Solve three of the following four problems. 
Indicate on which three you wish to be graded and email it to me at: mnk@rutgers.edu. 

Good Luck!

Problem 1:

1. Let $p$ and $q$ be real valued column $n$-vectors with $p' 1 = 0$. Prove the following:

$$|p' q| \leq \max_i \{q_i\} - \min_i \{q_i\}$$

$$= \max_{i,j} |q_i - q_j| \frac{1}{2} \sum_{i=1}^n |p_i|$$

$$= \max_{i,j} |q_i - q_j| \frac{1}{2} \sum_{i=1}^n |p_i|$$

2. Let $P = [p_{ij}]_{i,j=1,...,n}$ be an $n \times n$, stochastic matrix $^1$, $w = [w_i]$ be a real-valued column $n$-vector and let

$$v = Pw.$$ 

Show that:

$$\max_{i,j} |v_i - v_j| \leq h \max_{i,j} |w_i - w_j|$$

where

$$h = h(P) = \frac{1}{2} \max_{i,j} \sum_{k=1}^n |p_{ik} - p_{jk}| \in [0, 1].$$

3. Let $P^{n-1} = [p_{ij}^{(n-1)}]$, for $n \geq 1$, with $P^0 = I$ show that, for all $n$ and $k$:

$$\max_{i,j} |p_{ik}^{(n)} - p_{jk}^{(n)}| \leq h(P)^n$$

Note that if $P > 0$, i.e., all entries of $P$ are positive (or if there exists an $m > 1$ such that $P^m > 0$, which is true if the Markov chain with matrix $P$ is irreducible) then $h(P) < 1$ and then the last inequality implies that as $n \to \infty$ all rows of $P^n$ tend the same limit.

4. For fixed $k$ show that

(a) $\max_i p_{ik}^{(n)}$ is decreasing in $n$,

(b) $\min_i p_{ik}^{(n)}$ is increasing in $n$. $^2$ $^3$

---

$^1$ i.e., $P \geq 0$, $P 1 = 1$.

$^2$ It follows [since both sequences are bounded] that both have limits as $n \to \infty$. When $h(P) < 1$, all rows of $P^n$ tend to the same limiting probability vector. This property of all rows of $P^n$ tending to the same limiting probability distribution is called “strong ergodicity”.

$^3$ Notice that the argument to prove 4(a,b) uses the “backward” form: $P^n = P P^{n-1}$. Thus, when $h(P) < 1$, we can obtain the ergodicity of a finite homogeneous Markov chain, at geometric rate of convergence without the use of Perron-Frobenius theory of non-negative matrices.
Problem 2: (Strong Markov Property). Show that if $\tau$ is a stopping time for a time homogenous Markov Chain: $\{X_n\}_{n \geq 0}$ on a countable set $S$ then, for any $m \geq 1$, $j$, $i$, $i_{m-1},\ldots,i_0$:

$$P[X_{\tau + m + 1} = j | X_{\tau + m} = i, X_{\tau + m - 1} = i_{m-1},\ldots,X_{\tau} = i_0, \tau = n] = P[X_1 = j | X_0 = i].$$

Problem 3: (Work in a queueing system). The work in a queueing system at any time is defined as the sum of the remaining service times of all customers in the system at that time. For the $M/G/1$ queueing system in steady state compute the mean and variance of the work in the system.

Problem 4: (Optimal Stopping in financial markets.) In financial markets, options are traded and their values change in relationship to the price of the underlying stock. A call option gives the owner the right to purchase an asset, such as shares of common stock, at a fixed “strike” price $W$ at any time prior to a specified expiration date. We assume that a single call corresponds to a right for 100 shares.

A put option gives the holder the right to sell shares of an asset, such as shares of common stock, at a fixed “strike” price $W$ at any time prior to a specified expiration date.

Assume that the probabilities $p_{ij}$ that the next time instant’s $(t + 1)$ stock price is equal to $j$ given that the current (time $t$) stock price is $i$ are known and let $c_0$ denote the fixed (per transaction cost).

Formulate the problem of when to exercise a single call and a single put option as an optimal stopping problem, i.e., give:

1. State space and Actions
2. Rewards
3. System Dynamics
4. DP equations, in terms of the value function $v(t)$, $t = 0, 1, \ldots, N$, where $t$ is the time remaining to the expiration time.

---

4 i.e., given $\{\tau = n\}$ and $X_\tau = i$ the process $\{X_{\tau + m}\}_{m \geq 0}$ is probabilistically the same as the process $\{X_m\}_{m \geq 0}$ with $X_0 = i$

5 For example, one might purchase a call option to buy 100 shares of ASC company at $30 per share at or before May 30. Time is discrete $t = 0, \ldots, N$, where $N$ represents the expiration time of May 30. If at any time $t$, the stock price reached $31 per share, then one may exercise the option and buy 100 shares of ASC company stock for $3000$ and immediately sell them for $3100$ to make a profit of $100$ less transaction costs (commissions). When the stock price is below $30 per share, one would not exercise the option.

6 For example one might purchase a put option to sell 100 shares of ASC company at $30 per share at or before May 30. Time is discrete $t = 0, \ldots, N$, where $N$ represents the expiration time of May 30. If at any time $t$, the stock price reached $29$ per share, then one may exercise the option and sell 100 shares of ASC company stock for $3000$ having bought them for $2900$ to make a profit of $100$ less transaction costs (commissions). When the stock price is above $30$ per share, one would not exercise the option.

7 Note that $v(N)$ is the “fair price” of the option.