On Optimal Replacement under Semi-Markov Conditions

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Abstract: We study the following model for a system the state of which is continuously observed. The set of possible states is a finite set \( \{0, \ldots, L\} \), where larger values represent increased states of deterioration from the “new condition” represented by state 0, to the “totally inoperative condition” of state \( L \). Whenever the system enters a state a decision has to be made as to whether it is renovated or it is left unattended. In this paper we generalize the results of Derman [1] to the case in which the state sojourn times are distributed according to a general state dependent distribution. We also allow a renovation to result in a renovation state \( l \geq 0 \), i.e., the renovated system may not be as good as new. Whenever the system enters a state \( i < L \) and the decision to renovate is taken, then a cost \( c \) is incurred and its state, immediately, changes to a fixed state \( l \). If the system enters state \( L \) then it must be renovated at an increased cost \( c + A \). There is no cost whenever the decision to leave it unattended is taken in a state \( i < L \); in this case the next state will be state \( j \) with probability \( p_{ij} \) and the sojourn time in state \( i \) is a random variable with distribution function \( F_{ij}(\cdot) \). We provide necessary conditions under which optimal policies are of the “control limit” type. This analysis includes both the continuous time and discrete time cases, and examples are given for both cases.

Key Words: Markovian Decision Processes, Reliability, Nonlinear Optimization.

1 Introduction

In this article we study the following model for a system the state of which is continuously observed. The set of possible states is a finite set \( S = \{0, \ldots, L\} \), where larger values represent increased states of deterioration from the “new condition” (state 0) to the “totally inoperative condition” (state \( L \)). When the system changes state a decision has to be made as to whether it is renovated or it is left unattended. Whenever the system enters a state \( i < L \) and the decision to renovate is taken, then a cost \( c(> 0) \) is incurred and its state, immediately, changes to a fixed state \( l \).

If the system enters state \( L \) then it must be renovated at an increased cost \( c + A > c \). There is no cost if the decision to leave it unattended is taken when it enters a state \( i < L \). In this case the next state will be state \( j \) with probability \( p_{ij} \) and the sojourn time in state \( i \) is a random variable with distribution function \( F_{ij}(\cdot) \). We provide a necessary condition under which optimal policies are of the “control limit” type. The results herein generalize those of Derman [1] when the state sojourn times are distributed according to a general state dependent distribution and a renovation may not necessarily result in a “new system” i.e., the renovation state \( l \) maybe \( l > 0 \). We consider three cost criteria and determine the structure of the optimal policy in each case. The extension of the results of Derman [1] to the case in which the state space \( S \) is a Borel subset of the non-negative real numbers was given by Ross [4].

Related work in this area includes Abdel-Hameed and Y. Nakhi [6] who study a system subject to a sequence of shocks occurring randomly, where each shock causes a random amount of damage. The system fails when the total damage first exceeds a fixed threshold. Other related work is that of Grall et al [9], Sheu et al [8], Li and Pham [7] and Jin, Mashita, and Suzuki [5].

2 Semi Markov formulation and analysis.

Let \( a = 0 \) denote the action of no renovation and \( a = 1 \) denote the action to perform a renovation. The transition probabilities \( p_{ij} \) are such that \( p_{ij} = 0 \) if \( i > j \) and \( p_{ii} = 1 \) if \( i = L \) and \( j = L \). We consider three cost criteria. The long run average cost, which for a policy \( \pi \) is represented by \( v_\pi(i) \) where \( i \) is the initial state. The infinite horizon discounted cost \( V_{\alpha}^\pi(i) \) and the finite horizon discounted cost \( V_{\alpha}^\pi(i, N) \). Note that...
\[ V_\alpha^0(i, 0) = 0, \ i = 0, \ldots, L. \]

Define \( g(i, j, \alpha) \) as follows

\[ g(i, j, \alpha) = \int_0^\infty e^{-\alpha t} dF_{ij}(t). \quad (1) \]

Following the notation in Ross [3] the finite horizon discounted cost function can be written as

\[ V_\alpha(i, N) = \begin{cases} 
\min \left\{ V_\alpha^0(i, N), V_\alpha^1(i, N) \right\} & \text{if } i \neq L \\
A + V_\alpha^1(i, N) & \text{if } i = L
\end{cases} \]

where

\[ V_\alpha^0(i, N) = \sum_{j=0}^L p_{ij} g(i, j, \alpha) V_\alpha^\ast(j, N - 1), \]

and

\[ V_\alpha^1(l, N) = c + \sum_{j=0}^L p_{ij} g(l, j, \alpha) V_\alpha^\ast(j, N - 1). \]

We will establish the optimality of a control limit policy under the following conditions.

**Condition I:** For any increasing function \( h \) on \( S = \{0, 1, \ldots, L\} \), the function

\[ \xi(i) = \sum_{j=0}^L p_{ij} h(j) \]

is an increasing function of \( i \).

**Condition II** below is Condition B in Derman [1].

For each \( k = 0, 1, \ldots, L \), the function

\[ r_k(i) = \sum_{j=k}^L p_{ij} \]

is increasing in \( i \).

**Condition III:** For each \( i = 0, 1, \ldots, L \), the function \( g(i, j, \alpha) \) is a non-decreasing function of \( j \).

We first prove the following theorem.

**Theorem 1** Conditions I and III are equivalent to condition II.

**Proof:**

Let \( h'(j) = h(j) + g(i, j, \alpha) \), then conditions I and III together are equivalent to the following condition: for any increasing function \( h'(j) \) on \( S = \{0, 1, \ldots, L\} \), the function

\[ \xi'(i) = \sum_{j=0}^L p_{ij} h'(j), \quad (4) \]

is an increasing function of \( i \).

The equation (4) is condition A in [1]. Using an argument similar to that of the lemma in the same paper we can prove that the condition of equation (4) is equivalent to condition II and hence conditions I and III are also equivalent to condition II.

We next state and prove the main result of the paper.

**Theorem 2** If conditions I and III hold or equivalently if conditions II and III hold then the optimal policy for all three cost criteria considered is of the control limit type.

**Proof:** \( V_\alpha(i, 0) = 0 \) is a increasing function by definition. Using conditions II and III it follows that \( V_\alpha^0(i, 1) \) is increasing in \( i \). Using equation (2) it follows that there exists an \( i^*_1 = i^*_1(\alpha) \leq L - 1 \) such that an optimal policy: \( \pi^*_1, \alpha \) for \( V_\alpha(i, 1) \) is specified by the actions: \( \pi^*_1, \alpha(i) = 0 \), for \( i < i^*_1 \) and \( \pi^*_1, \alpha(i) = 1 \) for \( i > i^*_1 \). Also, from equation (2) and conditions II and III it follows that \( V_\alpha(i, 1) \) is an increasing function in \( i \). Using the same arguments and induction it follows that \( V_\alpha(i, N) \) is also increasing in \( i \) and that an optimal policy: \( \pi^*_N, \alpha \) for \( V_\alpha(i, N) \) is specified by the actions: \( \pi^*_N, \alpha(i) = 0 \), for \( i < i^*_N \) and \( \pi^*_N, \alpha(i) = 1 \) for \( i > i^*_N \), for some constant \( i^*_N = i^*_N(\alpha) \leq L - 1 \).

For the infinite horizon case note that the following hold, c.f. Ross [3].

\[ V_\alpha(i) = \lim_{N \to \infty} V_\alpha(i, N), \quad (5) \]

and

\[ V_\alpha(i) = \begin{cases} 
\min \left\{ V_\alpha^0(i), V_\alpha^1(i) \right\} & \text{if } i < L, \\
V_\alpha^1(L) & \text{if } i = L.
\end{cases} \quad (6) \]

where

\[ V_\alpha^0(i) = \sum_{j=0}^L p_{ij} V_\alpha^\ast(j, \alpha), \]

\[ V_\alpha^1(i) = \begin{cases} 
c + \sum_{j=0}^L p_{ij} V_\alpha^\ast(j, \alpha), & i < L, \\
c + A + \sum_{j=0}^L p_{ij} V_\alpha^\ast(j, \alpha), & i = L.
\end{cases} \]

From equation (5) and the fact that \( V_\alpha(i, N) \) are increasing in \( i \), it follows that \( V_\alpha(i) \) is increasing in \( i \).

Using this, equation (6), and conditions II and III we can conclude that there is a number \( i^* = i^*_\alpha \leq L - 1 \) such that

\[ V_\alpha(i) = \begin{cases} 
V_\alpha^0(i) & \text{if } i < i^*, \\
V_\alpha^1(i) & \text{if } i \geq i^*.
\end{cases} \quad (7) \]
i.e., the infinite horizon optimal policy \( \pi^* \) is specified by the actions \( \pi^*_a(i) = 0 \) (do not renovate) for \( i < i^*_a \) and \( \pi^*_a(i) = 1 \) (renovate) for \( i > i^*_a \).

For the average cost case let \( \nu(i) = \operatorname{sup}_\pi \pi(i) \); note that the following holds, c.f. Ross [3].

\[
v(i) = \lim_{\alpha \to \infty} \alpha * V_\alpha(i). \tag{8}
\]

Consider a sequence \( \{ \alpha_i \} \) with \( \lim_{\alpha \to \infty} \alpha_i = \infty \), with \( i^*_\alpha = i^* \) for all \( \alpha \). Since the total number of possible states is finite, such a sequence and \( i^* \) exist. Let \( \pi^\alpha \) be the control limit policy defined by \( i^* \) and let \( \pi \) is some policy that is not a control limit. Then,

\[
V^\pi_\alpha(i) \geq V^\pi^\alpha(i) = V^\pi(i) \quad \nu = 1, 2, \ldots \tag{9}
\]

and hence

\[
V^\pi = \lim_{\nu \to \infty} \alpha_i V^\pi^\alpha(i) \geq \lim_{\nu \to \infty} \alpha_i V^\pi_\alpha(i) = V^\pi^\alpha \tag{10}
\]

Hence the theorem.

**Observations**

1. If \( g(.) \) does not depend on \( j \), then for Condition I and Condition II to hold and for control limit policies to be optimal, it is required that \( g(.) \geq 0 \). This is always true since \( g(.) \) is the Laplace transform of a probability distribution function.

2. If the distribution function \( f(.) \) is not a function of \( t \), then the function \( g(i, j, \alpha) \) can be simplified to \( f(i, j)/\alpha \). Then Condition II will only require \( f(i, j) \) to be an increasing function in \( j \).

### 3 Examples

**3.1 Example 1**

Let the transition times between states \( i \) and \( j \) be exponentially distributed with parameter \( \mu_{ij} \) i.e. \( f(i, j) = \mu_{ij} e^{-\mu_{ij} t} \) and \( g(i, j, \alpha) = \frac{\mu_{ij}}{\mu_{ij} + \alpha} \) with, \( \mu_{ij} \leq \mu_{i+1j} \) and \( \mu_{ij} \leq \mu_{ij+1} \). Assume the transition probabilities to be \( p_{ij} = \frac{\mu_{ij}}{\mu_i} \) where, after *uniformization* (cf. [2]), we may take \( \mu_i = \sum_j \mu_{ij} = 1 \). Then conditions II and III are satisfied and by *theorem I*, the system has an optimal policy that is of the control limit type.

**3.2 Example 2**

Let the transition times between states \( i \) and \( j \) have a poisson distribution with parameter \( \lambda_{ij} \) i.e. \( f(i, j) = (\lambda_{ij} t)^n e^{-\lambda_{ij} t}/n! \) and \( g(i, j, \alpha) = e^{\lambda_{ij}(e^{\alpha} - 1)} \) with \( \lambda_{ij} \leq \lambda_{ij+1} \) and \( p_{ij} \) satisfying *Condition II*. Then conditions II and III are satisfied and by *theorem I*, the system has an optimal policy of the control limit type.

### 3.3 Example 3

Let the transition times between states \( i \) and \( j \) be distributed uniformly over \([b_i, a_j]\) i.e. \( f(i, j) = 1/(a_j - b_i) \) and using remark (2) \( g(i, j, \alpha) = 1/\alpha(a_j - b_i) \) where the parameter \( a_j \) depends only on the final state \( j \) and the parameter \( b_i \) depends only on the initial state \( i \). Consider the transition probabilities \( p_{ij} \) that satisfy *Condition II*. Then by *theorem I* we can say that the system has an optimal policy that is of the control limit type.

### 4 Numerical Evaluations

We performed computations using AMPL with the MINOS solver. The results obtained were tabulated and corresponding graphs are shown as figure 1 and figure 2. Both the graphs have total cost on the y-axis and values of \( i \) that were chosen as the control limit on the x-axis. The optimal control limit is the value of
i for which the total cost is minimum.

The graph in figure 1 represents the case where the sojourn time is exponentially distributed. The exponential parameter $\mu_{ij}$ was considered to be equal to $i$. The result shows that the optimum control limit is $i^* = 12$. The computation was done for $L = 50$ states with $C = 6$ and $A = 25$. The discount factor used was $\alpha = 0.25$.

The graph in figure 2 represents the case where the sojourn times have the Poisson distribution. with $\lambda_{ij} = 6$. The optimal control limit for this case was $i^* = 81$. The parameters for this computation were $L = 100$ states,$C = 6$, $A = 60$ and discount factor $\alpha = 0.1$.

The graph in figure 3 represents the case where the sojourn time is uniformly distributed. The interval $a_j - b_i$ was assumed to be equal to 10. The optimal control limit for this case was $i^* = 44$. The parameters for this computation were $l = 50$ states,$C = 6$, $A = 25$ and discount factor $\alpha = 0.25$.

5 Conclusions

Extending this work to a system with a structure such as the parallel system or the series system is an interesting area of research. Additionally, another area worth of further study is the case in which there is partial information on the state of the system which can be discerned only after inspection.

6 Acknowledgements

This research was supported by the Rutgers University Research Resources Committee.

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