Solutions to problems from Introduction to Probability and Statistics textbook (by Ross).

**Problem 10 (page 132)**

**Question**

The joint probability density function of $X$ and $Y$ is given by

$$f(x, y) = \frac{6}{7} \left( x^2 + \frac{xy}{2} \right), \quad 0 < x < 1, 0 < y < 2$$

(a) Verify that this is indeed a proper joint density function.

(b) Compute the density function of $X$.

(c) Find $P(X > Y)$.

**Solution**

(a)

First we notice that $f(x, y)$ is non-negative for all "allowed" combinations of $x$ and $y$. Next we check that it integrates to one,

$$\int_0^1 \int_0^1 f(x, y) \, dx \, dy = \int_0^1 \int_0^1 \frac{6}{7} \left( x^2 + \frac{xy}{2} \right) \, dx \, dy$$

$$= \int_0^1 \left[ \frac{6x^3}{21} + \frac{6yx^2}{28} \right]_0^1 dy$$

$$= \int_0^1 \left[ \frac{2x^3}{7} + \frac{3yx^2}{14} \right]_0^1 dy$$

$$= \int_0^1 \frac{2}{7} + \frac{3y}{14} \, dy$$

$$= \frac{2y}{7} + \frac{3y^2}{28} \bigg|_0^1$$

$$= 1$$
(b) For $0 < x < 1$ we have,

$$f_X(x) = \int_0^x f(x, y) \, dy$$

$$= \int_0^x \frac{6}{1} \left( x^2 + \frac{xy}{2} \right) \, dy$$

$$= \frac{6x^2y}{7} + \frac{3xy^2}{14} \bigg |_0$$

$$= \frac{12x^2}{7} + \frac{6x}{7}$$

(c) For $x$ to be larger than $y$, $y$ must be less than 1. So we have,

$$P(X > Y) = \int_{X>Y} f(x, y) \, dx \, dy$$

$$= \int_0^1 \int_y^1 \frac{6}{1} \left( x^2 + \frac{xy}{2} \right) \, dx \, dy$$

$$= \int_0^1 \left[ \frac{2x^3}{7} + \frac{3x^2y}{14} \right]_y^1 \, dy$$

$$= \int_0^1 \frac{2y^3}{7} + \frac{3y^2}{14} - \frac{7y^3}{14} \, dy$$

$$= \frac{2y}{7} + \frac{3y^2}{28} - \frac{7y^4}{56} \bigg |_0$$

$$= \frac{2}{7} + \frac{3}{28} - \frac{7}{56}$$

$$= \frac{16}{56} + \frac{6}{56} - \frac{7}{56}$$

$$= \frac{15}{56} \approx .268$$

Problem 16 (page 133)

Question

Suppose that $X$ and $Y$ are independent continuous random variables. Show that

(a) $P(X + Y \leq a) = \int_{-\infty}^{\infty} F_X(a - y) f_Y(y) \, dy$

(b) $P(X \leq Y) = \int_{-\infty}^{\infty} F_X(y) f_Y(y) \, dy$

where $f_Y$ is the density function of $Y$, and $F_X$ is the distribution function of $X$.

Solution

$X + Y \leq a \implies X \leq a - Y$
(a)

\[ P(X + Y \leq a) = \int_{x+y\leq a} f_X(x) \cdot f_Y(y) \, dx \, dy \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{a-y} f_X(x) \cdot f_Y(y) \, dx \, dy \]

\[ = \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{a-y} f_X(x) \, dx \, dy \]

\[ = \int_{-\infty}^{\infty} f_Y(y) \left[ F_X(x) \right]_{-\infty}^{a-y} \, dy \]

\[ = \int_{-\infty}^{\infty} f_Y(y) \left( F_X(a-y) - F_X(-\infty) \right) \, dy \]

\[ = \int_{-\infty}^{\infty} F_X(a-y) f_Y(y) \, dy \]

(b)

Similar to above, so I leave out some steps.

\[ P(X \leq Y) = \int_{x\leq y} f_X(x) \cdot f_Y(y) \, dx \, dy \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{y} f_X(x) \cdot f_Y(y) \, dx \, dy \]

\[ = \int_{-\infty}^{\infty} F_X(y) f_Y(y) \, dy \]

**Problem 25 (page 134)**

**Question**

A total of 4 buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let \( X \) denote the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let \( Y \) denote the number of students on her bus.

(a) Which of \( E[X] \) or \( E[Y] \) do you think is larger? Why?

(b) Compute \( E[X] \) and \( E[Y] \).

**Solution**

(a)

Discuss in class.

(b)

\[ E[X] = 40 \cdot \frac{40}{148} + 33 \cdot \frac{33}{148} + 25 \cdot \frac{25}{148} + 50 \cdot \frac{50}{148} \]

\[ = \frac{5814}{148} = \frac{2907}{74} \]

\[ \approx 39.3 \]
\[ E[Y] = 40 \cdot \frac{1}{4} + 33 \cdot \frac{1}{4} + 25 \cdot \frac{1}{4} + 50 \cdot \frac{1}{4} \]
\[ = 148 \cdot \frac{1}{4} \]
\[ = 37 \]

Problem 29 (page 135)

Question

Let \( X_1, X_2, \ldots, X_n \) be independent random variables having the common density function

\[ f(x) = \begin{cases} 
1, & 0 < x < 1 \\
0, & \text{otherwise}
\end{cases} \]

Find (a) \( E[Max(X_1, X_2, \ldots, X_n)] \) and (b) \( E[Min(X_1, X_2, \ldots, X_n)] \)

Solution

(a)

Let \( Y = Max(X_1, X_2, \ldots, X_n) \). Then,

\[ F_Y(y) = P(Y \leq y) = P(X_1 \leq y, X_2 \leq y, \ldots, X_n \leq y) \]
\[ = \prod_{i=1}^{n} P(X_i \leq y) \]
\[ = \prod_{i=1}^{n} F_{X_i}(y) \]
\[ = F_{X_1}(y)^n \]
\[ = y^n \]

So,

\[ E[Y] = \int_{0}^{1} y \cdot f_Y(y) \, dy \]
\[ = \int_{0}^{1} y \cdot ny^{n-1} \, dy \]
\[ = \int_{0}^{1} ny^n \, dy \]
\[ = \frac{ny^{n+1}}{n+1} \bigg|_{0}^{1} \]
\[ = \frac{n}{n+1} \]
Let \( Z = \text{Min}(X_1, X_2, \ldots, X_n) \). Then,

\[
F_Z(z) = P(Z \leq z) = 1 - P(Z > z) = 1 - P(X_1 > z, X_2 > z, \ldots, X_n > z)
\]

\[
= 1 - \prod_{i=1}^{n} P(X_i > z)
\]

\[
= 1 - \prod_{i=1}^{n} 1 - P(X_i \leq z)
\]

\[
= 1 - \prod_{i=1}^{n} 1 - F_X(z)
\]

\[
= 1 - (1 - z)^n
\]

So,

\[
E[Z] = \int_{0}^{1} z \cdot f_Z(z) \, dz
\]

\[
= \int_{0}^{1} z \cdot n(1 - z)^{n-1} \, dz
\]

Note: We can integrate using integration by parts. \( \int u \, dv = uv - \int v \, du \). In our case \( u = z^n \), \( dv = (1 - z)^{n-1} \, dz \), which yields

\[
= z(1 - z)^n - \int \frac{(1 - z)^{n+1}}{n+1} \, dz
\]

\[
= 0 - \left[ -\frac{1}{n+1} \right]_0^1
\]

\[
= \frac{1}{n+1}
\]

Problem 33 (page 135)

Question

Ten balls are randomly chosen from an urn containing 17 white and 23 black balls. Let \( X \) denote the number of white balls chosen. Compute \( E[X] \)

(a) by defining appropriate indicator variables \( X_i, i = 1, \ldots, 10 \) so that

\[
X = \sum_{i=1}^{10} X_i
\]

(b) by defining appropriate indicator variables \( Y_i, i = 1, \ldots, 17 \) so that

\[
X = \sum_{i=1}^{17} Y_i
\]

Solution

(a)

Let \( X_i \) be the indicator variable denoting whether the ith choice was white (1) or black (0). Then \( X = \sum_{i=1}^{10} X_i \), and \( P(X_i = 1) = \frac{17}{40} \). So,
\[ E[X] = E \left[ \sum_{i=1}^{10} X_i \right] \]
\[ = \sum_{i=1}^{10} E[X_i] \]
\[ = 10 \cdot E[X_i] \]
\[ = 10 \cdot \frac{17}{40} \]
\[ = 4.25 \]

*Note: The *ith ball is equally likely to be any of the 40 original balls, so its probability of being white is the same as for the first ball.*

(b)

Let \( Y_i \) be the indicator variable denoting whether the *ith white ball was chosen* (1) or not (0). Then \( X = \sum_{i=1}^{17} Y_i \), and to find the probability that the *ith white ball was chosen* we can write,

\[
P(Y_i = 1) = P(\text{White ball was chosen})
= P(\text{White ball was chosen first}) + P(\text{White ball was chosen second (and not first)}) + \ldots + P(\text{White ball was chosen the 10th time (and not the 9 previous times)})
= \frac{1}{40} + \frac{39}{40} \cdot \frac{1}{39} + \frac{39}{40} \cdot \frac{38}{39} + \ldots + \frac{39}{40} \cdot \frac{38}{39} \ldots \frac{31}{30} + \frac{39}{40} \ldots \frac{31}{30} \ldots \frac{3}{2} \frac{1}{2}
= \frac{10}{40}
\]

So

\[ E[X] = E \left[ \sum_{i=1}^{17} Y_i \right] \]
\[ = \sum_{i=1}^{17} E[Y_i] \]
\[ = 17 \cdot E[Y_i] \]
\[ = 17 \cdot \frac{10}{40} \]
\[ = 4.25 \]

**Problem 44 (page 137)**

**Question**

Let \( X_i \) denote the percentage of votes cast in a given election that are for candidate *i*, and suppose that \( X_1 \) and \( X_2 \) have a jdf

\[
f_{X_1, X_2}(x, y) = \begin{cases} 
3(x + y), & \text{if } x \geq 0, y \geq 0, \text{ and } 0 \leq x + y \leq 1 \\
0, & \text{if otherwise}
\end{cases}
\]

(a) Find the marginal densities of \( X_1 \) and \( X_2 \).
(b) Find \( E[X_i] \) and \( Var(X_i) \) for \( i = 1, 2 \).
Solution

(a) First we will examine $X_1$. The marginal distribution of $X_1$, $f_{X_1}(x)$ is the integral of the jdf, $f_{X_1,X_2}(x,y)$ over all possible values of $X_2(y)$. By the definition of the jdf above, we know that $f_{X_1,X_2}(x,y)$ will only be non-zero if $x \geq 0$, $y \geq 0$, and $0 \leq x+y \leq 1$. This implies that given some fixed value for $x$, $0 \leq y \leq 1-x$. Therefore we have,

$$f_{X_1}(x) = \int_0^{1-x} f_{X_1,X_2}(x,y) \, dy$$

$$= \int_0^{1-x} 3(x+y) \, dy$$

$$= 3xy + \frac{3}{2}y^2 \bigg|_0^{1-x}$$

$$= \frac{3}{2}(1-x^2) \text{ for } 0 \leq x \leq 1$$

Using similar arguments we can show that $f_{X_2}(y) = \frac{3}{2}(1-y^2) = f_{X_1}(y)$

(b) Since the marginal distributions of $X_1$ and $X_2$ are identical, $E[X_1] = E[X_2]$ and $Var(X_1) = Var(X_2)$. Starting with $E[X_1]$,

$$E[X_1] = \int_0^1 x \cdot f_{X_1}(x) \, dx$$

$$= \int_0^1 x \cdot \left( \frac{3}{2}(1-x^2) \right) \, dx$$

$$= \int_0^1 \frac{3x}{2} - \frac{3x^3}{2} \, dx$$

$$= \frac{3x^2}{4} - \frac{3x^4}{8} \bigg|_0^1$$

$$= \frac{3}{8}$$

And for $Var(X_1)$,

$$Var(X_1) = E[X^2] - E[X]^2$$

$$= E[X^2] - \frac{9}{64}$$

$$= \int_0^1 x^2 \cdot f_{X_1}(x) \, dx - \frac{9}{64}$$

$$= \int_0^1 \frac{3x^2}{2} - \frac{3x^4}{2} \, dx - \frac{9}{64}$$

$$= \frac{3x^3}{6} - \frac{3x^5}{10} \bigg|_0^1 - \frac{9}{64}$$

$$= \frac{1}{5} - \frac{9}{64}$$

$$= \frac{19}{320}$$
Problem 52 (page 139)

Question
If $X_1$ and $X_2$ have the same pdf, show that

\[ \text{Cov}(X_1 - X_2, X_1 + X_2) = 0 \]

Note: we are not assuming that $X_1$ and $X_2$ are independent

Solution
We know that \( \text{Cov}(X, Y) = E[X \cdot Y] - E[X] \cdot E[Y] \). Therefore we can write,

\[
\text{Cov}(X_1 - X_2, X_1 + X_2) = E[(X_1 - X_2) \cdot (X_1 + X_2)] - (E[X_1 - X_2] \cdot E[X_1 + X_2])
\]
\[
= E[X_1^2] + E[X_1X_2] - E[X_2X_1] - E[X_2^2] - (E[X_1]^2 + E[X_1]E[X_2] - E[X_2]E[X_1] - E[X_2]^2))
\]
\[
= E[X_1^2] - E[X_2^2] - E[X_1]^2 + E[X_2]^2
\]
\[
= E[X_1^2] - E[X_2]^2 - (E[X_2^2] - E[X_2]^2)
\]
\[
= \text{Var}(X_1) - \text{Var}(X_2)
\]
\[
= \text{Var}(X_1) - \text{Var}(X_2)
\]
\[
= 0
\]

Where the 2nd to last line follows because $X_1$ and $X_2$ have the same pdf, and therefore the same variance.