Habit Persistence or Durable Consumption

Economics 503: Mathematical Methods for Macroeconomics (Spring 2004)

Consider[1] the following Bellman Equation associated with the growth problem with habit persistence or durable consumption:

$$\text{For all } k \geq 0,\ c_{-1} \geq 0:$$

$$v(k, c_{-1}) = \max_{\{c \in [0, f(k)], k' \in [0, f(k)]\}} \{u(c, c_{-1}) + \beta v(k', c)\} \text{ subject to } c + k' = f(k);$$

where c and c_{-1} are current and past consumption; k and k' are current and future capital; $\beta \in (0, 1)$; the utility function $u(c, c_{-1})$ is bounded, continuously differentiable, strictly increasing in both arguments, jointly strictly concave in (c, c_{-1}), with $\lim_{c \to 0} u_1(c, c_{-1}) = +\infty$, for all $c_{-1} \geq 0$; the production function $f(k)$ is bounded, continuously differentiable, strictly increasing, strictly concave, with $f(0) = 0$. The model is interpreted as an economy with habit persistence if $u_{1,2} > 0$, or an economy with durable consumption if $u_{1,2} < 0$.

Equivalently, consider the following Bellman Equation:

$$\text{For all } k \geq 0,\ c_{-1} \geq 0:$$

$$v(k, c_{-1}) = \max_{\{k' \in [0, f(k)]\}} \{u(f(k) - k', c_{-1}) + \beta v(k', f(k) - k')\}.$$

i) Show that there exists a unique bounded and continuous value function $v(k, c_{-1})$ solving the previous Bellman Equation, and that the optimal policy correspondence $G(k, c_{-1})$ is compact-valued and u.h.c.

ii) Show that the value function is strictly increasing in both arguments.

iii) Show that the value function is jointly strictly concave in (k, c_{-1}) and the optimal policy correspondence is a continuous single-valued function.

iv) Show that the value function is continuously differentiable in the interior of the state space. Write the two envelope conditions and the first order condition.

v) Show that, unfortunately, the first order condition is not very informative, even if we make further assumptions on the utility function. What assumptions about the utility function and what information about the value function do we need to characterize the policy function as a function of k and c_{-1}?

Let us make, now, the further assumption that the utility function is additively separable in c and c_{-1}, so $u(c, c_{-1}) = m(c) + n(c_{-1})$.

vi) Show that the value function is additively separable in k and c_{-1}. Show that the policy function does not depend on c_{-1}. Reinterpret the problem as the standard growth problem with capital as the only state variable.

vii) Write the first order condition, show that the policy function is strictly increasing in k and find a bound for its slope.

Finally, suppose that $u(c, c_{-1}) = \ln(c) + \gamma \ln(c_{-1})$ and that $f(k) = Ak^{\alpha}$, where $\gamma > 0$, $A > 0$ and $\alpha \in (0, 1)$.

viii) Guess the value function, recalling that the value function for the standard growth problem ($\gamma = 0$) is $E + F \ln(k)$, for some strictly positive E and F.

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[1] The problem has been inspired by Exercises 1.4 and 1.5 of Sargent's Dynamic Macroeconomic Theory.