On the Maintenance of Systems Composed of Highly Reliable Components

Michael N. Katehakis; Cyrus Derman

Management Science, Vol. 35, No. 5. (May, 1989), pp. 551-560.

Stable URL:
http://links.jstor.org/sici?sici=0025-1909%28198905%2935%3A5%3C551%3AOTMOSC%3E2.0.CO%3B2-B

Management Science is currently published by INFORMS.

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at http://www.jstor.org/about/terms.html. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://www.jstor.org/journals/informs.html.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is an independent not-for-profit organization dedicated to creating and preserving a digital archive of scholarly journals. For more information regarding JSTOR, please contact support@jstor.org.

http://www.jstor.org/
Thu Jun 1 13:43:54 2006

MANAGEMENT SCIENCE
Vol. 35, No. 5, May 1989
Printed in U.S.A.

ON THE MAINTENANCE OF SYSTEMS COMPOSED OF HIGHLY RELIABLE COMPONENTS*

MICHAEL N. KATEHAKIS AND CYRUS DERMAN
Department of Industrial Engineering, Technical University of Crete, Hania 73100, Crete, Greece
Department of Industrial Engineering, Columbia University, New York, New York 10027
Department of Industrial Engineering, Columbia University, New York, New York 10027

We consider the dynamic repair allocation problem for a general multi-component system that is maintained by a limited number of repairmen. Component functioning and repair times are exponentially distributed random variables with known parameters. At most one repairman may be assigned to a failed component and it is possible to reassign a repairman from one failed component to another instantaneously. The objective is to determine repair allocation policies that maximize a measure of performance of the system such as the expected discounted system operation time or the availability of the system. We consider systems composed of highly reliable, i.e., small failure rates, components and study asymptotic techniques for the determination of optimal policies. In the final section we find asymptotically optimal policies for the series, parallel, and a system composed of parallel subsystems connected in series.
(FIRST PASSAGE TIMES; MARKOV DECISION PROCESSES)

1. Introduction

A situation that arises in the maintenance of systems which operate continuously and possess limited repair capacity can be modeled as follows. A system of known structure is composed of N components and it is maintained by R repairmen, where R is less than N. Since the number of available repairmen is less than the number of components, the performance of the system depends on the maintenance policy employed, i.e., the set of decisions on which failed components repairmen are assigned, whenever the number of failed components is greater than R. Thus, it is important to have maintenance policies that yield a maximum value to some relevant measure of system performance. Measures of performance that can be used in this situation are the expected discounted system operation time and the average system operation time (or availability of the system).

In this study we make the following assumptions. Each component and the system as a whole can be in only two states, functioning or failed. The functioning and repair times for the ith component are exponentially distributed random variables with known parameters μ_i and λ_i. Repaired components are as good as new. At most one repairman may be assigned to a failed component and it is possible to reassign a repairman from one failed component to another instantaneously. Components are independent, i.e., failure or repair of one has no effect on the others. Failures may take place even while the system is not functioning.

Under these assumptions optimal policies can be obtained, in principle, using methods from Markovian decision theory. However, the computational difficulties are prohibitive due to the very large number of possible states. Therefore, explicit solutions and approximations can provide valuable insight. An explicit solution has been obtained for the series system with N components maintained by a single repairman in Katehakis and Derman (1984); see also Derman *et al.* (1980), Nash and Weber (1982) and Smith (1978).

* Accepted by John P. Lehoczky, former Departmental Editor; received June 4, 1984. This paper has been with the authors 2 years for 3 revisions.

0025-1909/89/3505/0551$01.25
Copyright © 1989, The Institute of Management Sciences

In practice many systems are composed of highly reliable components. We have a simple model for such systems if we assume that the failure rate for the ith component is of the form $\rho\mu_i$, $1 \le i \le N$. Then, for small values of ρ all components are highly reliable. We obtain analytical characterizations and derive recursive formulas for the determination of policies that are optimal for small values of ρ. These asymptotically optimal policies can be used as approximations to optimal policies for systems composed of highly reliable components.

The approach used in this paper, with appropriate modifications, can be employed to study more general models such as the case in which component failures cannot take place while the system is not functioning, systems with dependent components such as the case in which the failure rate of the ith component can be assumed to be a known function of the status of some other relevant components, and models in which the repairmen are distinguishable.

The first study of such a model for repair allocation was done by Smith (1978). We extend the work of Smith in the following directions. We provide a formulation of the problem along the lines of Markovian decision theory. We treat the multi-repairmen case. We note that the recursive formulas for the determination of asymptotically optimal policies essentially constitute a Gauss successive approximations method for solving the general Markovian decision problem. We establish the existence of intervals of the form $(0, \rho^*)$ with the property that if they contain the failure rates of all components, then the asymptotically optimal policies under consideration are optimal, and in the final section we find asymptotically optimal policies for the series, parallel, and a system composed of parallel subsystems connected in series.

For the series system maintained by R, $R \ge 2$, repairmen it is asymptotically optimal to assign repairmen to failed components with the longest expected repair times first. Thus, we show that a result established in Katehakis and Derman (1984) for the series system does not hold in the case of more than one repairmen.

2. Problem Formulation

Under the assumptions made, at any time the status of all components is given by a vector $x = (x_1, \ldots, x_N)$ with $x_i = 1$ or 0 if the ith component is functioning or failed. Thus $S = \{0, 1\}^N$ is the set of all possible states. The structure of the system, i.e., the relation between the status of the components and that of the system, is given by a partition of the state space S into two sets G and B of "good" and "bad" states, where if $x \in G$ the system is functioning and if $x \in B$ the system is failed. Alternatively, this relation can be specified by the *structure function* ϕ defined on S, such that $\phi(x) = 1$ or 0 if $x \in G$ or $x \in B$.

Throughout, we assume that the system under consideration is *coherent*, i.e., we place the following restrictions on G and B (or, equivalently, on ϕ). (i) If $x \in G$ and $y \ge x$ (i.e., $y_i \ge x_i$, $1 \le i \le N$) then $y \in G$ and if $x \in B$ and $y \le x$ then $y \in B$. (ii) For any component i, $1 \le i \le N$, there exists a state $x \in G$ such that state $(0_i, x) \in B$, where we use the notation: $(\delta_k, x) = (x_1, \ldots, x_{k-1}, \delta, x_{k+1}, \ldots, x_N)$, $\delta = 0, 1$. For $x \in S$ we define: $C_0(x) = \{i \mid x_i = 0\}$, $C_1(x) = \{i \mid x_i = 1\}$. Given any finite set A, $|A|$ will denote the number of elements in it. A state $x \in B$ such that $y \in G$ for any $y \ge x$, $y \ne x$, is called a *cut*; it corresponds to a minimal set of components which by failing cause a system failure. The *size* of a cut x is the number $|C_0(x)|$.

Let $R(x) = \min\{R, |C_0(x)|\}$, i.e., $R(x)$ denotes the maximum number of components that can be under repair when the system is in state x.

The above assumptions lead to the following formulation of the problem along the lines of Markovian Decision Theory (see Ross 1970).

The state space is the set S defined above.

The set of all possible actions in state x is $A(x) = \{a\colon a \subseteq C_0(x),\ |a| = R(x)\}$. We exclude from consideration actions that leave repairmen idle while there are failed components since it is easy to show that policies that contain such actions can not be optimal (see Smith 1978).

When the system is in state x and action $a \in A(x)$ is chosen the following transitions are possible

(i) to state $(1_i, x)$, with rate λ_i, $i \in a$,

(ii) to state $(0_k, x)$, with rate $\rho\mu_k$, $k \in C_1(x)$.

When the system is in state x there is a reward rate $\phi(x)$ (or equivalently, a cost rate $\phi(\bar{x}) = 1 - \phi(x)$).

A nonidling deterministic policy π is a mapping that takes $x \in S$ into a subset $\pi(x) \in A(x)$. Let Π denote this finite set of policies. Notice that under any policy π in Π the evolution in time of the status of all components can be described by a continuous time, finite state, irreducible Markov chain $\{x^\pi(t) = (x_1^\pi(t), \ldots, x_N^\pi(t)),\ t \geq 0\}$, where $x_i^\pi(t) = x_i$ if the ith component is in state x_i at time t. It is known (Ross 1970, p. 114) that optimal policies with respect to the discounted system operation time or the average system operation time criteria exist in Π.

Let $v_\pi(x)$ (respectively $w_\pi(x)$) denote the expected discounted time that the system spends in nonfunctioning (respectively functioning) states, when the initial state is x and a policy $\pi \in \Pi$ is employed. For notational simplicity we suppress the dependence of $v_\pi(x)$ on the discount rate β, $\beta \in (0, \infty)$ and the parameter ρ. Also, let f_π (respectively g_π) denote the average time that the system spends in nonfunctioning (respectively functioning) states, under $\pi \in \Pi$. Since for π in Π, $\{x_\pi(t),\ t \geq 0\}$ is ergodic f_π, g_π are independent of the initial state.

It is easy to see that $v_\pi(x) + w_\pi(x) = 1/\beta$ for all π and x in S. Hence, a policy maximizes $w_\pi(x)$ if and only if it minimizes $v_\pi(x)$. In the sequel it is convenient to study policies that maximize $w_\pi(x)$, for small failure rates, by considering policies that minimize $v_\pi(x)$. An analogous argument allows us to do the same for the average system operation time criterion, since $f_\pi + g_\pi = 1$ for all π.

3. Asymptotically Optimal Policies

We first consider the expected discounted system operation time criterion. It is known (see Whittle 1983, p. 23 or Ross 1970, p. 120) that for any policy π in Π the corresponding values of $v_\pi(x)$, $x \in S$, can be obtained as the unique solution to the following system of linear equations:

$$v_\pi(x) = \frac{1}{\lambda(\pi(x)) + \rho\mu(x) + \beta}\{\bar{\phi}(x) + \sum_{i \in \pi(x)} \lambda_i v_\pi(1_i, x) + \sum_{j \in C_1(x)} \rho\mu_j v_\pi(0_j, x)\}, \qquad x \in S, \quad (1)$$

where we set $\lambda(a) = \sum_{i \in a} \lambda_i$, $\mu(x) = \sum_{j \in C_1(x)} \mu_j$ and a sum over an empty set is defined to be equal to zero.

It is known (Derman 1970) that there exists a deterministic policy π^* such that

$$v_{\pi^*}(x) \leq v_\pi(x) \qquad \forall x \in S, \qquad \forall \pi \neq \pi^*. \quad (2)$$

In the sequel we use (2) and Lemma 1 below in order to study policies that are optimal for small values of ρ.

LEMMA 1. *For any $x \in S$, $\beta \in (0, \infty)$ and π in Π, there exists a power series expansion of $v_\pi(x)$, of the form*

$$v_\pi(x) = \sum_{\nu=0}^{\infty} v_\pi^{(\nu)}(x)\rho^\nu. \quad (3)$$

PROOF. We can write the system of equations (2) in the following form

$$v_\pi(x) = \frac{1}{\lambda(\pi(x)) + \beta}[\bar{\phi}(x) + \sum_{i\in\pi(x)} \lambda_i v_\pi(1_i, x) + \rho \sum_{j\in C_1(x)} \mu_j(v_\pi(0_j, x) - v_\pi(x))], \quad (4)$$

$x \in S$. In matrix form (4) can be written as $v_\pi = b(\pi) + C(\pi)v_\pi + \rho D(\pi)v_\pi$.

Under an appropriate labelling of the states (e.g. any labelling with the property: y has higher label than x when $|C_0(x)| > |C_0(y)|$) the matrix $C = C(\pi)$ is triangular and all its elements are less than 1. Thus, $(I - C)^{-1}$ exists may be computed in a recursive fashion. Thus, $v_\pi = (I - C)^{-1}b + \rho(I - C)^{-1}Dv_\pi$ or in more compact form:

$$v_\pi = q(\pi) + \rho Q(\pi)v_\pi. \quad (5)$$

It follows from (5) that for any $k \geq 1$, we have:

$$v_\pi(x) = q(\pi) + \sum_{\nu=1}^{k} \rho^\nu (Q(\pi))^\nu c(\pi) + \rho^{k+1}(Q(\pi))^{k+1} v_\pi(x). \quad (6)$$

Now let $\|Q(\pi)\|$ denote a norm of the matrix $Q(\pi)$, then (6) implies that (3) holds for all $\rho \in (0, 1/\|Q(\pi)\|)$.

The next corollary provides a method for computing the coefficients $v_\pi^{(\nu)}(x)$ recursively for increasing $|C_0(x)|$.

COROLLARY 1. *For any $x \in S$, $\beta \in (0, \infty)$ and π in Π, the $v_\pi^{(\nu)}(x)$'s can be computed recursively from equations* (7), (8) *below*

$$v_\pi^{(0)}(1) = 0, \quad (7a)$$

$$v_\pi^{(0)}(x) = \frac{1}{\lambda(\pi(x)) + \beta}[\bar{\phi}(x) + \sum_{i\in\pi(x)} \lambda_i v_\pi^{(0)}(1_i, x)], \quad (7b)$$

$$v_\pi^{(\nu+1)}(x) = \frac{1}{\lambda(\pi(x)) + \beta}[\sum_{i\in\pi(x)} \lambda_i v_\pi^{(\nu+1)}(1_i, x) + \sum_{j\in C_1(x)} \mu_j(v_\pi^{(\nu)}(0_j, x) - v_\pi^{(\nu)}(x))], \quad (8)$$

$\nu \geq 0$, $x \in S$.

PROOF. It follows from (6) that $v_\pi^{(0)} = (I - C(\pi))^{-1}b(\pi)$ and since $C(\pi)$ is triangular, $v_\pi^{(0)}$ can be computed recursively by $v_\pi^{(0)} = b(\pi) + C(\pi)v_\pi^{(0)}$ which is (7). Similarly, $v_\pi^{(\nu+1)} = (I - C(\pi))^{-1}D(\pi)v_\pi^{(\nu)}$, hence $v_\pi^{(\nu+1)} = C(\pi)v_\pi^{(\nu+1)} + D(\pi)v_\pi^{(\nu)}$ which is (8).

REMARK 1. Note that equations (7), (8) constitute a Gauss Seidel iteration scheme for solving the system of linear equations (1). Thus the overall approach of determining policies that minimize the leading coefficients is essentially equivalent to employing a so-called pre-Gauss Seidel iteration for the under consideration Markovian decision problem, see Thomas *et al.* (1984) and references given there.

We next determine the leading coefficients of the power series (3). It turns out that the order of the leading coefficients is determined by the structure of the system. We first need to define the following quantities. Let

$$m(\phi) = \min\{|C_0(x)| \mid x \in B\}, \quad (9)$$

$$B_{m(\phi)} = \{x \in B \mid |C_0(x)| = m(\phi)\} \qquad \text{and} \quad (10)$$

$$I(x) = \min\{|C_0(y)| \mid y \leq x, y \in B\} - |C_0(x)|. \quad (11)$$

In the terminology of coherent structure theory, $m(\phi)$ is the size of a cut state of minimal size, $B_{m(\phi)}$ is the set of all such states and $I(x)$ is the minimum number of components that must fail when the system is in state x, in order to cause a system failure.

The next lemma summarizes properties of $I(x)$ that are easily verifiable from its definition and the fact that ϕ is a coherent structure.

LEMMA 2. *For any state x the following are true.*
(i) $I(1) = m(\phi) \geq I(x)$.
(ii) $I(0_i, x) \geq I(x) - 1$, $\forall i \in C_1(x)$.
(iii) *If* $\phi(x) = 1$ *then* $I(x) \geq 1$.
(iv) $\phi(x) = 0$ *if and only if* $I(x) = 0$.
(v) *If* $y \in B_{m(\phi)}$, $j_i \in C_0(y)$, $i = 1, 2, \ldots, k$, *and* $x = (1_{i_1}, \ldots, 1_{i_k}, y)$ *then* $I(x) = k$.

We can now prove the following.

LEMMA 3. *For any x in S and any policy* π *in* Π,

$$v_\pi^{(k)}(x) = 0 \qquad \textit{whenever} \qquad 0 \leq k \leq I(x) - 1. \tag{12}$$

PROOF. By induction on $k = 0, \ldots, I(x) - 1$ and subinduction on $|C_0(x)| = 0, \ldots, N$.
(i) For $k = 0$; we must show that $v_\pi^{(0)}(x) = 0$, for all x such that $I(x) - 1 \geq 0$. This follows by induction on $|C_0(x)|$.
(a) If $|C_0(x)| = 0$ then $x = 1$ and therefore $v_\pi^{(0)}(1) = 0$ by (7a).
(b) Assume that (i) holds for all x such that $|C_0(x)| = \nu$, $I(x) \geq 1$.
(c) Consider a state y such that $|C_0(y)| = \nu + 1$ and $I(y) \geq 1$.
Then, the induction hypothesis (i) (b), the observations that: $\bar{\phi}(y) = 0$ when $I(y) \geq 1$ and $|C_0(1_i, y)| = \nu$, and (7b) imply that $v_\pi^{(0)}(y) = 0$.
(ii) Assume that the lemma is true for $k = 0, 1, \ldots, k_0$ and for all x such that $I(x) - 1 \geq k_0$.
(iii) We next show that it holds for $k = k_0 + 1$ and for all x such that $I(x) - 1 \geq k_0 + 1$. Indeed:
(a) if $|C_0(x)| = 0$, i.e., $x = 1$, we have from (8):

$$v_\pi^{(k_0+1)}(1) = \frac{1}{\beta}\Big[\sum_{j=1}^{N} \mu_j(v_\pi^{(k_0)}(0_j, 1) - v_\pi^{(k_0)}(1))\Big]. \tag{13}$$

Note now that $I(0_j, 1) \geq I(1) - 1$ by Lemma 2(ii) and we have assumed that $k_0 + 1 \leq I(1) - 1$. Hence, $k_0 \leq I(0_j, 1) - 1$ and $k_0 < I(1) - 1$. Thus, the result follows from (13) since by the induction hypothesis (ii) we have that $v_\pi^{(k_0)}(0_j, 1) = v_\pi^{(k_0)}(1) = 0$.
(b) Assume that the lemma holds, for any x such that $|C_0(x)| = \nu$.
(c) Consider a state y such that $|C_0(y)| = \nu + 1$. Then $|C_0(1_i, y)| = \nu$ for $i \in \pi(y)$, and the induction hypothesis (iii) (b) implies that: $v_\pi^{(k_0+1)}(1_i, y) = 0$. Notice also that, as in (a) above, $k_0 \leq I(0_j, y) - 1$ and $k_0 < I(y) - 1$. Thus, the induction hypothesis (ii) implies that $v_\pi^{(k_0)}(0_j, y) = v_\pi^{(k_0)}(y) = 0$, for $j \in C_1(y)$. Now, it is easy to complete the induction step using (8).

A consequence of Lemmata 1 and 3 is the next

THEOREM 1. *For any* $x \in S$ *and any* π *in* Π, *there exist constants* $v_\pi^{(I(x))}(x)$ *in* $(0, \infty)$, *such that:*

$$v_\pi(x) = v_\pi^{(I(x))}(x)\rho^{I(x)} + o(\rho^{I(x)}), \tag{14}$$

where the $v_\pi^{(I(x))}(x)$*'s can be determined recursively as follows.*
(i) *For all states x such that* $I(x) = 0$,

$$v_\pi^{(0)}(x) = \frac{1}{\lambda(\pi(x)) + \beta}\Big[1 + \sum_{i \in \pi(x)} \lambda_i v_\pi^{(0)}(1_i, x)\Big]. \tag{15}$$

(ii) *For all x such that $I(x) \geq 1$,*

$$v_\pi^{(I(x))}(x) = \frac{1}{\lambda(\pi(x)) + \beta}\left[\sum_{i\in\pi(x)} \lambda_i v_\pi^{(I(x))}(1_i, x) + \sum_{j\in C_1(x)} \mu_j v_\pi^{(I(x)-1)}(0_j, x)\right]. \quad (16)$$

PROOF. Equations (15) and (16) follow from Corollary 1 and Lemma 3. To show that $v_\pi^{(I(x))}(x) > 0$ for all x, notice that this is true for all x such that $I(x) = 0$. The proof can be completed by induction on $I(x)$ using (16).

Theorem 1 shows that the order of the leading terms in the asymptotic power series expansion of $v_\pi(x)$ is the same for all deterministic policies. Thus, we can formally state the following.

PROPOSITION 1. *A policy π^* in Π minimizes the expected total discounted nonfunctioning time of the system for small values of ρ if and only if:*

$$v_{\pi^*}^{(I(x))}(x) = \min\{v_\pi^{(I(x))}(x), \pi \in \Pi\}, \qquad \forall x \in S. \quad (17)$$

In the absence of ties (17) determine unique asymptotically optimal actions for all states. Ties can be resolved by computing and minimizing higher order coefficients subject to minimizing all lower order coefficients.

REMARK 2. Notice that since $v_\pi^{(k)}(1) = 0$ for $k = 0, 1, \ldots, I(1) - 1$, the coefficients $v_\pi^{(k)}(x)$, for all $x \neq 1$, $k \leq I(x) - 1$, are the same with those in the asymptotic power series expansion of the expected discounted time that the system spends in failed states during the first passage time from state x to state 1 under policy π. Furthermore, it follows from (15), that $v_\pi^{(0)}(x)$ is equal to the expected discounted first passage time from state x to state 1 in the absence of failures. Thus, we have the following partial characterization of asymptotically optimal policies. If a policy is asymptotically optimal with respect to the expected discounted system operation time criterion then, it must assign repairmen to failed components in such a way that the expected discounted time that the system spends in failed states during the first passage time from any state $x \in B$ to state 1 is minimized.

REMARK 3. The approach used in this paper, i.e., to write (1) as (4) in order to obtain (3) and (7), (8), can be employed to study more general models. Specifically, in the following generalizations to the basic repair allocation problem under consideration, the results of Lemmata 1 to 3 and Theorem 1 remain valid after appropriate modifications.

1. The model with distinguishable repairmen. We assume that the time required to repair a the ith component with the jth repairman is an exponentially distributed random variable with parameter λ_i^j, $1 \leq j \leq R$.

In this case the main difference is in the action sets which are now defined as follows. $A(x) = \{a\colon a = \{(j, i_j), j = 1, \ldots, R(x), i_j \in C_0(x)\}\}$, with the additional property that if $(j, i_j), (k, i_k) \in A(x)$ and $k \neq j$ then $i_j \neq i_k$. The pair (j, i_j) specifies allocation of repairman j to component i_j.

2. The model in which failures cannot take place while the system is not functioning. In this case we only need to multiply μ_j by $\phi(x)$ in (1) while everything else remains the same.

3. A model with dependent components such as the case in which the failure rate of the ith component can be assumed to be a known function of the states of some other relevant components, e.g., when the system is in state x the failure rate of the ith component is a known function of the form $\rho\mu_i(x)$, where $\mu_i(x)$ are known functions of x for example $\mu_i(x) = \mu_i|C_0(x)|$.

We now turn to the problem of determining π in Π to maximize the availability of the system. This is equivalent to minimizing f_π. It is known (see Whittle 1983, p. 126

or Ross 1970, p. 156) that for any policy π in Π the corresponding to it value of f_π, can be obtained as the unique solution of the following system of linear equations.

$$\frac{f_\pi}{\lambda(\pi(x)) + \rho\mu(x)} + h_\pi(x) = \frac{1}{\lambda(\pi(x)) + \rho\mu(x)} \{\bar{\phi}(x) + \sum_{i \in \pi(x)} \lambda_i h_\pi(1_i, x) + \sum_{j \in C_1(x)} \rho\mu_j h_\pi(0_j, x)\}, \qquad x \in S - \{1\}, \tag{18a}$$

$$h_\pi(1) = 0, \tag{18b}$$

where $h_\pi(x)$ are the differential costs associated with π and we have arbitrarily chosen $h_\pi(1) = 0$.

Furthermore, there exists a policy π^* in Π such that:

$$f_{\pi^*} \leq f_\pi \qquad \text{for all } \pi \text{ in } \Pi. \tag{19}$$

We next prove the following.

LEMMA 4. *For any $x \in S$, $x \neq 1$, $\beta \in (0, \infty)$ and π in Π, there exist power series expansions of f_π, $h_\pi(x)$ of the form*

$$f_\pi = \sum_{\nu=0}^{\infty} f_\pi^{(\nu)} \rho^\nu, \tag{20}$$

$$h_\pi(x) = \sum_{\nu=0}^{\infty} h_\pi^{(\nu)}(x) \rho^\nu, \; x \neq 1. \tag{21}$$

PROOF. We can write the system of equations (18) in the following form:

$$f_\pi = \rho \sum_{j=1}^{N} \mu_j h_\pi(0_j, 1), \tag{22a}$$

$$h_\pi(x) = \frac{1}{\lambda(\pi(x))} [\bar{\phi}(x) - f_\pi + \sum_{i \in \pi(x)} \lambda_i h_\pi(1_i, x) + \rho \sum_{j \in C_1(x)} \mu_j (h_\pi(0_j, x) - h_\pi(x))], \tag{22b}$$

$$x \in S, \; x \neq 1.$$

The proof can be completed with arguments similar to those used in the proof of Lemma 1.

The next corollary provides a method for computing the coefficients $f_\pi^{(\nu)}$, $h_\pi^{(\nu)}(x)$, recursively for increasing $|C_0(x)|$.

COROLLARY 2. *For any $x \in S$, $\beta \in (0, \infty)$ and π in Π we have*

$$f_\pi^{(0)} = 0, \tag{23a}$$

$$h_\pi^{(0)}(x) = \frac{1}{\lambda(\pi(x))} [\bar{\phi}(x) + \sum_{i \in \pi(x)} \lambda_i h_\pi^{(0)}(1_i, x)], \tag{23b}$$

$$f_\pi^{(\nu+1)} = \sum_{j=1} \mu_j h_\pi^{(\nu)}(0_j, 1), \tag{24a}$$

$$h_\pi^{(\nu+1)}(x) = \frac{1}{\lambda(\pi(x))}\Big[-f_\pi^{(\nu+1)} + \sum_{i\in\pi(x)} \lambda_i h_\pi^{(\nu+1)}(1_i, x) + \sum_{j\in C_1(x)} \mu_j\big(h_\pi^{(\nu)}(0_j, x) - h_\pi^{(\nu)}(x)\big)\Big], \quad (24b)$$

$\nu \geq 0, x \in S, x \neq 1$.

The following theorem is analogous theorem 1. The proof is similar and it is omitted.

THEOREM 2. *For any $x \in S$, $x \neq 1$ and any π in Π, there exist constants $f_\pi^{(I(1))}$, $v_\pi^{(I(x))}(x)$ in $(0, \infty)$, such that:*

$$f_\pi = f_\pi^{(I(1))}\rho^{I(1)} + o(\rho^{I(1)}), \quad (25a)$$

$$h_\pi(x) = h_\pi^{(I(x))}(x)\rho^{I(x)} + o(\rho^{I(x)}). \quad (25b)$$

REMARK 4. As in Lemma 3 we obtain that the coefficients $f_\pi^{(I(x))}$ and $h_\pi^{(I(x))}(x)$ can be determined recursively as follows.

(i) For all states x such that $I(x) = 0$,

$$f_\pi^{(0)} = 0, \quad (26a)$$

$$h_\pi^{(0)}(x) = \frac{1}{\lambda(\pi(x))}\Big[1 + \sum_{i\in\pi(x)} \lambda_i h_\pi^{(0)}(1_i, x)\Big]. \quad (26b)$$

(ii) For all x such that $I(x) \geq 1$,

$$f_\pi^{(I(1))} = \sum_{j=1}^{N} \mu_j h_\pi^{(I(1)-1)}(0_j, 1), \quad (27a)$$

$$h_\pi^{(I(x))}(x) = \frac{1}{\lambda(\pi(x))}\Big[-f_\pi^{(I(x))} + \sum_{i\in\pi(x)} \lambda_i h_\pi^{(I(x))}(1_i, x) + \sum_{j\in C_1(x)} \mu_j h_\pi^{(I(x)-1)}(0_j, x)\Big], \qquad x = 1. \quad (27b)$$

Now it is easy to establish the following.

PROPOSITION 2. *A policy π^* in Π maximizes the availability of the system, for small values of ρ if and only if:*

$$h_{\pi^*}^{(I(x))}(x) = \min\{h_\pi^{(I(x))}(x), \pi \in \Pi\}, \qquad \forall x \in S, \qquad x \neq 1. \quad (28)$$

In the absence of ties, (28) determine a unique asymptotically optimal policy. Ties can be resolved by considering higher order coefficients as computed by (24).

REMARK 5. Notice that when the system is in a failed state x, $I(x) = 0$, and $h_\pi^{(0)}(x)$ as determined from equations (25b) is the expected time until the system is back in operation when the initial state is x, policy π is employed and there are no failures. Thus, we obtain the following, intuitively expected, partial characterization of policies that maximize the availability of the system, for small values of ρ. When the system is failed such policies must assign repairmen to failed components in such a way that the expected time until the system is back in operation, in the absence of failures, is minimized.

In the next theorem we show that asymptotically optimal policies are strictly optimal when all failure rates are sufficiently small.

THEOREM 3. *Let π^* be an asymptotically optimal policy, with respect to one of the criteria that have been considered. Then, there exists a $\rho_0 > 0$ such that π^* is optimal $\forall\rho \in (0, \rho_0)$.*

PROOF. We prove the theorem for the expected discounted operation time criterion only. The proof for the maximum availability criterion is similar and is omitted.

Recall that for any policy π in Π and for $\rho \in (0, 1/\|Q(\pi)\|)$ the $v_\pi(x)$'s possess convergent power series expansions. Since there are finite many policies in Π, it follows that the above power series representations of all $v_\pi(x)$'s are convergent for all π in Π in the interval $(0, \rho_1)$, where $\rho_1 = \min_{\pi\in\Pi}\{1/\|Q(\pi)\|\}$.

Now for any $x \in S$ and π_1, π_2 in Π, it follows (see Rudin 1976, p. 177) that the difference: $v_{\pi_1}(x) - v_{\pi_2}(x)$ may change sign a finite number of times. Thus the theorem follows from Proposition 2 and the fact that there are finite many policies in Π and states in S.

4. Applications

In the following examples we restrict our attention to determining policies which are asymptotically optimal with respect to the availability criterion.

4.1. *Series and Parallel Systems*

Consider first the N component series system maintained by R repairmen. The only functioning state is state $\mathbf{1} = (1, \ldots, 1)$. In Katehakis and Derman (1984) it was established that when $R = 1$ the optimal policy always assigns the repairman to the failed component with the smallest failure rate (SFR policy).

From Proposition 2, Remark 4 we know that an asymptotically optimal policy π^* minimizes the expected time to state $\mathbf{1}$ from any initial state x in the absence of failures. Thus, in the terminology of stochastic scheduling, an asymptotically optimal policy minimizes the expected makespan for allocating $|C_0(x)|$ tasks (repairs) on R identical processors (repairmen), for any state x.

It is easy to see that all policies in Π have the same makespan: $h_\pi^{(0)}(x) = \sum_{i\in C_0(x)} 1/\lambda_i$ and to show that the SFR policy is asymptotically optimal by considering higher order coefficients. For $R \geq 2$, it has been shown by Bruno *et al.* (1981) that an optimal policy assigns repairmen to failed components in $C_0(x)$ according to the LEPT (Longest Expected Processing Time First) rule. In the context of the series system an asymptotically optimal policy assigns repairmen to the failed components with the longest expected repair times. Notice now that this LEPT policy is optimal for sufficiently small failure rates (by Theorem 3). It follows from this example that in the general case the optimal policy does depend on the repair rates, and therefore the SFR policy is not optimal for $R \geq 2$.

For the parallel system the only failed state is state $\mathbf{0} = (0, \ldots, 0)$. Furthermore, $I(1_j, x) = I(x) + 1$ for all $x \neq 1$, thus it is easy to show, using Theorem 2, that the policy which always assigns repairmen to the failed components with the smallest repair rates is asymptotically optimal.

4.2. *Parallel Subsystems Connected in Series*

Consider a system that is composed of K subsystems and it is maintained by a single repairman. The ith subsystem is composed of N components with the same failure rates μ_i. Furthermore, we assume that all components have identical repair rates λ.

Since the subsystems have identical components, it is easy to see that the state of the system at any time can be adequately specified by a vector $\mathbf{z} = (z_1, \ldots, z_K)$, where z_i denotes the number of functioning components in subsystem i, $z_i = 0, 1, \ldots, N$. The structure of the system is specified by the sets $G = \{z \mid z_i \geq 1\ \forall i\}$, $B = \{z \mid z_i = 0$ for some $i\}$. Let $C_0(z) = \{i \mid z_i = 0\}$, $(n_j, z) = (z_1, \ldots, z_{j-1}, n, z_{j+1}, \ldots, z_K)$, $n = 0, 1, \ldots, N$, and define $(n_{j_1}, \ldots, n_{j_k}, z)$ recursively by $(n_{j_1}, n_{j_2}, z) = (n_{j_1}, (n_{j_2}, z))$. Note that $I(z) = \min\{z_i, i = 1, \ldots, K\}$. Finally, let $\mathcal{I}(z) = \{i \mid z_i = I(z)\}$ and $J((n_{j_1}, \ldots, n_{j_k}, x)) = \min\{z_i, i \neq j_1, \ldots, j_k\}$. Since subsystems have identical components a policy is specified up to the subsystem on which the repairman is assigned only. With this generalization of notation, using Theorem 2, we obtain that the asymptotically optimal policy is given by the following simple rule.

(i) If $z \in G$ assign the repairman to system j if and only if either $\mathcal{I}(z) = \{j\}$ or $\mu_j = \max\{\mu_i, i \in \mathcal{I}(z)\}$.

(ii) If $z \in B$ assign the repairman to subsystem j if and only if $\mu_j = \min\{\mu_i, i \in C_0(z)\}$.

The proof of (i) essentially involves establishing by induction on n that if $J((n_{i_1}, \ldots, n_{i_k}, z)) \geq n + 1$ then

$$h^{(n)}(n_{i_1}, \ldots, n_{i_k}, x) = n! \sum_{j=1}^{k} j\mu_{i_j}^{n} \lambda^{-(n+1)}, \qquad n = 1, \ldots, I(N, \ldots, N) - 1. \tag{29}$$

The proof of (ii) is easy to complete by induction on increasing $|\mathcal{I}(z)|$.[1]

[1] We wish to thank an anonymous referee for several useful observations and suggestions that have improved this paper considerably.

Work supported by USAF Contract AFOSR 87-0072 and NSF Grant ECS-85-07671.

References

BRUNO, J., P. DOWNEY AND G. N. FREDERICKSON, "Sequencing Tasks with Exponential Service Times to Minimize the Expected Flow Time or Makespan," *J. Assoc. Comput. Mach.*, 28 (1981), 100–113.

DERMAN, C. "*Finite State Markovian Decision Processes*," Academic Press, New York, 1970.

———, G. LIEBERMAN AND S. M. ROSS, "On the Optimal Assignment of Servers and Repairman," *J. Appl. Probab.*, 17 (1980), 577–581.

KATEHAKIS, M. N. AND C. DERMAN, "*Optimal Repair Allocation in a Series System*," *Math. Oper. Res.*, 9, 4 (1984), 615–623.

NASH, P. AND R. R. WEBER, "Dominant Strategies in Stochastic Allocation and Scheduling Problems," In *Deterministic and Stochastic Scheduling*, M. A. H. Dempster et al. (eds.), D. Reidel Publishing Company, Dordrecht, 1982, 343–353.

ROSS, S. M., *Applied Probability Models with Optimization Applications*, Holden Day, San Francisco, 1970.

RUDIN, W., *Principles of Mathematical Analysis*, 2nd ed., Mc Graw Hill, New York, 1976.

SMITH, D. R., "Optimal Repairman Allocation—Asymptotic Results," *Management Sci.*, 24 (1978), 665–674.

THOMAS, L. C., R. HARLEY AND A. C. LAVERCOMBE, "Computational Comparison of Value Iteration Algorithms for Discounted Markov Decision Processes," *Oper. Res. Lett.*, 2 (1983), 72–75.

WHITTLE, P., *Optimization Over Time. Vol.* 2, Wiley, New York, 1983.