Optimal Bidding in Sequential Procurement Auctions

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Abstract

We consider the problem of a firm that in each period procures items by participating in auctions and then it sells the acquired items by the end of the period, where any unsold items are salvaged. The objective of the firm is to have a bidding policy that maximizes the expected value of its profit over $N$ auctions.

In this model the firm’s valuations derive from the resale of acquired items via their sale price, acquisition cost, salvage value and shortage penalties. Under sensible assumptions, it is shown that the optimal bid is a decreasing function of the number of remaining auctions, an increasing function of the number of other auction participants and a decreasing function of the number of items at hand. We also establish monotonicity properties for the value function and we present computations.

1. Introduction

We consider the problem of a firm that in each period procures items by participating in $N$ auctions and then it sells the acquired items by the end of the period, where any unsold items are salvaged. The objective of the firm is to maximize the expected present value of its profit over the $N$ auctions. In this model the firm’s valuations derive from the resale of acquired items, their salvage value, shortage penalties and the number of other auction participants. We model this problem as a Markov decision process and establish monotonicity properties for the the optimal value function and the optimal bidding strategies.

An introduction to the study of multiple auctions can be found in the book by Milgrom [5]. Klemperer [4] has an introduction to the theory of auctions which
includes references to multiple object auctions. The first paper in sequential auctions was by Ortega-Reichert [8]. He considered a two-person two-auction scenario. He proved that the optimal bids in the first auction for the two auction scenario would be less than a corresponding single auction scenario, for the asymmetric Nash equilibrium sequence of pure bidding strategies. Rothkopf and Oren [7] characterized a sequential auction as a multi-stage control process where the state represented the competitor’s strategy and state transitions represented the competitors’ reaction to a strategy used by the bidder. The control was the bidders strategy. They showed that a bidder should bid less aggressively in initial auctions if he believes this will lead his competitors to bid less aggressively in future auctions. Milgrom and Weber [6] study a model where \( k \) identical objects are sold to \( n \) bidders where every bidder can only acquire one item. Prior to the auctions each bidder receives a private signal and after each auction the seller announces the winning bid. They proved that in this case a bidder does not deceive and change his bid. Further, Weber [9] considers models of sequential auctions for which the winning bid price is a martingale that on average does not increase or decrease with each auction. Englebrecht-Wiggans [2] showed that in the case of auctions of “stochastically equivalent” objects where bidders have independent valuations, the winning bid price decreases with each auction if the individual bidder valuations are bounded distributions.

The paper is organized as follows. In section 2 we define the problem as a Markovian decision process. In section 3 under sensible assumptions, it is shown that the optimal bid is a decreasing function of the number of remaining auctions, an increasing function of the number of other auction participants (“opponents”) and a decreasing function of the number of items at hand. We also establish monotonicity properties for the value function and present computations. In the final section 4, we present concluding remarks.

2. Problem formulation

In each time period, the buyer procures items through a sequence of \( N \) auctions which he then sells. The buyer’s demand \( D \) is a random variable with a known discrete distribution. Let \( f_D(d) = P(D = d) \), \( F_D(d) = P(D \leq d) \), and \( \bar{F}_D(d) = 1 - F_D(d) \). The sales price \( r \) is assumed to be known. When the sales price is a random variable \( R \) with a known distribution then we let \( r = E(R) < \infty \). As in a standard newsvendor model, excess demand is lost with a penalty and unsold items at the end of the period have same salvage value. Let \( \delta(x) \) denote the penalty associated with \( x \) units of excess demand and let \( s \) be the unit salvage value. We assume that \( s < r \).

In each auction the number of opposing bidders (opponents) \( m \) is known and each of the \( m + 1 \) bidders submits a sealed bid. At the end of each auction the winning bid is announced and one of the highest bidders wins the auction. The objective of the buyer is to maximize his expected profit.

It is assumed that the set of all bids available (to the buyer and all opponents) is a finite set \( \{a_0, a_1, \ldots, a_p\} \) where \( a_0 < a_1 < \ldots < a_p \). For simplicity we will
use the same symbol $a$ to represent both the bid price and the action of the buyer bidding amount $a$. We assume that $a_0 = 0$ represents the action of not bidding.

Let $p_m(a)$ denote the known probability that the buyer wins an auction when his bid is $a$ and there are $m$ opponents present, where $p_m(a_0) = 0$. For convenience let $\tilde{p}_m(a) = 1 - p_m(a)$.

Let $Z_n$ be the number of opponents participating in the $n^{th}$ auction. It is assumed that $Z_n$ for $n = 1, 2, \ldots, N$ is a discrete time Markov chain with transition probabilities:

\[ q_{mm'}(n) = P(Z_{n+1} = m' | Z_n = m), \]

and an initial distribution which is denoted for simplicity by:

\[ q_m(1) = P(Z_1 = m). \]

It is assumed that whenever there is a tie in an auction involving the buyer he loses. This assumption is made to simplify the exposition. Other tie breaking procedures like deciding the winner randomly will not change the analysis but would complicate the exposition.

We model this problem as a Markov Decision process.

1. The state space $\mathcal{X}$ in this case is the set \{(n, m, x), n = 0, \ldots, N, m = 1, \ldots, x = 0, 1, \ldots\}, where $n$ represents the number of remaining auctions, $m$ represents the number of bidders participating in the current auction, $x \geq 0$ represents the inventory level at the beginning of the current, $(N - n)^{th}$ auction. Note that:
   i) If $n = 0$ then $m = 0$.
   ii) State $(0,0,x)$ represents the state of the system when all auctions are over.
   iii) Possible states prior to the start of the $N$ auctions, are of the form $(N,m,0)$, for all $m = 1, \ldots$.

2. In any state $(n, m, x)$ the following action sets $A(n, m, x)$ are available.
   i) $A(0, 0, x) = \{a_0\}$.
   ii) $A(n, m, x) = \{a_0, \ldots, a_p\}$ for $n > 0$.

3. When an action $a \in A(n, m, x)$ is taken in state $(n, m, x)$ the following transitions are possible.
   i) If $n = 0$, then starting from state $(0, 0, x)$ the next state is $(0, 0, x)$ with probability 1.
   ii) If $n > 0$ then depending on whether or not the buyer wins the current auction the next state is $(n - 1, m', x + 1)$ with probability $p_m(a) q_{mm'}(N - n)$ or state $(n - 1, m', x)$ with $\tilde{p}_m(a) q_{mm'}(N - n)$.  

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4. When an action $a \in A(n, m, x)$ is taken in state $(n, m, x)$ the expected reward $r_a(n, m, x)$ is as follows.

i) \[ r_a(0, 0, x) = \sum_d^{\infty} r_d f_D(d) + \sum_{d=x+1}^{\infty} (r_x - \delta(d-x)) f_D(d) \]

ii) \[ r_a(n, m, x) = -a p_m(a) \text{ if } n > 0. \]

Let $a^*_{n,m,x}$ denote the optimal action in the state $(n, m, x)$. Let $v(n, m, x)$ denote the value function in state $(n, m, x)$ and $w(n, m, x; a)$ denote the expected future reward when action $a$ is taken in state $(n, m, x)$ and an optimal policy is followed thereafter. Note that $v(n, m, x) = w(n, m, x; a^*_{n,m,x})$.

For $n \geq 1$, let

$$u(n, m, x) = E(v(n, Z_{N-n}, x)|Z_{N-(n+1)} = m)$$

$$= \sum_{m' = 1}^{\infty} q_{mm'}(N-n)v(n-1, m', x).$$

The dynamic programming equations are

$$v(n, m, x) = \max_{a \in A} \{ w(n, m, x; a) \} \quad (1)$$

where for $n \geq 1$,

$$w(n, m, x; a) = r_a(n, m, x) + p_m(a)u(n-1, m, x+1) + \bar{p}_m(a)u(n-1, m, x)$$

and

$$w(0, 0, x; a) = r_a(0, 0, x).$$

3. The structure of the optimal bidding policy

In this section we derive structural properties of the optimal bidding policy under the following assumptions.

**Assumption A.** For any fixed $m$, $p_m(a)$ is an increasing function of $a$.

**Assumption B.** For any fixed $a$, $p_m(a)$ is a decreasing function of $m$.

**Assumption C.** There exists a function $G$ with $\sum_{i=-\infty}^{\infty} G(i) = 1$ such that:

$$q_{mm'}(n) = \begin{cases} 
G(m' - m) & \text{if } m' > 1, \\
\sum_{k=m-1}^{\infty} G(k) & \text{if } m' = 1. 
\end{cases} \quad (2)$$

**Assumption D.** $\delta(x)$ is an increasing convex function of $x$ and $\delta(x) = 0$ if $x \leq 0$.

We first state and prove the following.
Lemma 3.1. The expected reward function in state \((0, 0, x)\), \(r_a(0, 0, x)\) is an increasing function of \(x\) i.e.

\[
r_a(0, 0, x) \leq r_a(0, 0, x + 1). \tag{3}
\]

Proof. The proof is evident from the fact that the difference \(r_a(0, 0, x + 1) - r_a(0, 0, x)\) can be simplified to

\[
\sum_{x+1}^{\infty} r_{D}(d) + \sum_{d=0}^{x} s_{D}(d) + \sum_{d=x+1}^{\infty} (\delta(d - x) - \delta(d - x - 1)) f_{D}(d),
\]

which is non-negative because \(\delta(\cdot)\) is an increasing function. \(\square\)

The following theorems 3.1 and 3.2, contain the main results of the paper.

Theorem 3.1. Under assumptions A, B and C the following relationships hold for all \(n, m, x\).

\[
v(n, m, x) \leq v(n, m, x + 1), \tag{4}
\]

\[
v(n, m, x) \leq v(n + 1, m, x), \tag{5}
\]

\[
v(n, m, x) \geq v(n, m + 1, x). \tag{6}
\]

Proof. The proof is by induction on \(n\). We first show that Ineq. (4) holds. For \(n = 0\) the inequality \(v(0, 0, x) \leq v(0, 0, x + 1)\) is true from the definition of \(v(0, 0, x)\) and Lemma 3.1. For \(n = 1\) we show that \(v(1, m, x) \leq v(1, m, x + 1)\) by contradiction. If we assume the contrary, i.e., \(v(1, m, x) > v(1, m, x + 1)\), it implies that \(v(1, m, x) > w(1, m, x + 1; a_{1,m,x}^*)\). The last inequality simplifies to

\[
p_m(a_{1,m,x}^*)(v(0, 0, x + 2) - v(0, 0, x + 1)) + \bar{p}_m(a_{1,m,x}^*)(v(0, 0, x + 1) - v(0, 0, x)) < 0,
\]

which contradicts the previous step of the induction. Now, we assume that Ineq. (4) is true for \(n - 1\) and show that it holds for \(n\). The induction assumption of Ineq. (4) implies that \(u(n - 1, m, x) \leq u(n - 1, m, x)\). From this and the definition of \(w(n, m, x; a)\) we can conclude that \(w(n, m, x; a) \leq w(n, m, x + 1; a) \forall a\) which in turn implies the following:

\[
v(n, m, x) = \max_{a \in A} w(n, m, x; a) \leq \max_{a \in A} w(n, m, x + 1; a) = v(n, m, x + 1).
\]

This completes the induction.

Next, we prove that Ineq. (5) holds. For \(n = 0\) we show that the opposite inequality

\[
v(0, 0, x) > v(1, m, x), \tag{7}
\]

leads to a contradiction. Indeed, the dynamic programming equations Eq. (1) imply that

\[
v(1, m, x) = p_m(a_{1,m,x}^*)(v(0, 0, x + 1) + a_{1,m,x}^*) + \bar{p}_m(a_{1,m,x}^*)v(0, 0, x). \tag{8}
\]
Thus, $v(1, m, x)$ is a convex combination of $v(0, 0, x)$ and $v(0, 0, x + 1) + a^*_{1, m, x}$. From this and Ineq. (7) it follows that

$$v(0, 0, x) > v(0, 0, x + 1) + a^*_{1, m, x},$$

(9)

which in turn implies that $w(1, m, x; a_0) = v(0, 0, x) > v(1, m, x)$. The last inequality implies that $a^*_{1, m, x} = a_0$; this and Ineq. (9) imply that $v(0, 0, x) > v(0, 0, x + 1)$ which contradicts Ineq. (4), for $n = 0$.

The induction assumption in this case is $v(n - 1, m, x) \leq v(n, m, x)$. From the induction assumption and assumption C it follows that $u(n - 1, m, x; a) \leq w(n, m, x; a) \forall a$. The last inequality implies that

$$v(n, m, x) = \max_{a \in A} w(n, m, x; a) \leq \max_{a \in A} w(n - 1, m, x; a) = v(n - 1, m, x),$$

which completes the induction.

We now show that Ineq. (6) holds. For $n = 1$ we show that the opposite inequality $v(1, m, x) < v(1, m + 1, x)$ leads to a contradiction. The last inequality implies that $w(1, m, x; a^*_{1, m+1, x}) < v(1, m + 1, x)$. From this and Ineq. (7) it follows that

$$v(0, 0, x) > v(0, 0, x + 1) + a^*_{1, m, x}.$$  

(10)

The above implies that $v(0, 0, x) > v(0, 0, x + 1)$ which contradicts Ineq. (4), for $n = 0$.

We complete the induction of Ineq. (6) along similar lines. We assume that it holds for $n - 1$ and show that it holds for $n$. As above we show that the opposite inequality $v(n, m, x) < v(n, m + 1, x)$ leads to a contradiction. The last inequality implies that $w(n, m, x; a^*_{n, m+1, x}) < v(n, m + 1, x)$. Simplifying the previous inequality leads to

$$u(n, m, x + 1) + a^*_{n, m, x+1} < u(n, m, x).$$

(11)

From the definition of $v(n + 1, m, x)$ the last inequality implies that

$$w(n, m, x; a_0) = v(n, m, x) \geq v(n + 1, m, x).$$

(12)

The last inequality implies that $a^*_{n, m, x+1} = a_0$. This and Ineq. (11) imply that $u(n, m, x + 1) < u(n, m, x)$, which is a contradiction.

**Theorem 3.2.** Under assumptions A, B and C the following relationships hold, for all $n$, $m$, and $x$.

$$a^*_{n, m, x} \geq a^*_{n, m, x+1},$$

(13)

$$a^*_{n, m, x} \geq a^*_{n+1, m, x},$$

(14)

$$a^*_{n, m, x} \leq a^*_{n, m+1, x}.$$  

(15)
Proof. The proof is by induction on $n$. First we prove Ineq. (13) are true. For $n = 0$ the inequality is obviously true because $a_{0,0,x}^* = a_0$ for all $x$.

To complete the induction of Ineq. (13) we assume that $a_{n-1,m,x}^* \geq a_{n-1,m,x+1}^*$ and prove that $a_{n,m,x}^* \geq a_{n,m,x+1}^*$. To prove last inequality we assume that $a_{n,m,x+1}^* > a_{n,m,x}^*$ and show that this produces a contradiction. From the definitions of $v(n,m,x)$ and $w(n,m,x;a)$ we have $v(n,m,x) < w(n,m,x;a_{n,m,x+1}^*)$ and $v(n,m,x+1) < w(n,m,x+1;a_{n,m,x}^*)$. Simplifying and combining the results of the last two inequalities we obtain
\[
u(n-1,m,x+1) - u(n-1,m,x) > u(n-1,m,x+2) - u(n-1,m,x+1).
\]

This can be rewritten as
\[
\sum_{m'} q_{mm'} [2v(n-1,m',x+1) - v(n-1,m',x) - v(n-1,m',x+2)] > 0.
\]

The above inequality implies the following:
\[
\sum_{m'} q_{mm'} [v(n-1,m',x+1) - w(n-1,m',x;a_{n-1,m',x+1}^*)] > \sum_{m'} q_{mm'} [w(n-1,m',x+2;a_{n-1,m',x+1}^*) - v(n-1,m',x+1)].
\]

Notice that the induction assumption implies the following for all $m \geq 1$.
\[
u(n-2,m,x) + u(n-2,m,x+2) > 2u(n-2,m,x+1).
\]

Simplifying Ineq. (16) using assumption C and Ineq. (17) leads to the inequality
\[
\sum_{m'} q_{mm'} [u(n-2,m',x+1) - u(n-2,m',x+2)] > \sum_{m'} q_{mm'} [u(n-2,m',x+1) - u(n-2,m',x+2)]
\]

which is a contradiction because both sides of the strict inequality are identical.

We next show that Ineq. (14) holds. For $n = 1$ we need to prove that $a_{1,m,x}^* \geq a_{1,m,x}^*$. To prove this we assume $a_{1,m,x}^* < a_{2,m,x}^*$ and show that it produces a contradiction. We know that $v(1,m,x) > w(1,m,x;a_{2,m,x}^*)$ and $v(2,m,x) > w(2,m,x;a_{1,m,x}^*)$. Simplifying the inequalities and combining the results leads to $v(0,0,x+1) - v(0,0,x) < u(1,m,x+1) - u(1,m,x)$ or equivalently
\[
v(0,0,x+1) - v(0,0,x) < \sum_{m'} q_{mm'} [v(1,m',x+1) - v(1,m',x)].
\]

The above inequality implies that
\[
v(0,0,x+1) - v(0,0,x) < \sum_{m'} q_{mm'} [v(1,m',x+1) - v(1,m',x;a_{1,m',x+1}^*)].
\]
The last inequality simplifies to
\[ v(0, 0, x + 1) - v(0, 0, x) < v(0, 0, x + 1) - v(0, 0, x) \]
which is a contradiction. For the next step in the induction we assume that \( a_{n-1,m,x}^* \leq a_{n,m,x}^* \) and prove that \( a_{n,m,x+1}^* \leq a_{n+1,m,x}^* \). To prove the last inequality we assume that \( a_{n,m,x+1}^* > a_{n+1,m,x}^* \) and show that it produces a contradiction. From the definitions of \( v(n, m, x) \) and \( w(n, m, x; a) \) we have
\[ v(n, m, x) > w(n, m, x; a_{n+1,m,x}^*) \text{ and } v(n + 1, m, x) > w(n + 1, m, x; a_{n,m,x}^*). \]
Simplifying and combining the results of the last two inequalities we obtain
\[ u(n, m, x + 1) - u(n, m, x) < u(n - 1, m, x + 1) - u(n - 1, m, x). \]
The above inequality is equivalent to the following inequality.
\[ \sum_{m'} q_{mm'} [v(n, m', x + 1) - v(n, m', x)] < \sum_{m'} q_{mm'} [v(n - 1, m', x + 1) - v(n - 1, m', x)]. \]
The last inequality implies the following:
\[ \sum_{m'} q_{mm'} [w(n, m', x + 1; a_1^*) - v(n, m', x)] < \sum_{m'} q_{mm'} [v(n - 1, m', x + 1) - w(n - 1, m', x; a_2^*)]. \]
where \( a_1^* = a_{n,m',x}^* \) and \( a_2^* = a_{n-1,m',x+1}^* \). Simplifying the above inequality using assumption C and the induction assumption leads to the inequality
\[ \sum_{m'} q_{mm'} [u(n - 2, m', x + 1) - u(n - 2, m', x)] < \sum_{m'} q_{mm'} [u(n - 2, m', x + 1) - u(n - 2, m', x)], \]
which is a contradiction because both sides of the strict inequality are identical.

We next prove Ineq. (15). We first define the auxiliary quantities \( T_{n,m} \) below:
\[ T_{n,m} = \frac{a_{n,m,x}^* p_m(a_{n,m,x}^*) - a_{n,m+1,x}^* p_m(a_{n,m+1,x}^*)}{p_m(a_{n,m,x}^*) - p_m(a_{n,m+1,x}^*)}. \]
For \( n = 0 \) the inequality is obviously true because \( a_{0,0,x}^* = a_0 \) for all \( x \). For \( n = 1 \), we assume that \( a_{1,m,x}^* > a_{1,m+1,x}^* \) and show that it leads to a contradiction. The last inequality implies that \( v(1, m, x) > w(1, m, x; a_{1,m+1,x}^*) \text{ and } v(1, m + 1, x) > w(1, m + 1, x; a_{1,m,x}^*). \) Simplifying the inequalities we obtain \( v(0, 0, x + 1) - v(0, 0, x) > T_{1,m} \) and \( v(0, 0, x + 1) - v(0, 0, x) < T_{1,m+1}. \) This is a contradiction because we have \( T_{1,m} \geq T_{1,m+1} \) from assumption B together with \( a_{1,m,x}^* > a_{1,m+1,x}^* \).

To complete the induction of Ineq. (15) assume it holds for \( n - 1 \). To prove it holds for \( n \), assume that \( a_{n,m+1,x}^* < a_{n,m,x}^* \) and show that this produces a
contradiction. Since $a^*_{n,m,x}$ is the optimal action in state $(n, m, x)$ we have that $v(n, m, x) > w(n, m, x, a^*_{n,m+1,x})$. This simplifies to:

$$u(n - 1, m, x + 1) - u(n - 1, m, x) > T_{n,m}. \quad (18)$$

Similarly in state $(n, m + 1, x)$ we have $v(n, m + 1, x) > w(n, m + 1, x, a^*_{n,m,x})$ which simplifies to

$$u(n - 1, m + 1, x + 1) - u(n - 1, m + 1, x) < T_{n,m+1}. \quad (19)$$

We next show that the following inequality is true.

$$T_{n,m+1} < T_{n,m}. \quad (20)$$

Indeed, from the definitions of $T_{n,m}$ and $T_{n,m+1}$ Ineq. (20) simplifies to the following inequality

$$p_m(a^*_{n,m,x})p_m(a^*_{n,m-1,x}) < p_m(a^*_{n,m,x})p_m(a^*_{n,m-1,x}),$$

which is true under assumption B and the assumption $a^*_{n,m,x} > a^*_{n,m+1,x}$. Now, inequalities (18), (19), (20) together imply that

$$u(n - 1, m + 1, x + 1) - u(n - 1, m + 1, x) < u(n - 1, m, x + 1) - u(n - 1, m, x).$$

This implies that

$$\sum_{i=-m+1}^{\infty} \{G(i)[v(n - 1, m + i + 1, x + 1) - v(n - 1, m + i + 1, x)]\}$$

$$+ G(-m)(v(n - 1, 1, x + 1) - v(n - 1, 1, x))$$

$$< \sum_{i=-m+1}^{\infty} \{G(i)[v(n - 1, m + i, x + 1) - v(n - 1, m + i, x)]\}.$$ 

From Theorem 3.1 we have $v(n - 1, 1, x + 1) - v(n - 1, 1, x) \geq 0$. Hence, the above inequality implies that

$$\sum_{m' = 1}^{\infty} q_{mm'}[v(n - 1, m' + 1, x + 1) - v(n - 1, m' + 1, x)]$$

$$< \sum_{m' = 1}^{\infty} q_{mm'}[v(n - 1, m', x + 1) - v(n - 1, m', x)].$$

From the last inequality we obtain the following.

$$\sum_{m' = 1}^{\infty} q_{mm'}[w(n - 1, m' + 1, x; a^*_1) - v(n - 1, m', x + 1)]$$

$$< \sum_{m' = 1}^{\infty} q_{mm'}[v(n - 1, m' + 1, x) - w(n - 1, m', x; a^*_2)], \quad (21)$$

where $a^*_1 = a^*_{n-1,m'+1,x+1}$ and $a^*_2 = a^*_{n-1,m'+1,x}$. 

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Simplifying the induction assumption, as above, we obtain the following.
\[ u(n-2, m+1, x+1) - u(n-2, m, x+1) > u(n-2, m+1, x) - u(n-2, m, x). \] (22)

From assumption B we have \( p_m(a) \geq p_{m+1}(a) \), so, let \( p_m(a^*_1) = p_{m+1}(a^*_1) + \epsilon_1 \) and \( p_m(a^*_2) = p_{m+1}(a^*_2) + \epsilon_2 \), for some non negative \( \epsilon_1, \epsilon_2 \).

Simplifying Ineq. (21) using assumption C and Ineq. (22) leads to the inequality
\[ \sum_{m' = 1}^{\infty} q_{mm'}[u(n-2, m'+1, x+1) - u(n-2, m', x+1)] 
+ \epsilon_1[-a^*_1 + u(n-2, m'+1, x+1) - u(n-2, m'+1, x)] 
+ \epsilon_2[-a^*_2 + u(n-2, m', x+2) - u(n-2, m', x+1)] 
< \sum_{m' = 1}^{\infty} q_{mm'}[u(n-2, m'+1, x+1) - u(n-2, m', x+1)]. \]

From the proof of Ineq. (6) we know that the terms multiplying \( \epsilon_1 \) and \( \epsilon_2 \) are positive. Hence the above inequality implies
\[ \sum_{m' = 1}^{\infty} q_{mm'}[u(n-2, m'+1, x+1) - u(n-2, m', x+1)] 
< \sum_{m' = 1}^{\infty} \{ q_{mm'}[u(n-2, m'+1, x+1) - u(n-2, m', x+1)], \]

which is a contradiction because both sides of the strict inequality are identical.

**Computational Example.** The properties of \( a^*_{n,m,x} \) as stated in Theorem 3.2 are illustrated by the graphs of Fig. 1 for the case with \( N = 50, m^0 = 4 \) in all auctions and \( A = \{1 \ldots , 50\} \). The winning probability \( p(a) \) is calculated assuming the four opponents choose bids from \( A \) with equal probabilities, i.e., \( p_m(a) = ((a-1)/50)^m \). The optimal bid for three cases \( x = 1, x = 14 \) and \( x = 28 \) through all the auctions are presented.

### 4. Conclusions and discussion

We have studied the problem of a firm that in each period procures items by participating in auctions and then it sells the acquired items by the end of the period, where any unsold items are salvaged. The objective of the firm is to have a bidding policy that maximizes the expected value of its profit over \( N \) auctions. In this model the firm’s valuations derive from the resale of acquired items, their salvage value, shortage penalties and the number of other auction participants (“opponents”). We have modeled this problem as a Markov decision process. Under sensible assumptions, it is shown that the optimal bid is a decreasing function of the number of remaining auctions, an increasing function of the
number of other auction participants ("opponents") and a decreasing function of the number of items at hand. We also established monotonicity properties for the value function and we have presented computations.

i) The results can be extended to an infinite horizon version of the problem where in each period there are $N$ auctions and unsold items are salvaged, since rewards and costs in different periods are then statistically identical. The situation is more interesting when unsold items at the end of a period can be kept as inventory that can be used in subsequent periods. This is case is being investigated in current research. Further, this basic model can be extended in several ways. We are currently studying the following cases: i) Items can be sold after each auction. ii) The condition that the probability $p_m(a)$ of winning, for fixed $m$ and $a$, is constant through all the auctions can be relaxed. iii) Further, the case in which the probability of winning $p_m(a)$ is not known but it can be estimated during auctions using methods such those used by Katehakis and Robbins [3] and Burnetas and Katehakis [1].

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References
Figure 1: $a_{n,m,x}^* \text{ vs } n$