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Symmetry conditions for third order elastic moduli and implications in nonlinear wave theory

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Abstract. Several results are presented concerning symmetry properties of the tensor of third order elastic moduli. It is proven that a set of conditions upon the components of the modulus tensor are both necessary and sufficient for a given direction to be normal to a plane of material symmetry. This leads to a systematic procedure by which the underlying symmetry of a material can be calculated from the 56 third order moduli. One implication of the symmetry conditions is that the nonlinearity parameter governing the evolution of acceleration waves and nonlinear wave phenomena is identically zero for all transverse waves associated with a plane of material symmetry.

1. Introduction

The types of symmetry possessed by elastic materials can be usefully described in terms of the underlying planes of material symmetry. The idea of symmetry planes, as defined by Spencer [1] for example, has been adopted by Cowin and Mehrabadi [2] in their categorization of the elastic symmetries known to exist (the completeness of these symmetry classes was recently proved by Huo and Del Piero [3]). For instance, materials of orthotropic symmetry possess three orthogonal planes of symmetry. In general, the material symmetry is completely specified by the underlying symmetry planes, and vice versa.

The basic result of Cowin and Mehrabadi [2] is a set of conditions on the components of the elastic modulus tensor that are necessary and sufficient that a given direction be normal to a plane of symmetry. An analogous set of conditions for the components of the third order elastic modulus tensor are derived and discussed in Section 2. A simpler set of conditions are also given which are necessary but not sufficient, and are of practical use in determining the underlying material symmetry of a given set of 56 third order elastic constants.

The main result of the paper is derived in Section 4 and concerns the acoustic nonlinearity parameter β defined in Section 3. This is the parameter which governs the evolution of acceleration waves in elastic solids, and its

value determines whether or not a given initial disturbance develops into a shock. It is well known [4, 5, 6] that a set of infinitely many transverse wave modes is associated with every plane of material symmetry. By using one of the set of conditions derived in Section 2, it is shown that β is identically zero for this class of transverse waves, generalizing a previously known result of Green [7] that $\beta = 0$ in isotropic solids.

There is some ambiguity in the terminology used to discuss elastic coefficients. One approach is to view the moduli as the coefficients in an expansion of stress in terms of strain [8]. The first coefficient, which provides a linear stress-strain relation, is known as the tensor of first order moduli, and the coefficient of the term quadratic in strain is the tensor of second order moduli, etc. Alternatively, one can expand the energy as a power series in strain for materials possessing a strain energy function. Assuming the strain energy is zero for zero strain, i.e. the reference state is taken as the undeformed configuration, then the term linear in strain in the expansion is identically zero. The first non-zero contribution is quadratic in strain and the corresponding coefficient is known as the tensor of second order moduli, even though these moduli are closely related to the first order moduli of the previous definition. In fact they are identical if the constitutive relation expresses the second Piola-Kirchhoff stress tensor as a function of the Green-St. Venant strain tensor [8] and the material is hyperelastic. The term in the energy expansion that is cubic in the strain defines the so-called third order moduli, which are closely related to the second order moduli of the stress-strain definition.

The definitions of second and third order moduli used in this paper are those which follow from the expansion of the strain energy, and they will be discussed explicitly in Section 3. The precise origin of the moduli is immaterial to the discussions of Section 2, where the only relevant attribute used is the symmetry of the components.

A remark on notation: for the remainder of the paper all vectors are unit vectors in three dimensional space; lower case subscripts assume the values 1, 2 and 3; upper case subscripts the values 1, 2, . . . , 6; and the summation convention on repeated subscripts is taken for granted.

2. Necessary and sufficient conditions for the existence of a plane of symmetry

Given an elastic stiffness or compliance tensor, it is not immediately clear what symmetries, if any, the corresponding material possesses. This question of determining the symmetry was addressed and answered by Cowin and Mehrabadi [2] for the moduli of linear elasticity, i.e. the second order moduli. The components of the modulus tensor are C_{ijkl} relative to a rectangular basis,

and satisfy the symmetries

$$C_{ijkl} = C_{jikl}, \quad C_{ijkl} = C_{klij}. \quad (2.1)$$

These are a consequence of the symmetry of the strain tensor and the existence of a strain energy function, respectively, independent of whatever symmetry the material may have. The fundamental result of Cowin and Mehrabadi [2], in a simplified form due to Cowin [9] and Norris [6], is:

THEOREM 1. *The conditions*

$$C_{ijkl}q_jq_kq_l = (C_{abcd}q_aq_bq_cq_d)q_i, \quad (2.2)$$

$$C_{ijkl}r_jr_kq_l = (C_{abcd}q_aq_dr_b r_c)q_i, \quad (2.3)$$

for all directions \mathbf{r} perpendicular to \mathbf{q} , are necessary and sufficient that \mathbf{q} be normal to a plane of material symmetry.

The components of the tensor of third order elastic moduli referred to a rectangular basis are C_{ijklmn} , and are defined in Section 3 below. These also possess certain fundamental symmetries [10] analogous to (2.1)

$$C_{ijklmn} = C_{jiklmn}, \quad C_{ijklmn} = C_{klijmn} = C_{nmkl ij}. \quad (2.4)$$

It is clear that there are at most 56 independent third order moduli for a given material. In the presence of material symmetry, fewer moduli are involved, as few as 3 for the case of an isotropic material. Tables of the possible forms of the third order moduli tensors for all material symmetries were first given by Fumi [11, 12] and later by Brugger [13], and can now be found in several monographs [14], such as Thurston [10].

The following result is analogous to Theorem 1.

THEOREM 2. *The conditions*

$$C_{ijklmn}q_jq_kq_lq_mq_n = (C_{abcdef}q_aq_bq_cq_dq_eq_f)q_i, \quad (2.5)$$

$$C_{ijklmn}q_jq_kq_n r_l r_m = (C_{abcdef}q_aq_bq_cq_e r_d r_f)q_i, \quad (2.6)$$

$$C_{ijklmn}r_j r_k q_l q_m q_n = (C_{abcdef}q_a q_d q_e q_f r_b r_c)q_i, \quad (2.7)$$

$$C_{ijklmn}r_j r_k s_l s_m q_n = (C_{abcdef}r_b r_c s_d s_e q_a q_f)q_i, \quad (2.8)$$

for all \mathbf{r} and \mathbf{s} perpendicular to \mathbf{q} are both necessary and sufficient that \mathbf{q} is perpendicular to a plane of material symmetry.

Proof. It will first be shown that these conditions are necessary. The reflection operator associated with a given direction \mathbf{q} is defined by the tensor \mathbf{R} with components [1]

$$R_{ij} = \delta_{ij} - 2q_i q_j. \quad (2.9)$$

Thus,

$$R_{ij} q_j = -q_i, \quad (2.10)$$

$$R_{ij} r_j = r_i, \quad (2.11)$$

for any \mathbf{r} perpendicular to \mathbf{q} . The direction \mathbf{q} is defined to be the normal to a plane of material symmetry if the moduli satisfy [1, 2]

$$C_{ijklmn} = R_{ia} R_{jb} R_{kc} R_{ld} R_{me} R_{nf} C_{abcdef}. \quad (2.12)$$

Condition (2.5) follows by contracting (2.12) with $(q_i q_k q_l q_m q_n)$, using (2.9) and (2.10) to simplify the resulting expression. Similarly, (2.6), (2.7) and (2.8) follow by contracting (2.12) with $(q_j q_k q_n r_l r_m)$, $(r_j r_k q_l q_m q_n)$ and $(r_j r_k s_l s_m s_n)$, respectively, and then using (2.9)–(2.11). This proves that (2.5)–(2.8) are a set of necessary conditions.

In order to prove the sufficiency of conditions (2.5)–(2.8), it is helpful to take \mathbf{q} in the x_3 direction, with no loss in generality. The standard concise notation C_{IJK} will be used to denote the components C_{ijklmn} , where $I = 1, 2, 3, 4, 5,$ and 6 correspond to $ij = 11, 22, 33, 23, 13$ and 12 . Condition (2.5) then becomes

$$C_{i33333} = C_{333} \delta_{i3}, \quad (2.13)$$

implying,

$$C_{344} = C_{335} = 0. \quad (2.14)$$

Similarly, (2.6) implies

$$C_{i33m3r_l r_m} = (C_{333m3} r_l r_m) \delta_{i3}, \quad (2.15)$$

for any \mathbf{r} in the x_1 - x_2 plane. Taking $\mathbf{r} = (1, 0, 0)$ and letting $i = 1$ and 2 , gives

$$C_{444} = C_{445} = C_{455} = C_{555} = 0. \quad (2.16)$$

Condition (2.7) becomes

$$C_{ijk333} r_j r_k = (C_{3jk333} r_j r_k) \delta_{i3}. \quad (2.17)$$

Selecting $\mathbf{r} = (1, 0, 0)$, $(0, 1, 0)$ and $(1/\sqrt{2}, 1/\sqrt{2}, 0)$ for both $i = 1$ and $i = 2$, yields

$$C_{134} = C_{135} = C_{234} = C_{235} = C_{346} = C_{356} = 0. \quad (2.18)$$

Finally, condition (2.8) becomes

$$C_{ijklm3} r_j r_k s_l s_m = (C_{3jklm3} r_j r_k s_l s_m) \delta_{i3}, \quad (2.19)$$

for any pair \mathbf{r} and \mathbf{s} in the x_1 - x_2 plane. Taking combinations of $\mathbf{r} = (1, 0, 0)$, $(0, 1, 0)$ and $(1/\sqrt{2}, 1/\sqrt{2}, 0)$ with $\mathbf{s} = (1, 0, 0)$, $(0, 1, 0)$ and $(1/\sqrt{2}, 1/\sqrt{2}, 0)$, for $i = 1$ and 2 , yields

$$C_{114} = C_{115} = C_{124} = C_{125} = C_{146} = C_{156} = 0, \quad (2.20)$$

$$C_{224} = C_{225} = C_{246} = C_{256} = C_{466} = C_{566} = 0.$$

It may be verified by other means [10] that the 24 moduli indicated in (2.14), (2.16), (2.18) and (2.20), are the same 24 moduli that vanish when the material has monoclinic symmetry with the x_3 direction normal to the symmetry plane. That is, there are at most 32 nonzero third order moduli for a monoclinic solid. This completes the proof of Theorem 2.

There are, of course, simpler sets of necessary conditions than those of Theorem 2. The simpler conditions may be useful, for example, if one is given the components C_{ijklmn} and wishes to determine the underlying material symmetry. With regard to the analogous problem for the second order elastic constants, Cowin and Mehrabadi [2] showed that the less restrictive conditions

$$C_{ikkj} q_j = (C_{jkk1} q_j q_1) q_i, \quad (2.21)$$

$$C_{ijkk} q_j = (C_{jllk} q_j q_l) q_i, \quad (2.22)$$

are necessary in order that \mathbf{q} be normal to a plane of symmetry. These conditions may be derived by the method discussed below, or by using (2.2) and (2.3). The procedure for finding the symmetry using (2.21) and (2.22) is to first calculate the proper vectors of the tensors C_{ijkk} and C_{ikkj} . If these possess no common proper vectors, then the material has no underlying symmetry and is triclinic. On the other hand, any common proper vectors must be tested further to see if they also satisfy (2.2) and (2.3), and if they do they are normals to symmetry planes. This scheme, combined with Cowin and Mehrabadi's [2] classification of material symmetry in terms of symmetry planes, provides an algorithm for determining the symmetry from the 21 components C_{ijkl} .

It is not even necessary to compute the proper vectors of these tensors to determine whether or not the material possesses any symmetry. The following addendum to Theorem 1 relies on the fact [15] that two semisimple tensors commute if and only if they have a complete set of proper vectors in common.

COROLLARY. Define \mathbf{A} as the skew-symmetric commutator of the tensors C_{ijkk} and C_{ikkj} , i.e.,

$$A_{ij} = C_{ikmm}C_{knnj} - C_{innk}C_{kjmm}.$$

- (i) A necessary condition for the material to have orthorhombic or higher symmetry is that $\mathbf{A} = 0$.
- (ii) If \mathbf{A} is nonzero and its null vector, with components $e_{ijk}A_{jk}$, is a proper vector of both C_{ijkk} and C_{ikkj} , then the material may be monoclinic. Otherwise the material has no underlying symmetry.

The first statement is a consequence of the classification scheme of Cowin and Mehrabadi [2], combined with the completeness proof of Huo and Del Piero [3], which states that all symmetries higher than monoclinic require at least three symmetry planes.

The simpler set of necessary conditions (2.21) and (2.22) can be derived by noting that any tensor of lower order formed by contracting indices must satisfy the same symmetries as the original tensor. The same principle applies equally to the tensor of third order elastic moduli. For example, the second order tensor of components C_{ijkkl} must satisfy

$$C_{ijkkl} = R_{ia}R_{jb}C_{abkkll}, \quad (2.23)$$

where \mathbf{R} is defined in (2.9), if \mathbf{q} is to be normal to a plane of symmetry. Similar conditions can be stated for the second order tensors of components C_{ijklt} , C_{ikjkl} and C_{ikjlt} . Use of (2.9)–(2.11) then leads to the following as

simultaneous necessary conditions for \mathbf{q} to be normal to a symmetry plane,

$$C_{ijkkl}q_j = (C_{mjkkl}q_jq_m)q_i, \quad (2.24)$$

$$C_{ijkkl}q_j = (C_{mjkkl}q_jq_m)q_i, \quad (2.25)$$

$$C_{ikjkl}q_j = (C_{mkjkl}q_jq_m)q_i, \quad (2.26)$$

$$C_{ikjkl}q_j = (C_{mkjkl}q_jq_m)q_i. \quad (2.27)$$

These conditions can also be shown to follow from (2.5)–(2.8) by using arguments similar to those employed by Norris [6] in deriving (2.21) and (2.22) as consequences of (2.2) and (2.3).

Suppose that the 56 third order moduli are known relative to some rectangular coordinate system. Then the procedure to determine what, if any, symmetry the material possesses is to first find the common proper vectors of the four tensors with components C_{ijkkl} , C_{ijkkl} , C_{ikjkl} and C_{ikjkl} . If no common proper vectors exist, the material is triclinic. If there is one common proper vector, the material may be monoclinic, but not of higher symmetry. In order to ascertain whether the material is indeed monoclinic it must be checked that the single proper vector fully satisfies all the conditions (2.5)–(2.8). Similarly, if more than one common proper vector is found, then each must be checked in the same manner. Note that if there are two proper vectors in common, the highest symmetry the material can possess is again monoclinic since orthorhombic or higher symmetry requires the existence of at least three planes of symmetry. Conditions similar to those stated in the Corollary to Theorem 1 also exist for determining whether any symmetry is present. For example, a necessary condition that the material has at least orthorhombic symmetry is that each of the six commutators formed from C_{ijkkl} , C_{ijkkl} , C_{ikjkl} and C_{ikjkl} must be identically zero.

3. The nonlinearity parameter for elastic waves

The second and third order elastic moduli may be defined through the Taylor series expansion of the strain energy W with respect to an unstressed reference configuration,

$$W = \frac{1}{2}C_{ijkl}E_{ij}E_{kl} + \frac{1}{6}C_{ijklmn}E_{ij}E_{kl}E_{mn} + \dots \quad (3.1)$$

Here, \mathbf{E} is the Green-St. Venant strain tensor,

$$E_{ij} = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij}), \quad (3.2)$$

and \mathbf{F} the deformation gradient tensor,

$$F_{ij} = \frac{\partial x_i}{\partial X_j}, \quad (3.3)$$

which relates the current position of a material particle, \mathbf{x} , to the reference position of the same particle, \mathbf{X} . The non-symmetric Piola-Kirchhoff stress tensor, \mathbf{T} , can be derived from the stored energy as

$$T_{ij} = \frac{\partial W}{\partial F_{ij}}. \quad (3.4)$$

The Taylor series expansion for the stress with respect to the unstressed reference configuration is

$$T_{ij} = C_{ijkl}(F_{kl} - \delta_{kl}) + \frac{1}{2}B_{ijklmn}(F_{kl} - \delta_{kl})(F_{mn} - \delta_{mn}) + \dots, \quad (3.5)$$

where

$$B_{ijklmn} = C_{ijklnm} + C_{ijlm\delta_{kn}} + C_{jnkl\delta_{im}} + C_{jlmn\delta_{ik}}. \quad (3.6)$$

The second and third order moduli C_{ijkl} and C_{ijklmn} satisfy the symmetries (2.1) and (2.4), respectively; however, B_{ijklmn} do not satisfy all the symmetries in (2.4), specifically

$$B_{ijklmn} \neq B_{jklmni}. \quad (3.7)$$

We consider acceleration waves in hyperelastic materials with no prestrain, with quiescent conditions ahead of the wave, and such that the wavefront is a plane with constant normal \mathbf{n} . These are fairly strong restrictions upon the type of acceleration wave allowed, and are placed in perspective within the general theory of acceleration waves by Chadwick and Ogden [8]. This paper also provides an exhaustive discussion on the connection between the different definitions of elastic moduli. Let the speed of the wave be v , then the associated displacement polarization direction \mathbf{m} must satisfy [8, 16]

$$(C_{ijkl}n_jn_l - \rho v^2\delta_{ik})m_k = 0, \quad (3.8)$$

where ρ is the material density. The growth in amplitude of the acceleration

wave is dependent solely on the nonlinearity parameter β , defined as [8, 16]

$$\beta = \frac{\partial^2 T_{ij}}{\partial F_{kl} \partial F_{mn}} m_i m_k m_m n_j n_l n_n$$

$$= B_{ijklmn} m_i m_k m_m n_j n_l n_n. \quad (3.9)$$

The vanishing of β is physically significant as it means the acceleration wave will not develop shocks, irrespective of the initial sign of the acceleration wave amplitude [16].

It is known, for instance, that β vanishes identically for transverse waves in isotropic materials, a result established by Green [7]. This may be seen by noting that C_{ijkl} and C_{ijklmn} for an isotropic solid must be of the form [10]

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (3.10)$$

$$C_{ijklmn} = (C_{112} - 2C_{155}) \delta_{ij} \delta_{kl} \delta_{mn} + C_{155} [\delta_{ij} (\delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm})$$

$$+ \delta_{kl} (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) + \delta_{mn} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})]$$

$$+ C_{456} (e_{ikm} e_{jln} + e_{jkm} e_{iln} + e_{ilm} e_{jkn} + e_{ikn} e_{jlm}). \quad (3.11)$$

Here e_{ijk} are the components of the third order alternating tensor, $e_{123} = -e_{213} = 1$, etc.; λ and μ are the Lamé moduli of linear elasticity; and C_{112} , C_{155} and C_{456} are the third order elastic moduli. Green's [7] result that $\beta = 0$ for transverse waves in isotropic solids follows from (3.6) and (3.9)–(3.11) with $\mathbf{m} \cdot \mathbf{n} = 0$.

4. The nonlinearity parameter for transverse waves

The nonlinearity parameter may be expressed, via (3.6) and (3.9), with \mathbf{m} and \mathbf{n} defined as in Section 3,

$$\beta = C_{ijklmn} m_i m_k m_m n_j n_l n_n + 3C_{ijkl} m_i n_j n_k n_l. \quad (4.1)$$

This form for β shows the dependence upon both the second and third order moduli [8, 17]. If the wave motion is purely transverse, then \mathbf{m} is perpendicular to \mathbf{n} , and it follows from (3.8) and (4.1) that the dependence upon the second order elastic moduli disappears,

$$\beta = C_{ijklmn} m_i m_k m_m n_j n_l n_n. \quad (4.2)$$

It is clear from (2.2) and (3.8) that the direction normal to a plane of material symmetry supports a longitudinal wave, i.e. one for which \mathbf{m} and \mathbf{n} are parallel. The same direction therefore supports two transverse waves polarized in the plane of symmetry. In addition, (2.3) and (3.8) imply that a transverse wave polarized in the direction normal to a plane of symmetry can propagate in any direction in the symmetry plane. Let us denote the combined class of transverse waves, with $\infty + 2$ members, as the transverse waves associated with the plane of material symmetry. The existence of this class of transverse waves was noted by Fedorov [4].

The nonlinearity parameter for a wave in this class is given by (4.2), since it is a transverse wave. Furthermore, $C_{ijklmn}m_i m_k m_m n_j n_l n_n$ vanishes by virtue of (2.6), implying the result

THEOREM 3. *The nonlinearity parameter, β , vanishes for every transverse wave associated with a plane of material symmetry.*

Every plane is one of material symmetry in an isotropic solid, and so Green's [7] result that $\beta = 0$ for transverse waves in every isotropic solid is a simple consequence of Theorem 3. In a transversely isotropic solid, every plane containing the axis of symmetry is a plane of symmetry. Therefore $\beta = 0$ for all TH waves and for TV waves propagating both parallel and perpendicular to the symmetry axis in transversely isotropic materials. The application of the results of Theorem 3 to solids of material symmetries higher than triclinic, which has no symmetry, is apparent. In general, as long as the material is of monoclinic symmetry or higher, there will be infinitely many transverse waves for which $\beta = 0$. This does not mean that $\beta = 0$ for all transverse waves in elastic solids, since there are many transverse waves which are not associated with planes of material symmetry. Thus, there is at least one direction on every great circle of the unit sphere which supports a purely transverse wave. The proof of this surprising result is contained in a paper by Chadwick and Currie [18] on transverse waves in materials far more general than those considered here. We note in particular that the proof is independent of any considerations of material symmetry, and so there is no reason to expect that $\beta = 0$ for these transverse waves.

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