System Reliability Optimization Considering Uncertain Future Operating Conditions and Usage Stresses

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Abstract - This paper develops a new system reliability design optimization model based on a series-parallel system, where each subsystem is composed of components that are chosen to optimize the system reliability, while considering uncertainty of future usage conditions. In previous research, component reliability is generally assumed to be known with certainty because the usage and operating stresses are either not changing or they are known with certainty. However, this paper proposes a more general perspective in analyzing the uncertainty of actual system usage and operating stresses. In industry, predictions of component reliability often have inaccuracy or uncertainty due to unplanned variation, or changing environments and operating stresses. Sensitivity of particular component reliability occurs due to a relative increase/decrease of operating forces or stresses. A risk-neutral design is considered for the system reliability optimization model with a probability associated with defined possible future usage conditions. The optimal redundancy allocation problem for each subsystem is composed of multiple choices of components with system-level constraints. The system is designed to maximize expected system reliability while considering the uncertainty of future component usage conditions and stresses. Nonlinear integer programming and a neighborhood search heuristic was used to solve this new problem formulation.

1. Introduction

This paper pertains to system reliability optimization considering uncertainty of future usages and associated stress, loading, etc. for each component. The system and each component have a different stress response based on a given set of future circumstances. In this new model, it is known that system and component usage stresses will change, but it is not known what the future usage stresses will be. This stress response is referred to as the component sensitivity, and it has an impact on component reliability. Solving the optimal redundancy allocation maximizes the expected system reliability subjected to cost and weight constraints while considering uncertainty of future usage profile.

Previous system reliability optimization studies usually assume the components in the system have reliability values that are already known, i.e., deterministic. This assumption, however, is not always appropriate, especially in practice. Often times, component reliability is derived from subject matter expert knowledge or estimated by using statistical inference techniques on testing data. Moreover, when complex systems are being designed, the selection of components plays a critical role in the overall system reliability. Determining the optimal system design that maximizes the system reliability subject to important constraints (i.e., cost and space) has been studied in previous research, but the new model described in this paper extends this concept to system design when the future usage is unknown for the components to be selected.

The optimal design of a reliable system in the presence of uncertainty has been investigated in detail in previous research. Coit [1] developed a flexible procedure to approximate confidence intervals for system reliability when there is uncertainty in component reliability information. The method used was entitled SRCI, and it determined lower confidence bounds on the system reliability based on empirical estimates of component reliability. An optimization model considering risk profile of system designers and users has also been formulated [2]. This work was further advanced by Coit & Smith [3] who proposed the use of a genetic algorithm (GA) to maximize a lower-bound for system reliability [4], when there is uncertainty in the Weibull distribution parameters for component failure times.

Coit et al. [5] extended this work by assuming that component reliability is itself a random variable. Multiple objective optimization methods were then applied to maximize system reliability and to simultaneously minimize the associated variance. Pareto optimal solutions were found by varying the weights of the objective functions. Marseguerra et al. [6] investigated optimal network design in the presence of component reliability by combining Monte Carlo simulation methods along with a multiple objective GA approach to optimize system reliability.

The most recent work done on the redundancy allocation problem considering uncertainty was done by Coit & Tekiner-Mogulkoc [7,8]. This work used integer linear programming to select the optimal system design under component reliability estimation uncertainty by minimizing the coefficient of variance in series-parallel systems. Assuming uncertainty for failure threshold for the reliability analysis is extended to a degradation model as well [9]. Other relevant papers are [10,11,12].
The system considered in this paper is a series-parallel structure where each subsystem has a different number of components. The failure time for each component is distributed as Weibull distributions with changeable scale parameter depending on future usage.

1.1 Series-Parallel System

The system is comprised of s subsystems connected in series. Within each subsystem i, there are n_i components connected in parallel as shown in Figure 1. Figure 2 shows a simple example where there are 3 component choices for subsystem 1, two component choices for subsystem 2, and so on. Specifically in the example, the components in subsystem 1 are, x_11=2, x_12=0, x_13=1 and the components in subsystem 2 are x_21=1, x_22=4. In our model, the component selections for a particular subsystem are chosen by solving the optimal redundancy allocation problem, subject to cost and weight constraints.

![Figure 1. Series-parallel system with different number of components in each subsystem](image1)

![Figure 2. Series-parallel system with multiple choices of components in each subsystem](image2)

1.2 Component Weibull Distribution for Future Usage and Operating Stresses

In previous research, component reliability is generally assumed to be deterministic, and the optimal redundancy allocation is then solved as a deterministic linear or nonlinear optimization model. In practice, however, assuming component reliability values are known with certainty is not always feasible or appropriate, and can impact the quality of the final design configuration and system reliability. In this section, system reliability is described by a new model and problem formulation. In the new model, each component choice has different reliability functions and failure time distributions, which depend on the specific effect of usage conditions (e.g., impact of stress usage or operating conditions on the overall system). Let u_i = (u_{i1}, u_{i2},...,u_{il}) represent a vector of operating usage conditions and/or stresses. For example, for electronic components, temperature is a critical contributor to component failure (u_{i1}), and for a given future operating stress profile, the risk of failure increases along with increasing temperatures. For mechanical components, mechanical loading (u_{i2}) and stress (u_{i3}) are important factors.

![Figure 3. Probability on future usage profile](image3)

In our model, the component selections for a particular subsystem are chosen by solving the optimal redundancy allocation problem, subject to cost and weight constraints.

![Figure 4. Comparison of component Weibull distribution for failure time for different futures](image4)

As shown in Fig. 4, in different levels of component usage and stress, the sensitivity of operating stress of a component failure time distribution changes accordingly. Since it is unknown which future will occur, the component failure time distributions are also uncertain.

This new formulation has specific benefits compared to the other reliability optimization models that considered uncertainty. The concept of uncertain future usage is crucial to this work, as it is the main source of uncertainty in this formulation. The main advantage of this work is that it is practical and it can be applied to serve the needs of various industrial applications. The paper is organized by starting with the introduction and background of reliability optimization considering uncertainty. Section 2 defines reliability analysis and probability of future usage evaluation. The optimization for redundancy allocation is provided in Sections 3. Numerical examples are given in Section 4.

**Notation**

- \( R(t) \): System reliability at time \( t \);
- \( r_j(u;l) \): component reliability for \( j \)th component selection
- \( x_j \): type to be used in subsystem \( i \) in future usage \( l \)
- \( s \): number of the \( j \)th component selection type to be used in subsystem \( i \)
- \( m_i \): number of components selection types or choices for subsystem \( i \)
- \( p_l \): probability of future usage \( l \), \( l=1,2,...,v \)
- \( U \): random future usage profile vector, \( U = (U_1, U_2, ..., U_v) \)
\( u_l = \) future usage profile vector \( l \), \( u_l = (u_{1l}, u_{2l}, \ldots, u_{cl}) \),
\( U \in \{ u_1, u_2, \ldots, u_c \} \)
\( v = \) number of possible future usage profile
\( c = \) number of different operating usage and stress factors
\( a_{ijk} = \) Sensitivity coefficient for component type \( j^{th} \) in subsystem \( i \) of stress factor \( k \)
\( \eta_{ijk} = \) current scale parameter of Weibull distribution used for the reliability model of the \( j^{th} \) component selection type to be used in subsystem \( i \)
\( \eta_i(u_l) = \) Weibull scale parameter of component type \( j \) in subsystem \( i \) for future usage profile vector \( l \)
\( \beta_{ij} = \) Weibull shape parameter of component type \( j \) in subsystem \( i \)
\( w_{ij} = \) physical weight of component type \( j^{th} \), subsystem \( i \)
\( W = \) weight constraint
\( c_{ij} = \) cost of component type \( j^{th} \) at subsystem \( i \)
\( C = \) cost constraint
\( t = \) time \((t = 100 \text{ hours})\)

2. Reliability Formulation Analysis

A reliability formulation for the system is presented where the component failure times are distributed according to a Weibull distribution that considers uncertainty in future operating and usage stresses that result in changes to the Weibull distribution scale parameter. The uncertainty of the Weibull scale parameter is due to the unknown future usage conditions. Each of the components in the parallel configured subsystems is selected from a specified number of defined component choices. Once these components are selected for all subsystems, these subsystems are then connected in series.

2.1 Series-Parallel System Reliability with Component Choice Example

The structure of our proposed system is made up of \( s \) subsystems in series, where each subsystem has \( m_i \) component choices for the parallel configuration. Since the number of identical components of a particular type, \( x_j \), is connected in parallel within each subsystem, the system reliability is given as follows

\[
R(x, U; t) = \prod_{i=1}^{s} \left(1 - \sum_{j=1}^{m_i} (1 - r_{ij}(U; t))^{x_j} \right)
\]  
(1)

\( U \) is a random vector, so \( R(x, U; t) \) is also a random variable.

The Weibull scale parameter for component reliability is a function of operating usage and stress factors vector \( u_l \) and coefficient of operating usage and stress factors \( a_{ijk} \) to provide the relative effect on different future stresses. There are \( c \) different stress variables and corresponding coefficients of operating stress factors determined for the future \( l \) with a probability \( p_l \). \( \eta_{ijk} \) and \( r_{ij}(U; t) \) are given as follows

\[
\eta_{ijk}(u_l) = \eta_{ijk} \exp\left(-\sum_{k=1}^{v} a_{ijk} u_{kl} \right)
\]  
(2)

\[
r_{ij}(U; t) = \exp\left(-\frac{t}{\eta_{ijk}(u_l)} \right)^{\beta_{ij}}
\]  
(3)

The reliability model is sensitive to component reliability changing because each component and the system have potentially different operating stress impacts. The operating stress factors and coefficients for the stress factors for each component choice are required for each future usage since varying circumstance of each future usage can influence the selection of each component.

2.2 Risk Neutral System Reliability with Component Choices Considering Uncertain Future Usage

Obtaining the reliability model for our proposed system, determination of expected system reliability is based on possible future usages and their corresponding probability. The expected system reliability with random vector \( U \) and the number of components for each choice are defined as follows.

Expected system reliability can then be determined as

\[
E_U[R(x, U; t)] = \sum_{i=1}^{v} p_i \prod_{j=1}^{m_i} (1 - \sum_{j=1}^{m_i} (1 - r_{ij}(U; t))^{x_j})
\]  
(4)

Operating stress factors vector \( u_l \) and coefficient of usage and stress factors \( a_{ijk} \) are introduced in Equation (5). The expectation of system reliability as a function of decision variables \( x_j \), equation can obtain as follows

\[
E_U[R(x, U; t)] = \sum_{i=1}^{v} p_i \prod_{j=1}^{m_i} (1 - \exp(-t/\eta_{ijk}(u_l) \sum_{k=1}^{v} a_{ijk} u_{kl})^{\beta_{ij}})
\]  
(5)

The risk is related to the probability of the unknown future usage stress factors. Risk is much higher than in the deterministic case because of the uncertainty and the expected value of system reliability is used to perform system design optimization for a risk-neutral decision-maker.

3. Optimization Model

The objective is to maximize expected system reliability by determining the component choices, and the number of components for each choice in each subsystem. Expected system reliability is maximized considering cost and weight of components and system-level constraints. This is a new formulation for the system reliability optimization problem, and it provides for unique perspectives on reliability design.

3.1 Series-Parallel System Optimization with Component Choice Example

For series-parallel systems, the redundancy allocation problem is formulated for a system with decision variable \( x_j \). There are \( m_i \) choices of components. The risk-neutral mathematical formulation of the redundancy allocation problem is to maximize expected system reliability considering uncertain future usage as is follows.

In this optimization problem, possible future usages are related to probability \( p_i \) to compute the expected value of the system reliability. The decision variables \( x_j \) are selected for each component choice \( j \) at subsystem \( i \) to maximize the objective function considering cost and weight constraints.
max \( E_{u}[R(x, U; t)] = \sum_{i=1}^{n} p_i \prod_{j=1}^{m} (1 - \prod_{j=1}^{m} (1 - r_j(u_i; t))^s_j) \)

s.t. \( \sum_{i=1}^{s} \sum_{j=1}^{m} c_{ij} x_{ij} \leq C \)
\( \sum_{i=1}^{s} \sum_{j=1}^{m} w_{ij} x_{ij} \leq W \)
\( x_{ij} \in \{0,1,2,3,...\}, i = 1,2,...,s \)

A heuristic approach was used to solve the problem by relaxing integer constraints to initially solve a continuous nonlinear optimization problem, and then a Neighborhood heuristic search is applied to determine an integer solution.

### 3.2 Ceiling and Floor Neighborhood Search Heuristic Algorithm

After determining an optimal non-integer solution by solving a continuous nonlinear optimization model, a heuristic is applied to determine a recommended integer solution. For searching the number of components in each subsystem, a heuristic approach is presented based on Ceiling and Floor Greedy Neighborhood search to determine a final solution.

The Floor Neighborhood Search proceeds as follows:

1. Based on the optimal continuous solution, a lower neighborhood of redundancy allocation integer solutions is generated around the optimum continuous solution.

2. The neighborhood solution provides an estimated lower bound of the solution where we can obtain integer number of components. We consider different solutions by increasing a component to each subsystem to obtain the objective function value. Select the subsystem to added with the maximum value in crease.

3. The number of components for every subsystem is determined as the decision variable that can maximize objective function without violating constraints. Constraints are computed and checked for violations.

4. When iterating solutions that do not violate constraints, the algorithm continues to increase and iterate to the second largest objective function increase, and so on. If they violate a constraint, the previous solution is selected.

The Ceiling Neighborhood Search is as follows:

1. Based on the optimal continuous solution from part A, a neighborhood of redundancy allocation integer solution is generated around the optimum solution.

2. The neighborhood solution provides an estimated upper bound of the solution where we can obtain integer number of components. We consider different solutions by decreasing a component once each subsystem to obtain the objective function value. Select the subsystem of minimum value out.

3. The number of component is determined as the decision variable that can maximize the objective function. Constraints are computed and checked for violations.

4. If checking whether solutions violate constraints, the algorithm continues to decrease and iterate to the second, and so on. If they does not violate a constraint, take the optimal number of components

### 4. Illustrative Example

In the example, there are 14 subsystems in the proposed system \((s=14)\) and reliability \(r_j(t=100)\) is given [5], the Weibull shape parameter is specified as \(\beta_{ij} = 1.5\) for all components. Four factors of usage and operating stress, \(k\), occur in each future \(l\) with a probability \(p_{ikl}\). There are four possible futures \(l=1,2,3,4\) and component Weibull scale parameter variables is given by Equation (7).

\[ \eta_{ij}(u_i) = \eta_{ij} \exp((-\alpha_{ijk} u_i + \alpha_{ij} u_{ij} + \alpha_{ijk} u_{ij} + \alpha_{ijk} u_{ij})) \]

\(\alpha_{ijk}\) is a coefficient for the operating stress factors and each component can have different effects or sensitivity to stress/load profile. Different coefficient factors for each component are computed from test data based on testing conducted over a range of conditions. Table 1 presents the \(\alpha_{ijk}\) coefficients for the example problem with \(m_i = 1\) for each subsystem.

### Table 1: Usage and operating stress coefficients

<table>
<thead>
<tr>
<th>(i)</th>
<th>(a_{ijk})</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-4</td>
<td>(a_{11k})</td>
<td>0.6</td>
<td>0.45</td>
<td>0.2</td>
<td>0.6</td>
</tr>
<tr>
<td>5-8</td>
<td>(a_{11k})</td>
<td>0.5</td>
<td>0.35</td>
<td>0.4</td>
<td>0.2</td>
</tr>
<tr>
<td>9-11</td>
<td>(a_{11k})</td>
<td>0.7</td>
<td>0.5</td>
<td>0.2</td>
<td>0.6</td>
</tr>
<tr>
<td>12-14</td>
<td>(a_{11k})</td>
<td>0.4</td>
<td>0.4</td>
<td>0.6</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Future usage probability associated with operating condition factors \(u_i\) vector and coefficient of operating stress factors are shown in Fig. 5. The current operating and usage stress vector is given \(u_0 = 0\).

![Figure 5. Future usage probability (ui) is the operating usage and stress vector, l=1,2,…v](image)

From Figure 5, there is a specified probability that the system and all associated components will be exposed to future usage 1 with probability 0.1 and operating stress profile vector \(u_i\). Similarly for future usage 2, the influence of each operating stress factor to a component is dictated by \(u_i\) with probability 0.2 and so on.

### 4.1. Series-Parallel System Reliability Example

In this section, example problems are solved to demonstrate numerical results. MATLAB optimization toolbox was used to do optimization and perform Ceiling and Floor Greedy Neighborhood search algorithm to solve the mathematical formulation in Section 3.

In the example, the cost and weight constraints are given as \(C = 600\), \(W = 1000\). The Weibull shape parameter \(\beta_{ij}\) for all components is 1.5. The parameters of uncertain future usages are given in Table 1. Probability for each future usage occurrence is \((0.1,0.2,0.4,0.03)\) and only one component choice is used in every subsystem, \(m_i = 1\), \(t=100\), and \(x_{ij}\) is decision variable for all \(i\).

Table 2 and 3 show the component reliabilities of each future usage experiencing different operating stresses in each
subsystem $r_{ij}(u,t)$ and the system reliabilities of each future usage probability $p_i$ in each subsystem $R(u,t)$ with decision variables $x_{ij}$.

Table 2: Component reliability $r_{ij}(u,t)$ for different future operating and usage stresses

<table>
<thead>
<tr>
<th>i</th>
<th>$r_{ij}(u,t)$</th>
<th>$r_{ij}(u,t)$</th>
<th>$r_{ij}(u,t)$</th>
<th>$r_{ij}(u,t)$</th>
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<tbody>
<tr>
<td>1</td>
<td>0.223409</td>
<td>0.376291</td>
<td>0.824132</td>
<td>0.854886</td>
</tr>
<tr>
<td>2</td>
<td>0.482079</td>
<td>0.621639</td>
<td>0.910132</td>
<td>0.926511</td>
</tr>
<tr>
<td>3</td>
<td>0.099080</td>
<td>0.221434</td>
<td>0.742036</td>
<td>0.785178</td>
</tr>
<tr>
<td>4</td>
<td>0.070613</td>
<td>0.177548</td>
<td>0.710298</td>
<td>0.757845</td>
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<tr>
<td>5</td>
<td>0.621351</td>
<td>0.681921</td>
<td>0.897904</td>
<td>0.919292</td>
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<tr>
<td>6</td>
<td>0.925618</td>
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<td>0.982516</td>
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<tr>
<td>7</td>
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<td>0.422996</td>
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<td>0.827668</td>
<td>0.863680</td>
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</table>

Table 3: Sub-system reliability functions $R_i(u,t)$

<table>
<thead>
<tr>
<th>i</th>
<th>$R_1(u,t)$</th>
<th>$R_2(u,t)$</th>
<th>$R_3(u,t)$</th>
<th>$R_4(u,t)$</th>
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For this example, there is only one choice of component for every subsystem. Although that component can be duplicated multiple times to form a parallel subsystem, the objective function of maximizing the expected system reliability considers cost and weight constraints.

The number of components for redundancy allocation for the reliability optimization of this formulation problem is shown in Table 4. We obtain a solution having an expected system reliability of 0.9392 for the ceiling greedy neighborhood search and 0.9385 for the floor greedy neighborhood search.

5. Discussion

This paper presents the alternative model to consider the actual variation of component reliability caused by uncertain future usage and operating conditions and stresses. We can optimize expected system reliability by considering practical data corresponding to real stress circumstances. This reliability redundancy allocation problem is illustrated for which the failure density function of components is Weibull-distributed with uncertain scale parameter corresponding to uncertain future usages. The actual reliability analysis is considered for the system application taking into account operating stresses. The effective neighborhood search plays a major role in integer nonlinear solutions.

Table 4: Solution of optimization 1 and optimization 2

<table>
<thead>
<tr>
<th>Subsystem</th>
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<th>Subsystem</th>
<th>Number</th>
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6. References


