Cash-Flow Based Dynamic Inventory Management

Junmin Shi∗, Michael N. Katehakis † and Benjamin Melamed ‡

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Abstract

Often firms such as retailers or whole-sellers when managing interrelated flows of cash and inventories of goods, have to make financial and operational decisions simultaneously. Specifically, goods are acquired by capital (cash) expenditure in the procurement phase of operations, while in the selling stage income, that contributes to the firm’s cash balance, is generated by the sales of the acquired goods. Therefore, it is critical to the firm’s success to manage these two (cash and material) flows in an efficient manner.

We model a firm that uses its capital position (i.e., its available cash or an external loan if so desired) to invest on product inventory, that is considered to consist of identical items. The remaining capital (if any) can be deposited to a bank account for interest. The lead time for replenishment is zero and demands are assumed to be independent and identically distributed over periods. The objective is to maximize the expected total wealth at the end of planning horizon.

We show that the optimal order policy for each period is characterized by two threshold values which is referred to as \((\alpha_n, \beta_n)\)-policy, under which the Newsvendor orders up to \(\alpha_n\) if the total asset is less than \(\alpha_n\) (an over-utilization case); orders up to \(\beta_n\) if the total asset is greater than \(\beta_n\) (an under-utilization case); otherwise, orders exactly the affordable units with capital (a full-utilization case). Each threshold value is increasing in the total value of asset and capital. For single period problem, we show that the \((\alpha, \beta)\) optimal policy brings a positive expected value even with zero initial asset and capital. For multiple period problem, we propose two myopic ordering policy which respectively provide upper and lower bounds for each threshold values. Based on the upper-lower bounds, an efficient algorithm is provided to locate those two constants. Finally, some numerical studies provide more insights of the problem.

Keywords and Phrases: News-vendor, external fund availability, capital-asset portfolio.

1 Introduction

Most business organizations such as retailers are encountered with the financial and material decisions simultaneously of managing interrelated flows of cash and material. In the procurement process, goods are acquitted by capital, while in the selling process, the goods are sold which in turn contributes to cash reserves. Therefore, it is imperative to business success to manage those two flows efficiently. This paper studies a
single-item single/multiple-period inventory system under both operational (inventory replenishment) and financial decisions. In particular, the retailer takes its available capital or an external loan if needed to invest on product inventory. The remaining capital (if any) can be deposited to a bank account for interest. The lead time for replenishment is zero and demands are assumed to be independent and identically distributed over periods. The objective is to maximize the expected total wealth level at the end of a (finite or infinite) planning horizon.

It is a fashion to treat a product as a special financial instruments so that a generally defined portfolio composed of products and regular financial instruments can be studied by the well developed finance/investment principles such as Modern Portfolio Theory. There are a big number of related literature. We refer the readers to Corbett et al. (1999) and related references therein. Conversely, holding cash or stocks may be considered as special inventories and a transaction as replenishing. This analogy is sophisticated and should be handled carefully. The real nature of the relationship between inventories and finance, together with the theoretical and empirical consequences is discussed by Girlich (2003).

There are a number of studies in the operations management have addressed the interface of production and financial decisions. Xu and Birge (2004) provides a comprehensive literature review and develops models to make production and financing decisions simultaneously in the presence of demand uncertainty and market imperfections. Early studies of problems in which inventory and financial decisions were made simultaneously were done by Li et al. (1997) and Buzacott and Zhang (2004). Their models allowed different interest rates on cash balance and outstanding loans. These papers also demonstrated the importance of joint consideration of production and financing decisions in a start-up setting in which the ability to grow the firm is mainly constrained by its limited capital and dependence on bank financing. Dada and Hu (2008) assumes that the interest rate is charged by the bank endogenously and the newsvendor’s problem is modeled as a multi-period problem that explicitly examines the cost when bankruptcy risks are significant. Accordingly, such single-period model could be used as a building block for considering such models when liquidity or working capital is an issue. This paper studies a game between bank and inventory manager through which a comparative statics of the equilibrium are presented and a non-linear loan schedule is proposed. But those three papers are limited to single period model.

This paper presents and studies a discrete-time model in which inventory decisions for a single product in the presence of random demand are made by taking into account cash flow issues related to sale generated profits as well as borrowing costs to finance purchases. In the current literature the topics of the inventory policy and the financial policy of a firm are often treated separately except for Li et al. (1997), Buzacott and Zhang (2004), Dada and Hu (2008), Chao et al. (2005) and Chao et al. (2008). (2008) and a few others. As a fact, there is considerable interaction between the inventory policy at operational level and cash flow at finance/accounting level. Thus, we consider a firm or retailer that in each period has to decide on how many units to order taking into account not only inventory on hand but also capital availability and possible borrowing costs.

Regarding inventory flow we make the standard newsvendor assumptions. In particular, at the beginning of each period, the firm decides on an order quantity and the corresponding replenishment order materializes with zero lead time. During the remainder of the period, no further replenishment takes place. At the end of each period, incoming demand is aggregated over that period, and the total period demand draws down the on-hand inventory. In each period, if the demand exceeds the on-hand inventory, then the excess demand is lost subject to amount of lost-sale penalty. All the left-over products at the end of a $t < N$ period (i.e.,
inventory or stock) are carried over to its next period subject to a holding cost. At the final period \( N \) we consider two cases: excess items are either salvaged at a positive value or disposed off at a cost.

Regarding cash flow we make the following assumptions. In each period the firm’s excess cash on hand is deposited in a bank account and yields some interest over each period. Deposited cash may be withdrawn at any time without a withdrawal restriction to finance a replenishment order. However, if at any period the cash on-hand is insufficient to cover the cost of an order, then the firm can borrow an additional amount from an external loan, at some interest rate, to finance the desired order quantity. At the end of the period the firm pays off the bank as much of the outstanding loan as its on-hand cash position allows it, any remaining, positive or negative, cash amount is carried over to the next period at zero additional interest, reward or cost respectively. All interest payments (both firm receivables on bank deposits and firm payables on bank loans) are computed as simple interest over each period. In each period, all cash realized from provisioning the demand of the period is credited to the firm’s cash on hand.

The goal in this paper is to dynamically optimize the order quantities (as the operational decision) and financing cash (as the financial decision) simultaneously in each period so as to maximize the expected value to the retailer of this cash flow based operations, at the end of a finite time horizon.

The main result of the paper is to show that the optimal order policy is determined by a sequence of constants \( \alpha_n \) and \( \beta_n \) for each period which is referred to as \( (\alpha_n, \beta_n) \)-policy. The major results are presented in Theorems 1 and 2.

Our study is closely related to Chao et al. (2008). In the paper by Chao et al. (2008), the authors study a classic dynamic inventory control problem of a self-financing retailer who periodically replenishes its stock from a supplier and sells it to the market subject to random demands. The inventory replenishment decisions of the retailer are constrained by cash flow, which is updated periodically with purchasing and/or selling operations in each period. The retailer’s objective is to maximize its expected terminal wealth at the end of the planning horizon. The authors provide the explicit structure on how the optimal inventory control strategy depends on the cash flow and characterize the optimal replenishment policy as a capital-dependant base stock policy where the base stock level is uniquely determined by the total value of cash and asset at the beginning of the period. Our study differs from Chao et al. (2008) in the following ways: (1) for self-financing vendor, we consider a loan which provides the retailer with flexibility to order more quantity, while Chao et al. (2008) restricts the order quantity subject to its available capital (budget) imposed to the retailer’s decision; (2) Although both models assume lost sales of excess demand, our model has penalty cost incorporated in cost evaluation; (3) Realizing the holding cost is a significant cost component to material flow, our model includes inventory holding cost as an important part of cost function.

The remaining of this paper is organized as follows. In Section 2, a single period model is developed and its optimal policy is derived. Section 3 extends the analysis for multiple period system and derives the optimal policy via dynamic programming approach. Finally, Section 6 concludes the paper.

## 2 The Single Period Model

We first introduce necessary notation and assumptions.

At the beginning of the period, the “asset-cash” state of the system can be summarized by a vector \((x, y)\), where \(x\) denotes the amount of on-hand inventory (number of product units) and \(y\) denotes the amount of product that can be purchased using all the available capital (i.e., \(y\) is the capital position measured in “product units”). Note that, \(X = cx\) and \(Y = yc\) represent respectively value of on-hand inventory and
available cash position available at the beginning of the period. Throughout this paper, we allow $x$ and $y$ to be negative, in which case a negative $x$ represents a backorder quantity and a negative $y$ represents an amount of initial loan: $-Y = -yc$.

Let $D$ denote the single period demand. For simplicity, we assume that $D$ is a non-negative continuous random variable with a probability density function $f(z)$ and cumulative distribution $F(z)$. Let $p$, $c$, $s$, denote respective the selling price, the ordering cost and the salvage price per unit of material. Note that we allow a negative $s$ in which case $s$ represents a disposal cost, per unit, e.g. vehicle tires, etc. Further, let $i$ denote the interest rate for deposits, and $\ell$ the interest rate for a loan. The decision variable is the order quantity $q \geq 0$.

To avoid trivialities we assume that $i < \ell$ and and that it is possible to realize a profit by using a loan, i.e., $(1 + \ell)c < p$. This assumption is equivalently written as:

$$\ell < \frac{p}{c} - 1. \quad (1)$$

Note also, that the above assumptions implies $i < \frac{p}{c} - 1$ since $i < l$, which says that investing on inventory is preferable to depositing all the available capital $Y$ to the bank.

At the beginning of the period it is possible to purchase products with available capital $y$ (when $y = Y/c > 0$) but it is not possible to convert any of the available on hand inventory $x$ into capital. Thus, when at the beginning of a period an order of size $q \geq 0$ is placed while the asset-cash state is $(x,y)$, and if the demand during the period is $D$, then

1. The cash flow from sales of items (the realized revenue from inventory) at the end of the period is given by

$$R(D,q,x) = p \cdot \min\{q + x, D\} + s \cdot [q + x - D]^+$$

$$= p \cdot [q + x - (q + x - D)^+] + s \cdot [q + x - D]^+$$

$$= p(q + x) - (p - s) \cdot [q + x - D]^+ \quad (2)$$

where $[z]^+$ denotes the positive part of real number $z$, and the second equality holds by $\min\{z,t\} = z - [t - z]^+.$

2. The cash flow from capital at the end of the period can be computed when we consider the following two scenarios:

   i) If the order quantity $0 \leq q \leq y$, then the amount $y - q$ will be left in the bank and it will yield a positive flow of $c(y - q)(1 + i)$ at the end of the period.

   ii) Otherwise, if $q > y$ (even if $q = 0 > y$) then a loan amount of $c(q - y)$ will be incurred during the period and it will result in a negative cash flow of $c(q - y)(1 + \ell)$ at the end of the period.

Consequently, the cash flow from the bank (positive or negative) can be written as

$$K(q,y) = c(y - q) \left[ (1 + i)1_{\{q \leq y\}} + (1 + \ell)1_{\{q > y\}} \right] \quad (3)$$

Note that the cash flow from inventory, $R(D,q,x)$, is independent of $y$. Also, the cash flow from capital, $K(q,y)$, is independent of the initial on-hand inventory, $x$ and the demand size $D$. Also, note that the ordering cost, $qc$, has been accounted for in Eq. (3) while the remaining capital, if any, has been invested in the bank and its value at the end of period is given by $K(q,y)$.
Thus, for any given initial state \((x, y)\), the conditional expected value of total asset at the end of the period is given by

\[
G(q, x, y) = E[R(D, q, x)] + K(q, y). \tag{4}
\]

Substituting Eqs.\((2) - (3)\) into Eq. \((4)\) yields

\[
G(q, x, y) = p(q + x) - (p - s) \int_0^{q+x} (q + x - z)f(z)dz \\
+ c \cdot (y - q) \left[ (1 + i)^1_{\{q \leq y\}} + (1 + \ell)^1_{\{q > y\}} \right]. \tag{5}
\]

We next state and prove the following.

**Lemma 1.** The function \(G(q, x, y)\) is continuous in \(q, x\) and \(y\), and it has the following properties.

\[i)\text{ It is concave in } q \in [0, \infty), \text{ for all } x, y \text{ and all } s < p.\]

\[ii)\text{ It is increasing and concave in } x, \text{ for } s \geq 0.\]

\[iii)\text{ It is increasing and concave in } y, \text{ for all } s < p.\]

**Proof.** The continuity follows immediately from Eq. \((5)\). We next prove the concavity via examining the first-order and second-order derivatives. To this end, differentiating Eq. \((5)\) yields via Leibniz’s integral rule

\[
\frac{\partial}{\partial q}G(q, x, y) = \begin{cases} 
p - c \cdot (1 + i) - (p - s)F(q + x) & \text{if } q < y, \\
p - c \cdot (1 + \ell) - (p - s)F(q + x) & \text{if } q > y.
\end{cases} \tag{6}
\]

Therefore, for \(q > y\) or \(q < y\)

\[
\frac{\partial^2}{\partial q^2}G(q, x, y) = -(p - s)f(q + x). \tag{7}
\]

Then the concavity in \(q\) readily follows since \(\frac{\partial^2}{\partial q^2}G(q, x, y) \leq 0\) by Eq. \((7)\).

The increasing property of \(G(q, x, y)\) in \(x\) and \(y\) can be shown by taking the first order derivatives using Eq. \((5)\):

\[
\frac{\partial}{\partial x}G(q, x, y) = pF(q + x) + sF(q + x) > 0, \tag{8}
\]

\[
\frac{\partial}{\partial y}G(q, x, y) = c \cdot \left[ (1 + i)^1_{\{q < y\}} + (1 + \ell)^1_{\{q > y\}} \right] > 0. \tag{9}
\]

The joint concavity of \(G(q, x, y)\) in \(x\) and \(y\) can be established by computing the second order derivatives below using again Eq. \((5)\):

\[
\frac{\partial^2}{\partial x^2}G(q, x, y) = -(p - s)f(q + x) < 0, \tag{10}
\]

\[
\frac{\partial^2}{\partial y^2}G(q, x, y) = 0, \tag{11}
\]

\[
\frac{\partial^2}{\partial x \partial y}G(q, x, y) = 0. \tag{12}
\]

Thus the Hessian matrix is negative semi-definite and the proof is complete. ☐
Remarks.

1. It is important to point out that $G(q,x,y)$ might not increase in $x$ if $s < 0$. In particular, if $s$ represents a disposing cost, i.e., $s < 0$, the right side of Eq. (8) might be negative, which implies that $G(q,x,y)$ is decreasing for some high values of $x$.

For the special case with $s < 0$, it is of interest to locate the critical value, $x'$ such that $G(q,x,y)$ is decreasing for $x > x'$. To this end, we set Eq. (8) to be zero, which yields

$$(p - s)F(q + x) = p. \quad (13)$$

Therefore,

$$x' = F^{-1}\left(\frac{p}{p - s}\right) - q, \quad (14)$$

where $F^{-1}(\cdot)$ is the inverse function of $F(\cdot)$. Eq. (14) shows that a higher disposing cost, $-s$, implies a lower threshold for $x'$ above.

2. Lemma 1 implies that higher values of initial assets, $x$, $y$ or their sum, will yield a higher expected revenue $G(q,x,y)$. Further, for any fixed assets $(x,y)$ there is a unique optimal order quantity $q^*$ such that

$$q^*(x,y) = \arg \max_{q \geq 0} G(q,x,y).$$

We next introduce the critical values of $\alpha$ and $\beta$ as follows:

$$\alpha = F^{-1}(a), \quad (15)$$
$$\beta = F^{-1}(b), \quad (16)$$

where

$$a = \frac{p - c[1 + i]}{p - s}, \quad (17)$$
$$b = \frac{p - c[1 + \ell]}{p - s}. \quad (18)$$

It is easy to see that $a \leq b$, since $0 \leq i \leq \ell$ by assumption. This implies that $\alpha \leq \beta$, since $F^{-1}(z)$ is increasing in $z$. The critical value $\beta$ can be interpreted as the optimal order quantity for the classical Newsvendor problem corresponding to the case of sufficiently large $Y$ of our model, in which case no loan is involved, but the unit “price” $c(1 + i)$ has been inflated to reflect the opportunity cost of cash not invested in the bank at interest $i$. Similarly, $\alpha$ can be interpreted as the optimal order quantity for the classical Newsvendor problem corresponding to the case $Y = 0$ of our model, i.e., all units are purchased by a loan at an interest $\ell$.

Note also that in contrast to the classical Newsvendor model, the critical values $\alpha$ and $\beta$ above, are now functions of the corresponding interest rates and represent opportunity costs that take into account the value of time using the interest factors $1 + i$ and $1 + \ell$.

We can now state and prove the following theorem regarding the optimality of the $(\alpha, \beta)$ ordering policy.
Figure 1: Functional Structure for the Derivative of $G(q, x, y)$ with Respect to $q$

**Theorem 1.** For any given initial cash-asset state $(x, y)$, the optimal order quantity is

$$q^*(x, y) = \begin{cases} 
(\beta - x)^+, & \beta \leq x + y; \\
y, & \alpha \leq x + y < \beta; \\
\alpha - x, & x + y < \alpha,
\end{cases} \quad (19)$$

where $\alpha$ and $\beta$ are given by Eq. (15) and (16), respectively.

**Proof.** For any given initial state $(x, y)$, Lemma 1 implies that there exists a unique optimal order quantity $q^*(x, y)$ such that the profit function $G(q, x, y)$ is maximized. To prove Eq. (19), we investigate the first order derivative of the profit function given by Eq. (6). Figure 1 illustrates its functional structure with respect to three cases for different values of $x + y$.

a) If $x + y < \alpha$, then $G(q, x, y)$ is strictly increasing in $q$ as long as $q + x \leq \alpha$, and decreasing thereafter, while $\partial G(q, x, y)/\partial q = 0$ for $q + x = \alpha$, cf. Figure 1 (a). It follows that in this case the optimal quantity $q^*$ is such that $q^* + x = \alpha$.

b) If $\alpha \leq y < \beta$, then the profit function $G(q, x, y)$ is strictly increasing in $q$ until $q = y$, and decreasing thereafter cf. Figure 1 (b). Then, the optimal quantity is $q^*(x, y) = y$.

c) If $x + y \geq \beta$, then the profit function $G(q, x, y)$ of $x + q$ is strictly increasing until $\beta$, and decreasing thereafter, cf. Figure 1 (c). Then, the optimal quantity after ordering is the one such that $q + x$ is close to $\beta$ as much as it could be. Therefore, the optimal order quantity is $(\beta - x)^+$.

This completes the proof.

Note that the optimal ordering quantity to a classical Newsvendor model [cf. Zipkin (2000) and many others], can be obtained from Theorem 1 as the solution to the extreme case with $i = \ell = 0$ when we obtain the optimal order quantity is given by:

$$\alpha = \beta = F^{-1} \left( \frac{p - c}{p - s} \right).$$

We further elucidate the structure of the $(\alpha, \beta)$ optimal policy below where we discuss the utilization level of the initially available capital $Y$. 


1. **(Over-utilization)** When \( x + y < \alpha \), it is optimal to order \( q^* = \alpha - x = y + (\alpha - x - y) \). In this case \( y = Y/c \) units are bought using all the available fund \( Y \) and the remaining \((\alpha - x - y)\) units are bought using a loan of size: \( c(\alpha - x - y) \).

2. **(Full-utilization)** When \( \alpha \leq x + y < \beta \), we would order \( q^* = y = Y/c \) with all the available fund of \( Y \). In this case, there is no investment in the fund market and no loan.

3. **(Under-utilization)** When \( x + y \geq \beta \), it is optimal to order \( q^* = (\beta - x)^+ \). In this case if in addition \( x < \beta \), we would order \( \beta - x \) using \( c \cdot (\beta - x) \) units of the available fund \( Y \), and invest the remaining cash in the fund market. However, if in addition \( x \geq \beta \) then \( q^* = 0 \) and it is optimal not to order any units and invest all the amount of \( Y \) in the fund market.

The above ideas are illustrated in Figure 2 for the case in which \( x = 0 \), by plotting the optimal order quantity \( q^* \) as a function of \( y \). Note that for \( y \in (0, \alpha) \) there is over utilization of \( y \); for \( y \in [\alpha, \beta) \) there is full utilization of \( y \) and for \( y \in [\beta, \infty) \) there is under utilization of \( y \).

We next define the function

\[
V(x, y) = \max_{q \geq 0} G(q, x, y).
\]

and state and prove the following lemma which will be used in the next section.

**Theorem 2.** For any initial state \((x, y)\),

i) \( V(x, y) \) is given by

\[
V(x, y) = \begin{cases} 
px - (p - s)L(x) + cy(1 + i), & x > \beta; \\
p\beta - (p - s)L(\beta) + c(x + y - \beta)(1 + i), & x \leq \beta, \beta \leq x + y; \\
p(x + y) - (p - s)L(x + y), & \alpha \leq x + y < \beta; \\
p\alpha - (p - s)L(\alpha) + c(x + y - \alpha)(1 + l), & x + y < \alpha,
\end{cases}
\]

where \( L(x) = \int_0^x (x - z)f(z)dz \);

ii) the function \( V(x, y) \) is increasing in \( x \) and \( y \), and jointly concave in \((x, y)\), for \( x, y \geq 0 \).

**Proof.** Part (i) follows from Theorem 1.

For part (ii) the increasing property of \( V \) can be justified straightforwardly. For the concavity of \( V \), note that by Lemma 1, \( G(q, x, y) \) is concave in \( q, x \) and \( y \). Taking the maximization of \( G \) over \( q \) and using Proposition A.3.10 of Zipkin (2000), p436, and Eq. (20) we have that the concavity in \( x \) and \( y \) is preserved and the proof is complete.

From investment perspective, it is of interest to see the possibility of speculation. The following result shows that the operational strategy given in Theorem 1 is of positive value with zero value of investment. Specifically, when the Newsvendor has zero initial inventory assets and capital, i.e., \( x = 0 \) and \( y = 0 \), the optimal Newsvendor operation has a positive expected final asset value.

**Corollary 1.** The following is true

\[
V(0, 0) = (p - s) \int_0^\alpha zf(z)dz > 0.
\]

**Proof.** The result can be readily proved by setting \( x = y = 0 \) in Eq. (21).
Figure 2: The Optimal Order Quantity when \( x = 0 \)

3 The \( N \)-period problem

In this section, we extend the results of the previous section and consider the finite horizon version of the problem, with \( N \geq 2 \) periods. As in the single period, at the beginning of a period \( n = 1, \ldots, N \), let the “asset-cash” state of the system be summarized by a vector \( (x_n, y_n) \), where \( x_n \) denotes the amount of on-hand inventory (number of product units) and \( y_n \) denotes the amount of product that can be purchased using all the available capital (i.e., \( y_n \) is the capital position measured in “product units”). Note again that, \( X_n = c x_n \) and \( Y_n = y_n c \) represent respectively value of on-hand inventory and available cash position, in terms of $, available at the beginning of period \( n \). Let \( q_n \) denote the order quantity the Newsvendor uses in the beginning of period \( n = 1, \ldots, N \). We assume the lead time of replenishment is zero. Throughout all periods \( t = 1, \ldots, N - 1 \), any unsold units are carried over in inventory to be used in subsequent periods subject to a constant holding cost per unit per period. At the end of the horizon, i.e., period \( t = N \), all the leftover inventory (if any) will be salvaged (or disposed of) at a constant price (cost) per unit.

Let \( p_n, c_n, h_n \) denote the selling price, ordering cost and holding cost per unit in period \( n \), respectively. Let \( s \) denote the salvage price (or disposal cost) per unit at the end of period \( N \). Let \( i_n \) and \( \ell_n \), with \( i_n \leq \ell_n \), be the interest rates for deposit and loan in period \( n \), respectively.

Finally, let \( D_n \) denote the demand of period \( n \). We assume that demands of different periods are independent. Let \( f_n(z) \), \( F_n(z) \) denote respectively the probability density function, the cumulative distribution function, of \( D_n \). The system state at the beginning of period \( n \) is characterized by \( (x_n, y_n) \). The order quantity \( q_n = q_n(x_n, y_n) \) is decided at the beginning of period \( n \) as a function of \( (x_n, y_n) \). It is readily shown that the state \( (x_n, y_n) \) process under study is a Markov decision process (MDP) with decision variable \( q_n \) [cf. Ross
Then, the dynamic states of the system are formulated as follows, for \( n = 1, 2, \ldots, N - 1 \)

\[
x_{n+1} = [x_n + q_n - D_n]^+ \\
y_{n+1} = [R_n(D_n, q_n, x_n) + K_n(D_n, q_n, y_n)]/c_{n+1}
\]

where

\[
R_n(D_n, q_n, x_n) = p_n \cdot (x_n + q_n) - (p_n + h_n) [x_n + q_n - D_n]^+ \\
K_n(D_n, q_n, y_n) = c_n \cdot (y_n - q_n) \left[(1 + i_n)1_{\{q_n \leq y_n\}} + (1 + \ell_n)1_{\{q_n > y_n\}} \right]
\]

In particular, at the end of period \( N \), the revenue from inventory is

\[
R_N(D_N, q_N, x_N) = p_N \cdot \min\{x_N + q_N, D_N\} - h_N [x_N + q_N - D_N]^+ = p_N[q_N + x_N - (p_N - s)[q_N + x_N - D_N]^+ + (1 + i_N)1_{\{q_N \leq y_N\}} + (1 + \ell_N)1_{\{q_N > y_N\}} \]

where \( h_N = -s \), and the revenue from the bank is

\[
K_N(D_N, q_N, y_N) = c_N \cdot (y_N - q_N) \left[(1 + i_N)1_{\{q_N \leq y_N\}} + (1 + \ell_N)1_{\{q_N > y_N\}} \right]
\]

For a risk-neutral Newsvendor, the objective is to maximize the expected value of the total asset at the end of period \( N \), that is,

\[
\max_{q_1, q_2, \ldots, q_N} \mathbb{E}[R_N(D_N, q_N, x_N) + K_N(D_N, q_N, y_N)]
\]

where \( x_N \) and \( y_N \) are sequentially determined by decision variables \( q_n \), \( n \leq N \). Accordingly, we have the following dynamic programming formulation:

\[
V_n(x_n, y_n) = \sup_{q_n \geq 0} \mathbb{E}[V_{n+1}(x_{n+1}, y_{n+1})|x_n, y_n], \quad n = 1, 2, \ldots, N - 1
\]

where the expectation is taken with respect to \( D_n \), and \( x_{n+1}, y_{n+1} \) are given by Eqs. (32), (33), respectively. For the final period \( N \), we have:

\[
V_N(x_N, y_N) = \sup_{q_N \geq 0} \mathbb{E}[R_N(D_N, q_N, x_N) + K_N(D_N, q_N, y_N)]
\]

Note that for period \( N \), the optimal solution is given by Theorem 1.

In the sequel it is convenient to work with the quantities \( p'_n = p_n/c_{n+1} \), \( h'_n = h_n/c_{n+1} \) and \( c'_n = c_n/c_{n+1} \) and to take \( z_n = x_n + q_n \) as the decision variable instead of \( q_n \). Here, \( z_n \) refers to the available inventory after replenishment, and it is restricted by \( z_n \geq x_n \) for each period \( n \).

Then, the DP model defined by Eqs. (28)-(29) can be presented as:

\[
V_n(x_n, y_n) = \sup_{z_n \geq x_n} \mathbb{E}[V_{n+1}(x_{n+1}, y_{n+1})|x_n, y_n], \quad n = 1, 2, \ldots, N - 1
\]

\[
V_N(x_N, y_N) = \sup_{z_N \geq x_N} \mathbb{E}[R_N(D_N, q_N, x_N) + K_N(D_N, q_N, y_N)]
\]

where the cash-asset states dynamics are given by

\[
x_{n+1} = [z_n - D_n]^+; \\
y_{n+1} = p'_n \cdot z_n - (p'_n + h'_n) [z_n - D_n]^+ + c'_n \cdot (x_n + y_n - z_n) \left[(1 + i_n)1_{\{z_n \leq x_n \} + (1 + \ell_n)1_{\{z_n > x_n \}} \right].
\]
Note also that the DP equations can be presented as:

\[ V_n(x_n, y_n) = \max_{z_n \geq x_n} G_n(z_n, x_n, y_n), \]

where

\[ G_n(z_n, x_n, y_n) = E[V_{n+1}(x_{n+1}, y_{n+1})|x_n, y_n], \]

for \(0 \leq x_n \leq z_n\).

The Hessian Matrix (if it exists) of a function \(G = G(x, y)\) will be denoted by \(H^G(x, y)\). For example, the Hessian Matrix of \(V_n(x_n, y_n)\) is denoted by

\[ H^{V_n}(x_n, y_n) = \begin{bmatrix}
\frac{\partial^2 V_n}{\partial x_n \partial y_n} & \frac{\partial^2 V_n}{\partial x_n \partial y_n} \\
\frac{\partial^2 V_n}{\partial y_n \partial x_n} & \frac{\partial^2 V_n}{\partial y_n \partial y_n}
\end{bmatrix}. \]

We first state and prove the following result.

**Lemma 2.** For \(n = 1, 2, \ldots, N\),

1. The function \(G_n(z_n, x_n, y_n)\) is increasing in \(x_n\) and \(y_n\), and it is concave in \(z_n\) and \((x_n, y_n)\).
2. The function \(V_n(x_n, y_n)\) is increasing and concave in \((x_n, y_n)\).

**Proof.** We prove the result by induction. In particular, in each iteration, we will prove properties (1) and (2) by recursively repeating two steps: deducing the property of \(G_n\) from the property of \(V_{n+1}\) and obtaining the property of \(V_n\) from the property of \(G_n\). Throughout the proof, for a matrix or a vector \(w\), we denote its transpose by \(w^T\).

1. For \(V_N\), we have a one period problem. In this case, the result for function \(G_N(z_N, x_N, y_N)\) is obtained by Lemma 1 with \(z_n = x_n + q_n\) and the result for \(V_N(x_N, y_N)\) is given by Lemma ?? of the single period problem.

2. For \(n = 1, 2, \ldots, N - 1\), we prove the results recursively using the following two steps:

**Step 1.** We show that \(G_n(z_n, x_n, y_n)\) is increasing in \(y_n\) and concave in \(z_n\) and \((x_n, y_n)\) if \(V_{n+1}(x_{n+1}, y_{n+1})\) is increasing in \(y_{n+1}\) and concave \((x_{n+1}, y_{n+1})\).

We first compute the partial derivatives that will be used in the sequel for any given \(z_n\). From Eq. (32) we have:

\[ \frac{\partial x_{n+1}}{\partial z_n} = \frac{\partial x_{n+1}}{\partial x_n} = 1_{\{z_n > D_n\}}, \quad (37) \]

\[ \frac{\partial x_{n+1}}{\partial y_n} = 0. \quad (38) \]

Similarly, from Eq. (33) we obtain:

\[ \frac{\partial y_{n+1}}{\partial z_n} = p'_n 1_{\{z_n < D_n\}} - h'_n 1_{\{z_n > D_n\}} - c_n \left[ (1 + i_n) 1_{\{z_n < x_n + y_n\}} + (1 + \ell_n) 1_{\{z_n > x_n + y_n\}} \right], \quad (39) \]

and

\[ \frac{\partial y_{n+1}}{\partial x_n} = p'_n 1_{\{z_n < D_n\}} - h'_n 1_{\{z_n > D_n\}}, \quad (40) \]

\[ \frac{\partial y_{n+1}}{\partial y_n} = c'_n \left[ (1 + i_n) 1_{\{z_n < x_n + y_n\}} + (1 + \ell_n) 1_{\{z_n > x_n + y_n\}} \right]. \quad (41) \]

From Eqs. (37)-(41), it readily follows that the second order derivatives of \(x_{n+1}\) and \(y_{n+1}\) with respect to \(z_n, x_n\) and \(y_n\) are all zero.
In the sequel we interchange differentiation and integration in several places, this is justified by the *Lebesgue’s Dominated Convergence Theorem* [cf. Bartle (1995)].

The increasing property of function $G_n(z_n, x_n, y_n)$ in $y_n$ can be established by taking the first order derivative of Eq. (5) with respect to $y_n$. Then,

$$
\frac{\partial}{\partial y_n} G_n(z_n, x_n, y_n) = E \left[ \frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial x_{n+1}} \frac{\partial x_{n+1}}{\partial y_n} + \frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial y_{n+1}} \frac{\partial y_{n+1}}{\partial y_n} \right]
$$

$$
= E \left[ \frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial y_{n+1}} \frac{\partial y_{n+1}}{\partial y_n} \right] \geq 0,
$$

where the second equality holds since $\frac{\partial x_{n+1}}{\partial y_n} = 0$, by Eq. (38), and the inequality holds by Eq. (41) and the induction hypothesis that $V_{n+1}$ is increasing in $y_{n+1}$.

To prove the concavity of $G_n(z_n, x_n, y_n)$ in $z_n$, we next show that $\frac{\partial^2 G_n(z_n, x_n, y_n)}{\partial z_n^2} \leq 0$. To this end we compute the first and second order derivatives as follows:

$$
\frac{\partial}{\partial z_n} G_n(z_n, x_n, y_n) = E \left[ \frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial x_{n+1}} \frac{\partial x_{n+1}}{\partial z_n} + \frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial y_{n+1}} \frac{\partial y_{n+1}}{\partial z_n} \right]
$$

(42)

and

$$
\frac{\partial^2}{\partial z_n^2} G_n(z_n, x_n, y_n) = E \left[ \frac{\partial x_{n+1}}{\partial z_n}, \frac{\partial y_{n+1}}{\partial z_n} \right] \cdot H^{V_{n+1}} \cdot \left[ \frac{\partial x_{n+1}}{\partial z_n}, \frac{\partial y_{n+1}}{\partial z_n} \right]^T,
$$

(43)

where

$$
H^{V_{n+1}} = \begin{bmatrix}
\frac{\partial^2 V_{n+1}}{\partial x_{n+1} \partial x_{n+1}} & \frac{\partial^2 V_{n+1}}{\partial x_{n+1} \partial y_{n+1}} \\
\frac{\partial^2 V_{n+1}}{\partial y_{n+1} \partial x_{n+1}} & \frac{\partial^2 V_{n+1}}{\partial y_{n+1} \partial y_{n+1}}
\end{bmatrix}
$$

is the Hessian matrix of $V_{n+1}(x_{n+1}, y_{n+1})$. Now the induction hypothesis regarding $V_{n+1}$, implies that $H^{V_{n+1}}$ is negative semi-definite, i.e., $w H^{V_{n+1}} w^T \leq 0$ for any 1 by 2 vector $w$, thus the result follows using Eq. (43).

To prove the concavity of $G_n(z_n, x_n, y_n)$ in $(x_n, y_n)$, we compute its Hessian matrix and show that it is negative semi-definite. To this end we compute the first and second order partial derivatives of $V_{n+1}$ with respect to $x_n$ and $y_n$ for any given $z_n$, as follows:

$$
\frac{\partial G_n}{\partial x_n} = E \left[ \frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial x_{n+1}} \frac{\partial x_{n+1}}{\partial x_n} + \frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial y_{n+1}} \frac{\partial y_{n+1}}{\partial x_n} \right],
$$

(44)

$$
\frac{\partial G_n}{\partial y_n} = E \left[ \frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial x_{n+1}} \frac{\partial x_{n+1}}{\partial y_n} + \frac{\partial V_{n+1}(x_{n+1}, y_{n+1})}{\partial y_{n+1}} \frac{\partial y_{n+1}}{\partial y_n} \right],
$$

(45)

and

$$
\frac{\partial^2 G_n}{\partial x_n^2} = E \left[ J_{n+1}^{(1)} \right],
$$

(46)

$$
\frac{\partial^2 G_n}{\partial x_n \partial y_n} = E \left[ J_{n+1}^{(2)} \right],
$$

(47)

$$
\frac{\partial^2 G_n}{\partial y_n^2} = E \left[ J_{n+1}^{(3)} \right],
$$

(48)
where, by Eqs. (37) - (41), the terms involved with the second order derivatives of $x_{n+1}$ and $y_{n+1}$ with respect to $x_n$ and $y_n$ have vanished and where for notational convenience we have defined:

$$J^{(1)}_{n+1} = \left[ \frac{\partial x_{n+1}}{\partial x_n}, \frac{\partial y_{n+1}}{\partial x_n} \right] \cdot H^{V_{n+1}}, \left[ \frac{\partial x_{n+1}}{\partial x_n}, \frac{\partial y_{n+1}}{\partial x_n} \right]^T,$$

$$J^{(2)}_{n+1} = \left[ \frac{\partial x_{n+1}}{\partial y_n}, \frac{\partial y_{n+1}}{\partial y_n} \right] \cdot H^{V_{n+1}}, \left[ \frac{\partial x_{n+1}}{\partial y_n}, \frac{\partial y_{n+1}}{\partial y_n} \right]^T,$$

$$J^{(3)}_{n+1} = \left[ \frac{\partial x_{n+1}}{\partial y_n}, \frac{\partial y_{n+1}}{\partial y_n} \right] \cdot H^{V_{n+1}}, \left[ \frac{\partial x_{n+1}}{\partial y_n}, \frac{\partial y_{n+1}}{\partial y_n} \right]^T.$$

Thus, the Hessian matrix of $G_n$ in terms of $(x_n, y_n)$ is:

$$H^{G_n} (x_n, y_n) = \begin{bmatrix} \frac{\partial^2 G_n}{\partial x_n \partial x_n} & \frac{\partial^2 G_n}{\partial x_n \partial y_n} \\ \frac{\partial^2 G_n}{\partial y_n \partial x_n} & \frac{\partial^2 G_n}{\partial y_n \partial y_n} \end{bmatrix},$$

with its elements given by Eqs. (46) - (48). To prove it is negative semi-definite, we consider the quadratic function below for any real $z$ and $t$,

$$[z, t] \cdot H^{G_n} \cdot [z, t]^T = \frac{\partial^2 G_n}{\partial x_n \partial x_n} z^2 + 2 \frac{\partial^2 G_n}{\partial x_n \partial y_n} \cdot z \cdot t + \frac{\partial^2 G_n}{\partial y_n \partial y_n} \cdot t^2$$

$$= E \left[ J^{(1)}_{n+1} z^2 + 2 J^{(2)}_{n+1} z t + J^{(3)}_{n+1} t^2 \right].$$

If we define the $1 \times 2$ vector $w = w(n, z, t)$ as follows:

$$w = z \cdot \left[ \frac{\partial x_{n+1}}{\partial x_n}, \frac{\partial y_{n+1}}{\partial x_n} \right] + t \cdot \left[ \frac{\partial x_{n+1}}{\partial y_n}, \frac{\partial y_{n+1}}{\partial y_n} \right],$$

then Eq. (50) can be further written as

$$[z, t] \cdot H^{G_n} \cdot [z, t]^T = E \left[ w \cdot H^{V_{n+1}} \cdot w^T \right].$$

Since by the induction hypothesis $H^{V_{n+1}}$ is negative semi-definite, we have

$$w \cdot H^{V_{n+1}} \cdot w^T \leq 0$$

and this implies that the right side of Eq. (52) is non-positive. Thus, the proof for Step 1 is complete.

**Step 2.** We show that $V_n(x_n, y_n)$ is concave in $(x_n, y_n)$ if $G_n(z_n, x_n, y_n)$ is concave in $z_n$ and $(x_n, y_n)$.

Since $G_n(z_n, x_n, y_n)$ is concave in $z_n$ and $(x_n, y_n)$, then $V_n(x_n, y_n) = \max_{z_n \geq x_n} G_n(z_n, x_n, y_n)$ is concave in $x_n, y_n$ by the fact that concavity is reserved under maximization [cf. Proposition A.3.10 in Zipkin (2000), p436].

Thus the induction proof is complete.

We next present and prove the main result of this section.

**Theorem 3.** *(The $(\alpha_n, \beta_n)$ ordering policy).

For period $n = 1, 2, \cdots, N$ with given state $(x_n, y_n)$ at the beginning of the period, there exist positive constants $\alpha_n = \alpha_n(x_n, y_n)$ and $\beta_n = \beta_n(x_n, y_n)$ with $\alpha_n \leq \beta_n$, which define the optimal order quantity as follows:

$$q^*(x_n, y_n) = \begin{cases} (\beta_n - x_n)^+, & x_n + y_n \geq \beta_n; \\
y_n, & \alpha_n \leq x_n + y_n < \beta_n; \\
\alpha_n - x_n, & x_n + y_n < \alpha_n. \end{cases}$$

(53)
Further, $\alpha_n$ is uniquely identified by
\[
E \left[ \left( \frac{\partial V_{n+1}}{\partial x_{n+1}} - (p_n' + h_n') \frac{\partial V_{n+1}}{\partial y_{n+1}} \right) 1_{\{x_n > D_n\}} \right] = [c_n'(1 + \ell_n) - p_n'] E \left[ \frac{\partial V_{n+1}}{\partial y_{n+1}} \right],
\]
and $\beta_n$ is uniquely identified by
\[
E \left[ \left( \frac{\partial V_{n+1}}{\partial x_{n+1}} - (p_n' + h_n') \frac{\partial V_{n+1}}{\partial y_{n+1}} \right) 1_{\{y_n > D_n\}} \right] = [c_n'(1 + i_n) - p_n'] E \left[ \frac{\partial V_{n+1}}{\partial y_{n+1}} \right].
\]

**Proof.** Given state $(x_n, y_n)$ at the beginning of period $n = 1, 2, \ldots, N$, we consider the equation:
\[
\frac{\partial}{\partial z_n} G_n(z_n, x_n, y_n) = 0,
\]
where $\partial G_n(z_n, x_n, y_n)/\partial z_n$ is given by Eq. (42). Substituting Eqs. (37) and (39) into Eq. (42) we consider the following cases:

(1) for $z_n \leq x_n + y_n$,
\[
\frac{\partial G_n}{\partial z_n} = E \left[ \frac{\partial V_{n+1}}{\partial x_{n+1}} 1_{\{x_n > D_n\}} + \frac{\partial V_{n+1}}{\partial y_{n+1}} (p_n' 1_{\{z_n < D_n\}} - h_n' 1_{\{z_n > D_n\}} - c_n'(1 + i_n)) \right]
\]

(2) for $z_n > x_n + y_n$,
\[
\frac{\partial G_n}{\partial z_n} = E \left[ \frac{\partial V_{n+1}}{\partial x_{n+1}} 1_{\{z_n > D_n\}} + \frac{\partial V_{n+1}}{\partial y_{n+1}} (p_n' 1_{\{z_n < D_n\}} - h_n' 1_{\{z_n > D_n\}} - c_n'(1 + \ell_n)) \right]
\]

where for each case above, random variables $x_{n+1}$ and $y_{n+1}$ within the expectations are given by Eqs. (32) and (33), respectively.

The results follow easier by setting the right sides of Eqs. (57) and (58) equal to zero and simple simplifications. Note that $\partial G_n(z_n, x_n, y_n)/\partial z_n$ is monotonically decreasing in $z_n$ due to its concavity shown in part (1) of Lemma 2, therefore, there are unique solutions to these equations.

Theorem 3 establishes that the optimal ordering policy is determined by two threshold values. More importantly, these two threshold values $\alpha_n$ and $\beta_n$ can be obtained recursively by solving the implicit equations Eqs. (54) and (55), respectively. Given the current state of computer technology, these calculations can be easily done and the results can be implemented in practice.

**Remark.** The study of Chao et al. (2008) assumes that borrowing is not allowed and thus the Newsvendor is firmly limited to order at most $y_n$ units for period $n$. For this model, it was shown that the optimal policy is determined, in each period, by one-critical value. Our results presented in Theorem 3 contain this study as a special case. This can be seen if we set $l_n$ to be sufficiently large. In this case, $\alpha_n$ becomes zero and $\beta_n$ is the critical value of Chao et al. (2008).

**Corollary 2.** For any period $n = 0, 1, 2, \ldots, N$ and its initial state $(x_n, y_n)$, the following results hold.
(i) For $n < N$, the critical constants of $\alpha_n$ and $\beta_n$ are only determined by $x_n + y_n$, i.e., they are of the form: $\alpha_n = \alpha_n(x_n + y_n)$ and $\beta_n = \beta_n(x_n + y_n)$. But for the last period $N$, $\alpha_N$ and $\beta_N$ are independent of $x_n$ and $y_n$.
(ii) Further, $\alpha_n(x_n + y_n)$ and $\beta_n(x_n + y_n)$ are both decreasing in $x_n + y_n$. 

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Proof. For period $N$, the independence of $x_N$ or $y_N$ is obvious since this is a single period. For period $n < N$, let us revisit Eqs. (57) and (58). Note that $x_{n+1}$ is independent of $(x_n, y_n)$ by Eq. (32) while $y_{n+1}$ is dependent of $x_n + y_n$ by Eq. (33). Therefore, $\alpha_n$ and $\beta_n$ implicitly given by Eqs. (57) and (58) are dependent of $x_n + y_n$ only, and thus completes the proof for part (i). To prove part (ii), we may increase $x_n + y_n$, then for any $D_n$ and $z_n$, $y_{n+1}$ increases accordingly by Eq. (33). Note further that $V_{n+1}$ is concave in $y_{n+1}$ in view of Lemma 2. Hence, the derivative term $\partial V_{n+1}/\partial y_{n+1}$ decreases while $x_n + y_n$ increases. To maintain the equalities in Eqs. (57) and (58), we need reduce $\alpha_n$ and $\beta_n$, respectively. This finally completes the proof.

Theorem 4. For the stationary case in which for all $n = 1, \ldots, N$ we have $F_n = F$, $p_n = p$, $c_n = c$, $p_n = p$, $\ell_n = \ell$, $i_n = i$, (and $p'_n = p', h'_n = h', c'_n = c'$) the following is true. If the total asset values $x_n + y_n = x + y$ are identical for each period $n$, then the optimal $(\alpha_n, \beta_n)$ ordering policy satisfies:

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_N;$$
$$\beta_1 \geq \beta_2 \geq \cdots \geq \beta_N,$$

where $\alpha_n = \alpha_n(x + y)$, $\beta_n = \beta_n(x + y)$ and

$$\alpha_N = F^{-1}\left(\frac{p - c(1 + \ell)}{p - s}\right),$$
$$\beta_N = F^{-1}\left(\frac{p - c(1 + \ell)}{p - s}\right).$$

Proof. In what follows, we only prove $\alpha_n \geq \alpha_{n+1}$. A similar argument can be applied to prove $\beta_n \geq \beta_{n+1}$. In light of Eq. (54), $\alpha_n$ is uniquely determined as the solution to the following equation

$$E\left[\frac{\partial V_{n+1}}{\partial x_{n+1}} 1_{(\alpha_n > D_n)} + \frac{\partial V_{n+1}}{\partial y_{n+1}} (p'_n 1_{(\alpha_n < D_n)} - h'_n 1_{(\alpha_n > D_n)} - c'(1 + \ell_n))\right] = E\left[\frac{\partial V_{n+1}}{\partial x_{n+1}} 1_{(\alpha_n > D)} + \frac{\partial V_{n+1}}{\partial y_{n+1}} (p' 1_{(\alpha_n < D)} - h' 1_{(\alpha_n > D)} - c'(1 + \ell))\right] = 0,$$  \hspace{1cm} (59)

where the first equality hold by the stationary assumption. To complete the proof, we next show that for any $x$ and $y$, the following inequalities hold.

$$\frac{\partial V_n(x, y)}{\partial y} \geq \frac{\partial V_{n+1}(x, y)}{\partial y};$$  \hspace{1cm} (60)
$$\frac{\partial V_n(x, y)}{\partial x} \leq \frac{\partial V_{n+1}(x, y)}{\partial x}.$$  \hspace{1cm} (61)

Inequalities (60) and (61) can be established using algebra and induction, however we think the following intuitive explanation is worth stating instead. First note that they can be interpreted respectively as the statement(s): the marginal contribution of capital asset $y$ (inventory asset $x$) of period $n$ is greater (less) than that of period $n + 1$. This is due to the time value of the capital asset, since in period $n$, one may put all the capital $y$ in the savings account to obtain a return of $y(1 + i)$ (or hold $x$ subject to holding cost). Therefore, capital asset $y$ (inventory asset $x$) in period $n$ has more (respectively less) value for the same amount in period $n + 1$. The marginal contribution of $y$ (respectively $x$) in period $n$ is no less (respectively no greater) than that in period $n + 1$.

In light of Eq. (59), one can write the following for period $n - 1$,

$$E\left[\frac{\partial V_n}{\partial x_{n-1}} 1_{(\alpha_{n-1} > D)} \right] + E\left[\frac{\partial V_n}{\partial y_{n}} (p' 1_{(\alpha_{n-1} < D)} - h' 1_{(\alpha_{n-1} > D)} - c'(1 + \ell))\right] = 0.$$

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Consequently, To prove $\alpha_{n-1} \geq \alpha_n$, we follow a contradiction argument. Suppose $\alpha_{n-1} < \alpha_n$, then by Eq. (61) one has

$$\mathbb{E}\left[\frac{\partial V_{n+1}}{\partial x_{n+1}} 1_{\{\alpha_n > D\}}\right] \geq \mathbb{E}\left[\frac{\partial V_n}{\partial x_n} 1_{\{\alpha_{n-1} > D\}}\right],$$

which implies

$$\mathbb{E}\left[\frac{\partial V_{n+1}}{\partial y_{n+1}} (p' 1_{\{\alpha_n < D\}} - h' 1_{\{\alpha_n > D\}} - c'(1 + \ell))\right] \leq \mathbb{E}\left[\frac{\partial V_n}{\partial y_n} (p' 1_{\{\alpha_{n-1} < D\}} - h' 1_{\{\alpha_{n-1} > D\}} - c'(1 + \ell))\right] \leq 0.$$ 

by Eq.(60), the above is contradict with $\alpha_{n-1} < \alpha_n$ and this completes the proof.

Finally, period $N$ can be treated as a single period problem and consequently $\alpha_N$ and $\beta_N$ can be obtained by Eqs. (15)-(18).

4 Myopic Policies and Threshold Bounds

As shown by Theorem 3, there is a complex computation involved in the calculation of $\alpha_n$ and $\beta_n$. In what follows, we study two myopic ordering policies that are relatively simple to implement. Such myopic policies optimize a given objective function with respect to any single period and ignore multi-period interactions and cumulative effects. We introduce two types of myopic policies. Myopic policy (I) assumes the associated cost for the leftover inventory $\hat{s}_n$ is only the holding cost, i.e., $\hat{s}_n = -h_n$. Myopic policy (II) assumes that the leftover inventory cost $\tilde{s}_n$ is not only the holding cost but it also includes its value in the next period, i.e., $\tilde{s}_n = c_{n+1} - h_n$. It is shown that myopic policy (I) (respectively myopic policy (II) ) corresponds to a policy of section 3 that uses lower bounds, $\hat{\alpha}_n$ and $\hat{\beta}_n$ (respectively upper bounds, $\tilde{\alpha}_n$ and $\tilde{\beta}_n$) for the two threshold values, $\alpha_n$ and $\beta_n$.

4.1 Myopic Policy (I) and Lower Threshold Bounds

Myopic policy (I) is the one period optimal policy obtained when we change the periodic cost structure by assuming that only the holding cost is assessed for any leftover inventory i.e., we assume the following modified “salvage value” cost structure:

$$\hat{s}_n = \begin{cases} -h_n, & n < N, \\ s, & n = N. \end{cases}$$

(63)

Let further,

$$\hat{\alpha}_n = \frac{p_n - c_n [1 + \ell_n]}{p_n - \hat{s}_n};$$

(64)

$$\hat{\beta}_n = \frac{p_n - c_n [1 + i_n]}{p_n - \hat{s}_n};$$

(65)

and the corresponding critical values are respectively given by

$$\hat{\alpha}_n = F_n^{-1}(\hat{\alpha}_n);$$

(66)

$$\hat{\beta}_n = F_n^{-1}(\hat{\beta}_n).$$

(67)
For $n = 1, \ldots, N$, the order quantity below defines the myopic policy (I):

$$
\hat{q}_n(x_n, y_n) = \begin{cases} 
(\hat{\beta}_n - x_n)^+, & x_n + y_n \geq \hat{\beta}_n; \\
y_n, & \hat{\alpha}_n \leq x_n + y_n < \hat{\beta}_n; \\
\hat{\alpha}_n - x, & x + y < \hat{\alpha}_n.
\end{cases}
$$  \hspace{1cm} (68)

The next theorem establishes the lower bound properties of the myopic policy (I).

**Theorem 5.** The following are true:

i) For the last period $N$, $\alpha_N = \hat{\alpha}_N$ and $\beta_N = \hat{\beta}_N$.

ii) For any period $n = 1, 2, \ldots, N - 1$,

$$
\alpha_n \geq \hat{\alpha}_n,
$$

$$
\beta_n \geq \hat{\beta}_n.
$$

**Proof.** We only prove the result for $\alpha_n$. The same argument can be applied to prove the result for $\beta_n$.

In view of Eq. (57), $\alpha_n$ is uniquely given as the solution to the equation below,

$$
E \left[ \frac{\partial V_{n+1}}{\partial y_{n+1}} \left( p_n' \mathbf{1}_{\{\alpha_n < D_n\}} - h_n' \mathbf{1}_{\{\alpha_n > D_n\}} - c_n'(1 + \ell_n) \right) \right] = -E \left[ \frac{\partial V_{n+1}}{\partial x_{n+1}} \mathbf{1}_{\{\alpha_n > D_n\}} \right].
$$  \hspace{1cm} (69)

Since $\frac{\partial V_{n+1}}{\partial x_{n+1}} \geq 0$ by Lemma 2 part (2), the equation above is negative, which implies

$$
E \left[ \frac{\partial V_{n+1}}{\partial y_{n+1}} \left( p_n' \mathbf{1}_{\{\alpha_n < D_n\}} - h_n' \mathbf{1}_{\{\alpha_n > D_n\}} - c_n'(1 + \ell_n) \right) \right] \leq 0.
$$  \hspace{1cm} (70)

Further note that for any realization of demand $D_n = d > 0$, the two terms of the left hand side of Eq. (70):

$$
\frac{\partial V_{n+1}(x_{n+1}(d), y_{n+1}(d))}{\partial y_{n+1}(d)}
$$

and

$$
p_n' \mathbf{1}_{\{\alpha_n < d\}} - h_n' \mathbf{1}_{\{\alpha_n > d\}} - c_n'(1 + \ell_n)
$$

are both increasing in $d$. Specifically, the first term is increasing by the concavity of $V_{n+1}$ [cf. Lemma 2 part (2)] and Eq. (33). Then, by Lemma 3 and Eq. (70), one has,

$$
E \left[ \frac{\partial V_{n+1}}{\partial y_{n+1}} \right] E \left[ p_n' \mathbf{1}_{\{\alpha_n < D_n\}} - h_n' \mathbf{1}_{\{\alpha_n > D_n\}} - c_n'(1 + \ell_n) \right] \leq 0
$$  \hspace{1cm} (71)

Since $\frac{\partial V_{n+1}}{\partial y_{n+1}} \geq 0$ by Lemma 2 part (2), the above inequality implies

$$
E \left[ p_n \mathbf{1}_{\{\alpha_n < D_n\}} - h_n \mathbf{1}_{\{\alpha_n > D_n\}} - c_n(1 + \ell_n) \right] \leq 0
$$

which, after simple algebra, is equivalent to

$$
p_n - c_n \cdot (1 + \ell_n) - (p_n - s_n) F_n(\alpha_n) \leq 0.
$$

The above further simplifies to

$$
F_n(\alpha_n) \geq \frac{p_n - c_n \cdot (1 + \ell_n)}{p_n - s_n}.
$$

By Eqs. (64) and (66), the right hand side in the above inequality is $F_n(\hat{\alpha}_n)$. Thus, we have $F_n(\alpha_n) \geq F_n(\hat{\alpha}_n)$, which completes the proof for $\alpha_n \geq \hat{\alpha}_n$ by the increasing property of $F_n(\cdot)$.
4.2 Myopic Policy (II) and Upper Bounds

Myopic policy (II) is the one period optimal policy obtained when we change the periodic cost structure by assuming that not only the holding cost is assessed but also the cost in the next period for any leftover inventory i.e., we assume the following modified “salvage value” cost structure:

\[
\tilde{s}_n = \begin{cases} 
  c_{n+1} - h_n, & n < N; \\
  s, & n = N. 
\end{cases}
\]  

(72)

One can interpret the new salvage values \(\tilde{s}_n\) of Eq. (72) as representing a fictitious income from inventory liquidation (or pre-salvage at full current cost) at the beginning of the next period \(n + 1\), i.e., it corresponds to the situation that the Newsvendor can salvage inventory at the price \(c_{n+1}\) at the beginning of the period \(n+1\). Note that the condition \(c_n(1+\ell_n) + h_n \geq c_{n+1}\) is required if inventory liquidation is allowed. Otherwise, the Newsvendor will stock up at an infinite level and sell them off at the beginning of period \(n + 1\). Such speculation is eliminated by the aforementioned condition.

Let further,

\[
\tilde{a}_n = \frac{p_n - c_n[1 + \ell_n]}{p_n - \tilde{s}_n}, \\
\tilde{b}_n = \frac{p_n - c_n[1 + i_n]}{p_n - \tilde{s}_n},
\]  

(73)

(74)

and the corresponding critical values which are given by

\[
\tilde{\alpha}_n = F_n^{-1}(\tilde{a}_n), \\
\tilde{\beta}_n = F_n^{-1}(\tilde{b}_n).
\]  

(75)

(76)

For \(n = 1, \ldots, N\), the order quantity below defines the myopic policy (II):

\[
\tilde{q}_n(x_n, y_n) = \begin{cases} 
  (\tilde{\beta}_n - x_n)^+, & x_n + y_n \geq \tilde{\beta}_n; \\
  y_n, & \tilde{\alpha}_n \leq x_n + y_n < \tilde{\beta}_n; \\
  \tilde{\alpha}_n - x, & x + y < \tilde{\alpha}_n.
\end{cases}
\]  

(77)

Let \(V_n(x_n, y_n)\) denote the optimal expected future value when the inventory liquidation option is available only at the beginning of period \(n + 1\) (but not the rest of the periods \(n + 2, \ldots, N\)) given the initial state \((x_n, y_n)\) of period \(n\). For notational simplicity let \(\xi_{n+1} = \xi_{n+1}(x_n, y_n, z_n, D_n) = x_{n+1} + y_{n+1}\) represent the total capital and inventory asset value in period \(n + 1\) when the Newsvendor orders \(z_n \geq x_n\) in state \((x_n, y_n)\) and the demand is \(D_n\).

Prior to giving the upper bounds of \(\alpha_n\) and \(\beta_n\), we present the following result.

**Proposition 1.** For any period \(n\) and given its initial state \((x_n, y_n)\), function \(V_n(x_n - d, y_n + d)\) is increasing in \(d\) where \(0 \leq d \leq x\).

**Proof.** It is sufficient to prove that for an arbitrarily small value of \(d > 0\), \(V_n(x_n, y_n) \leq V_n(x_n - d, y_n + d)\). To this end, consider the initial state to be \((x_n - d, y_n + d)\). In this case, the Newsvendor can always purchase \(d\) units without any additional cost to reset the initial state to be \((x_n, y_n)\). This means \(V_n(x_n, y_n) \leq V_n(x_n - d, y_n + d)\), and thus completes the proof.

\[\square\]
In view of Proposition 1, \( V_n^L \) can be written as

\[
V_n^L(x_n, y_n) = \max_{z_n \geq x_n} \mathbb{E}[V_{n+1}(0, \xi_{n+1}) | x_n, y_n].
\]  
(78)

It is straightforward to show \( \mathbb{E}[V_{n+1}(0, \xi_{n+1}) | x_n, y_n] \) is concave in \( z_n \). Therefore, \( V_n^L \) has an optimal policy determined by a sequence of two threshold values \( \alpha_n^L \) and \( \beta_n^L \).

**Proposition 2.** The following are true: \( \alpha_n^L \geq \alpha_n \) and \( \beta_n^L \geq \beta_n \), for all \( n \).

We omit a rigorous (by contradiction) mathematical proof of the above proposition and instead we provide the following intuitively clear explanation that holds for both \( \alpha_n \) and \( \beta_n \). Note that inventory liquidation at period \( n+1 \) provides the Newsvendor with more flexibility i.e., the Newsvendor can liquidate the initial inventory \( x_{n+1} \) into cash so that the Newsvendor holds cash \( \xi_{n+1} = x_{n+1} + y_{n+1} \) only. Further, note that the Newsvendor will chose to stock up to a higher level of inventory when liquidation is allowed. Indeed, if the Newsvendor ordered more in period \( n \), all the leftover inventory after satisfying the demand \( D_n \) can be salvaged at full cost \( c_{n+1} \) at the beginning of the next period \( n+1 \). In other words, the Newsvendor will take the advantage of inventory liquidation to stock a higher level than that corresponding to the case in which liquidation is not allowed in the current period \( n \). The advantage of doing so is twofold: (1) more demand can be satisfied so more revenue can be generated and (2) there is no extra cost while liquidation of the leftover inventory is allowed.

The next theorem establishes the upper bound properties of the myopic policy (II).

**Proposition 3.** For any period \( n = 1, 2, \ldots, N - 1 \), the critical constants of the optimal policy given in Eqs. (54)-(55) and its myopic optimal policy given in Eqs. (75)-(76) satisfy

\[
\hat{\alpha}_n \geq \alpha_n; \quad \hat{\beta}_n \geq \beta_n.
\]

For the last period \( N \), \( \alpha_N = \hat{\alpha}_N \) and \( \beta_N = \hat{\beta}_N \).

**Proof.** For period \( N \), the result readily follows from the optimal solution of single period model. We only prove for \( \hat{\alpha}_n \geq \alpha_n \) as a similer argument (with replacing \( \ell_n \) with \( i_n \)) can be applied to prove \( \hat{\beta}_n \geq \beta_n \).

By Proposition 2, we have \( \alpha_n \leq \alpha_n^L \) and \( \alpha_n^L \) is determined by taking derivative of Eq. (78) and setting it equal to zero, that is

\[
\mathbb{E} \left[ \frac{\partial V_{n+1}(0, \xi_{n+1})}{\partial \xi_{n+1}} \left( 1\{\alpha_n^L > D_n\} + p'_n 1\{\alpha_n^L < D_n\} - h'_n 1\{\alpha_n^L > D_n\} - c'_n (1 + \ell_n) \right) \right] = 0.
\]  
(79)

For any realization of the demand \( D_n = d > 0 \) the term

\[
\frac{\partial V_{n+1}(0, \xi_{n+1}(d))}{\partial \xi_{n+1}(d)}
\]

is decreasing in \( d \) by the concavity of \( V_{n+1} \) [cf. Lemma 2 part (2)] and the fact that \( \xi_{n+1} \) is increasing in \( d \) by Eqs. (32)-(33).

In addition the term

\[
1\{\alpha_n > d\} + p'_n 1\{\alpha_n < d\} - h'_n 1\{\alpha_n > d\} - c'_n (1 + \ell_n)
\]

\[
= p'_n - (p'_n + h'_n - 1)1\{\alpha_n > d\} - c'_n (1 + \ell_n),
\]
is increasing in \( d \).

By Eq. (79) and Lemma 3, one has

\[
\mathbb{E} \left[ \frac{\partial V_{n+1}(0, \xi_{n+1})}{\partial \xi_{n+1}} \right] \cdot \mathbb{E} \left[ 1_{\{\alpha^*_n > D_n\}} + p'_n 1_{\{\alpha^*_n < D_n\}} - h'_n 1_{\{\alpha^*_n > D_n\}} - c'_n (1 + \ell_n) \right] = 0.
\]

Since \( \partial V_{n+1}(0, \xi_{n+1})/\partial \xi_{n+1} \geq 0 \) by Lemma 2 part (2), the above inequality implies

\[
\mathbb{E} \left[ 1_{\{\alpha^*_n > D_n\}} + p'_n 1_{\{\alpha^*_n < D_n\}} - h'_n 1_{\{\alpha^*_n > D_n\}} - c'_n (1 + \ell_n) \right] \geq 0 \tag{80}
\]

which, after simple algebra, is equivalent to

\[
p_n - c_n \cdot (1 + \ell_n) - (p_n + h_n - c_{n+1})F_n(\alpha^L_n) \geq 0.
\]

The above further simplifies to

\[
F(\alpha^L_n) \leq \frac{p_n - c_n \cdot (1 + \ell_n)}{p_n + h_n - c_{n+1}}.
\]

By Eqs. (74) and (75), the right hand side in the above inequality is \( F_n(\hat{\alpha}_n) \). Thus, we have \( F_n(\hat{\alpha}_n) \geq F_n(\alpha^L_n) \), which means \( \hat{\alpha}_n \geq \alpha^L_n \). Thus, the proof for \( \hat{\alpha}_n \geq \alpha_n \) is complete, since \( \alpha^L_n \geq \alpha_n \) by Proposition 2. \( \square \)

### 4.3 An Algorithm to Compute \((\alpha_n, \beta_n)\)

With the aid of the lower and the upper bounds presented in §4.1 and §4.2, we develop the following algorithm for a computational-simplification purpose.

**Algorithm**: The thresholds \( \alpha_n \) and \( \beta_n \) can be obtained via

\[
\alpha_n = \arg \max \left\{ \mathbb{E} \left[ V_{n+1}(x_{n+1}, y_{n+1}) \mid x_n + y_n \right] : z_n \in (\hat{\alpha}_n, \tilde{\alpha}_n) \right\}, \tag{81}
\]

\[
\beta_n = \arg \max \left\{ \mathbb{E} \left[ V_{n+1}(x_{n+1}, y_{n+1}) \mid x_n + y_n \right] : z_n \in (\hat{\beta}_n, \tilde{\beta}_n) \right\}, \tag{82}
\]

where \( x_{n+1} \) is given by Eq. (32) and \( y_{n+1} \) is given by Eq. (33). Note that the calculations involved in Eqs. (81)-(82) are optimization within bounded spaces and we can employ an efficient search procedure based on Eq. (54) for \( \alpha_n \) and Eq. (55) for \( \beta_n \). Those bounds simplify the computational space and thus expedite the calculation process.

### 5 Numerical Studies

In this section, we provide some numerical studies for the case of Uniform and Exponential demand distributions. Specifically, Subsection 5.1 conducts the study for single period problem, while Subsection 5.2 deals with a three-period system.

#### 5.1 Single Period Model

As shown in Section 2, one major reason for the two threshold values \( \alpha \) and \( \beta \) is the two distinct financial rates, \( i \) and \( l \). It is of interest to see how sensitive the variation between the two threshold values with respect to the difference between \( i \) and \( \ell \). In this section, we experiment the single period model with Uninform demand distribution of \( D \sim U(0, 100) \) and Exponential demand distribution of \( D \sim \text{Exp}(50) \). We
Figure 3: $\alpha$ of Single Period Newsvendor Problem

set the selling price as $p = 50$; cost $c = 20$; salvage cost per unit $s = 10$. We fix the interest rate as $i = 2\%$ and change the loan rate $\ell$ from $2\%$ to $50\%$. It shows that the value of $\beta$ does not change with respect to $\ell$. For any $\ell$, $\beta = 74.00$ for Uniform demand, while $\beta = 67.35$ for Exponential demand.

Figure 3 depicts the value of $\alpha$ with respect to $\ell$ for each demand distribution. For both demand distributions, $\alpha$ is decreasing in $\ell$. The threshold values, $\alpha$ and $\beta$, of Uniform demand are larger than those of Exponential demand. This can be explained by the difference between their cdf functions.

Figure 4 depicts the ratio of $\beta/\alpha$ with respect to $\ell$. This numerical study shows that the difference between $\alpha$ and $\beta$, measured by $\beta/\alpha$ is not significantly sensitive to the difference between $i$ and $\ell$, measured by $\ell/i$. Specifically, while $\ell/i = 25$, $\beta/\alpha = 1.48$ for Uniform demand, and $\beta/\alpha = 1.94$ for Exponential demand.

5.2 Three-Period Model

In this experiment, we consider a three-period problem and apply the algorithm presented in §4.3 to calculate the optimal solutions for each period. We assume iid Uniform demand distributions, $D \sim U(0, 100)$, for each period and set the selling price as $p = 50$; cost $c = 20$; salvage cost per unit $s = 10$ and holding cost $h = 5$. We fix the interest rate as $i = 2\%$ and the loan rate $\ell = 15\%$. This numerical study shows the sensitivities of the optimal order quantity and the optimal expected total wealth associated with each period with respect to the initial capital at the beginning of the period. For each period, we assume a zero initial inventory, $x_n = 0$ but increase the initial capital $Y_n$ from 380 to 1780, i.e., $y_n$ from 19 to 89.

Figure 5 depicts the optimal order quantity in each period, where the zigzag shape can be explained by the rounding calculations to approximate $y$ by $\lfloor Y/c \rfloor$. The same explanation is applicable for Figure 6. First, it shows that the optimal order quantity may decrease as time goes on given the same initial state. Second, for the last period, the structure of the the optimal order quantity obtained in this numerical study repeats
Figure 4: $\beta/\alpha$ of Single Period Newsvendor Problem
Figure 5: Optimal Order Quantities of Three Periods V.S. Initial Capital

here of Figure 2 presented via analysis.

Figure 6 depicts the optimal total wealth starting in each period. First, it shows that the optimal total
wealths at the end of time horizon increase given the same initial state as more periods considered in the
time horizon. This can be explained by the value of time period and the value of optimal operation in each
period. Second, starting with each period, the expected total wealth is increasing and concave in $y_n$, which
can be explained by Lemma 2.

6 Conclusions and Discussion

In this paper, we studied the optimal inventory policy for a single-item inventory system within a financial
market which allows capital loan and interest earning. We showed that the optimal order policy for each
period is characterized by two constants, so-called $(\alpha_n, \beta_n)$-policy. In addition, we provided two myopic
policies each of which give a lower bound and a upper bound of the threshold values. With the two bounds,
we developed an algorithm to compute the two threshold values $\alpha_n$ and $\beta_n$.

There are various possible trends in research to follow up with our current study.

a) To include a fixed ordering cost, it is of interest to study the optimal ordering policy;

b) In current study, the loan function is assumed to be a linear function, $l(x) = (1 + \ell)x$ with flat
load rate $\ell$. It can be of more complicated forms in practice. With a fixed loan cost, for example
$l(x) = k \cdot \delta(x) + (1 + \ell)x$, where $k$ is a positive constant. For another example, $l(x)$ can be a piecewise
Figure 6: Optimal Expected total Wealth of Three Periods V.S. Initial Capital
function with various loan rate $\ell_i$ for various loan range $(x_i, x_{i+1}]$, where $i = 1, 2, 3, \ldots$. It is a possible direction to generalize our model to be with a loan generic non-linear function $l(x)$.

c) The deposit credit function is assumed to be a simple linear function, $D(x) = (1 + r)x$ with a flat interest rate $r$. It can be of more complicated form in practice as discussed above for loan rate. For another example, $D(x)$ can be a piecewise function with various interest rates $r_i$ for the corresponding deposit amount range $(d_i, d_{i+1}]$, where $i = 1, 2, 3, \ldots$. It is also of interest to generalize of our model to incorporate a generic deposit credit function $D(x)$.

d) Study issues of risk, i.e. bankruptcy probabilities, cf. Babich et al. (2007).

7 Appendix

Lemma 3. For real functions $f(x)$ and $g(x)$,

(a) if both $f(x)$ and $g(x)$ are monotonically increasing or decreasing, then

$$E[f(X) \cdot g(X)] \geq E[f(X)] \cdot E[g(X)],$$

where the expectation is taken with respect to the random variable $X$.

(b) If $f(x)$ is increasing (decreasing), while $g(x)$ is decreasing (increasing), then

$$E[f(X) \cdot g(X)] \leq E[f(X)] \cdot E[g(X)].$$

Proof. To prove (a), we only give the proof for the case that $f(x)$ and $g(x)$ are increasing. The same argument can be applied for the case of decreasing $f(x)$ and $g(x)$.

Let $X'$ be another random variable which is iid of $X$. Since $f(x)$ and $g(x)$ are increasing, we always have

$$[f(X) - f(X')][g(X) - g(X')] \geq 0.$$ 

Taking expectations with respect to $X$ and $X'$ yields

$$E[(f(X) - f(X'))[g(X) - g(X')]] = E[f(X)g(X) + f(X')g(X') - f(X')g(X) - f(X)g(X')]$$

$$= E[f(X)g(X)] + E[f(X')g(X')] - E[f(X')]E[g(X)] - E[f(X)]E[g(X')]$$

$$= 2E[f(X)g(X)] - 2E[f(X)]E[g(X)] \geq 0.$$ 

The result of part (a) readily follows from the above.

In a similar vein, we can prove part (b) via changing the direction of the inequality above.

References


