (4.23),

\[ \frac{1}{\psi(z)} = \frac{V_{\rho}(z)}{(z - \xi)(z - \theta)}. \]  

(A.15)

where \( \xi \) and \( \theta \) are singularities of \( \frac{1}{\psi(z)} \). Taking the inverse Laplace transform of Eq. (A.15) yields
\[ \mathcal{L}^{-1} \left[ \frac{1}{\psi(z)} \right] (x) = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{\gamma-iR}^{\gamma+iR} e^{zx} \frac{V_\rho(z)}{(z - \xi)(z - \theta)} \, dz \]

for any real \( \gamma > \xi \). To see that, define for \( R > \gamma - \theta \) a counter-clock contour path \( C_R = H_R \cup L_R \) (see Figure 4), where

\[ H_R = (x, iy) : (x - \gamma)^2 + y^2 = R^2, \quad \gamma - R \leq x \leq \gamma \]

\[ L_R = (x, iy) : x = \gamma, \quad -R \leq y \leq R \].

Figure 4. Contour integral for the inverse Laplace transform

Hence, the contour integral can be written as

\[ \frac{1}{2\pi i} \int_{C_R} e^{zx} \frac{1}{\psi(z)} \, dz = \frac{1}{2\pi i} \int_{H_R} e^{zx} \frac{1}{\psi(z)} \, dz + \frac{1}{2\pi i} \int_{\gamma-iR}^{\gamma+iR} e^{zx} \frac{1}{\psi(z)} \, dz \], \hspace{1cm} (A.16)

and by the Residue Theorem and Eq. (A.15), the left-hand side of Eq. (A.16) becomes

\[ \frac{1}{2\pi i} \int_{C_R} e^{zx} \frac{1}{\psi(z)} \, dz = \frac{V_\rho(\xi)}{\xi - \theta} e^{\xi u} + \frac{V_\rho(\theta)}{\theta - \xi} e^{\theta u} \]. \hspace{1cm} (A.17)

Eq. (A.14) now follows by substituting Eq. (A.17) into Eq. (A.16) and sending \( R \uparrow \infty \), since the first term on the right-hand side of Eq. (A.17) vanishes in view of the fact that for any \( x > 0 \), and we have [cf. Saff and Snider (1993)],

\[ \lim_{R \to \infty} \int_{H_R} e^{zx} \frac{1}{\psi(z)} \, dz \leq \lim_{R \to \infty} \frac{\pi R}{\psi(-R)} e^{Rx} = 0. \]
Next, in view of Eq. (A.14), the convolution term in Eq. (4.21) becomes

$$\langle G_\xi * \eta_\xi \rangle(u) = \frac{V_{\rho(\xi)}(\xi)}{\xi - \theta} e^{\xi u} \int_0^u G_\xi(x) e^{-\xi x} \, dx + \frac{V_{\rho(\xi)}(\theta)}{\theta - \xi} e^{\theta u} \int_0^u G_\xi(x) e^{-\theta x} \, dx , \quad (A.18)$$

where $G_\xi(x)$ is given by Eq. (4.22). Now, substituting Eqs. (4.18) and (A.18) into Eq. (4.21) yields

$$\Phi_\xi(u) = \frac{\xi \tilde{g}(\xi)}{r} + \frac{V_{\rho(\xi)}(\xi)}{\xi - \theta} e^{\xi u} \int_0^u G_\xi(x) e^{-\xi x} \, dx + \frac{V_{\rho(\xi)}(\theta)}{\theta - \xi} e^{\theta u} \int_0^u G_\xi(x) e^{-\theta x} \, dx , \quad (A.19)$$

Finally, Eq. (4.26) follows by substituting Eq. (4.22) into Eq. (A.19) and simplifying.

\[ Q.E.D. \]

### A.8 Proof of Theorem 4.

For $r = 0$, Eq. (4.12) becomes

$$c_\rho(0) = \frac{1}{\rho} \tilde{g}(\xi_0) .$$

Eq. (7.5) readily follows by substituting $c_\rho(0) = \frac{1}{\rho} \tilde{g}(\xi_0)$ and $E[\tau_1 \mid I(0) = 0] = \frac{1}{\rho \xi_0}$ (cf. Eq. (7.4) ) into Eq. (7.3) and simplifying.

\[ Q.E.D. \]
Appendix B

B.1 Proofs of Table 1 Formulas

**Constant Demand Size.** Consider the first distribution row of Table 1, where $D = d > 0$ is a constant, so that

$$\tilde{f}_D(z) = \exp\{-zd\}. \quad (B.1)$$

The corresponding $\xi^*$ follows by substituting Eq. (B.1) into Eq. (5.6); the corresponding $\rho^*$ follows by substituting this $\xi^*$ and Eq. (B.1) into Eq. (5.3); and the corresponding $\Phi_{\rho^*}(0)$ follows by substituting these $\xi^*$ and $\rho^*$ into Eq. (5.5).

**Exponentially-Distributed Demand Size.** Consider the second distribution row of Table 1, where $D \sim \text{Exp}(\beta)$. Substituting Eq. (4.38) into Eq. (5.6) yields

$$\xi^* = \arg\min_{\xi > 0} \left\{ \frac{h}{\xi} - \frac{\beta \lambda K_0}{\xi + \beta} \right\}. \quad (B.2)$$

Finally, the corresponding $\xi^*$ is obtained from Eq. (B.2) by taking the first derivative with respect to $\xi$; the corresponding $\rho^*$ follows by substituting this $\xi^*$ into Eq. (5.3); and the corresponding $\Phi_{\rho^*}(0)$ follows by substituting this $\xi^*$ into Eq. (5.5).

**Uniformly-Distributed Demand Size.** Consider the third distribution row of Table 1, where $D \sim U(a, b)$, so that

$$\tilde{f}_D(z) = e^{\frac{-az}{b-a}} - e^{\frac{-bz}{b-a}} \quad (b-a)z. \quad (B.3)$$

The corresponding $\xi^*$ follows by substituting Eq. (B.3) into Eq. (5.6); the corresponding $\rho^*$ follows by substituting this $\xi^*$ and Eq. (B.3) into Eq. (5.3); and the corresponding $\Phi_{\rho^*}(0)$ follows by substituting these $\xi^*$ and $\rho^*$ into Eq. (5.5).

**Gamma-Distributed Demand Size.** Consider the fourth distribution row of Table 1, where $D \sim \Gamma(\alpha, \beta)$, so that
\[ \hat{f}_D(z) = \left(1 + \frac{z}{\beta} \right)^{-\alpha}. \] (B.4)

The corresponding \( \xi^* \) follows by substituting Eq. (B.4) into Eq. (5.6); the corresponding \( \rho^* \) follows by substituting this \( \xi^* \) and Eq. (B.4) into Eq. (5.3); and the corresponding \( \Phi_{\rho^*}(0) \) follows by substituting these \( \xi^* \) and \( \rho^* \) into Eq. (5.5).

\[ \text{□} \]

### B.2 Proofs for Table 2 Formulas

**Constant Demand Size.** Consider the first distribution row of Table 2, where \( D = d \). The corresponding \( \xi^* \) follows by substituting Eq. (B.1) into Eq. (5.8); the corresponding \( \rho^* \) follows by substituting this \( \xi^* \) and Eq. (B.1) into Eq. (5.3); and the corresponding \( \Phi_{\rho^*}(0) \) follows by substituting these \( \xi^* \) and \( \rho^* \) into Eq. (4.35).

**Exponentially-Distributed Demand Size.** Consider the second distribution row of Table 2, where \( D \sim \text{Exp}(\beta) \). Substituting Eq. (4.38) into Eq. (5.8) yields

\[ \xi^* = \arg\min_{\xi > 0} \left\{ \frac{h}{\xi} - \frac{\lambda K_1}{\xi + \beta} \right\}. \] (B.5)

The corresponding \( \xi^* \) is obtained from Eq. (B.5) by taking the first derivative with respect to \( \xi \); the corresponding \( \rho^* \) follows by substituting this \( \xi^* \) into Eq. (5.3); and the corresponding \( \Phi_{\rho^*}(0) \) follows by substituting this \( \xi^* \) into Eq. (4.35).

**Uniformly-Distributed Demand Size.** Consider the third distribution row of Table 2, where \( D \sim U(a, b) \). The corresponding \( \xi^* \) follows by substituting Eq. (B.3) into Eq. (5.8); the corresponding \( \rho^* \) follows by substituting this \( \xi^* \) and Eq. (B.3) into Eq. (5.3); and the corresponding \( \Phi_{\rho^*}(0) \) follows by substituting these \( \xi^* \) and \( \rho^* \) into Eq. (4.35).

**Gamma-Distributed Demand Size.** Consider the fourth distribution row of Table 2, where \( D \sim \Gamma(\alpha, \beta) \). The corresponding \( \xi^* \) follows by substituting Eq. (B.4) into Eq. (5.8); the corresponding \( \rho^* \) follows by substituting this \( \xi^* \) and Eq. (B.4) into Eq. (5.3); and the corresponding \( \Phi_{\rho^*}(0) \) follows by substituting these \( \xi^* \) and \( \rho^* \) into Eq. (4.35).

\[ \text{□} \]