Characterizing Decoding Robustness under Parametric Channel Uncertainty

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***Abstract*— This paper characterizes the robustness of decoding under parametric channel uncertainty. We consider discrete channel models described by probabilistic graphical models, such as finite state channels. Using recent advances from the field of tropical geometry, we are able to partition the channel parameter space into cells corresponding to all channels that identically decode a given output codeword. The partition, based on a combinatorial object called the *Newton polytope* of the graphical model, can be efficiently computed in polynomial time (in terms of the number of parameters in the channel model). This partitioning of the channel parameter space provides two key results: 1) Given a nominal (vector) channel parameter, one can easily gauge the robustness of the decoding as a function of deviations from this nominal parameter; 2) Rather than obtaining one decoding for a single parametrization, one can obtain a list of decodings for a family of channel parametrizations at the decoder.**

I. INTRODUCTION

Success of a communication system lies in how well it can encode and decode data. While many of today's commonly employed channel decoding techniques are built on the premise that the decoder has knowledge of the true channel parameters, this is almost never the case in reality. In this paper, we attempt to characterize the robustness of decoding under parametric uncertainties in discrete channels described by probabilistic graphical models. The focus of the paper is not on modeling the uncertainties themselves, but rather on examining the relationship between decoding outputs and variations in the channel parametrization. Recent advances in the field of tropical geometry [1] enable us to partition the channel parameter space into cells consisting of "equivalent" channels. By equivalent we mean that for a given output codeword, all the channels described by the parameters in a given cell will produce the identical maximum a posteriori (MAP) decoding. This partitioning of the channel parameter space, which is a function of the family of graphical models and the output codeword, can be computed in time that is polynomial in the dimension of the channel parametrization and provides tremendous insight into the robustness of a given decoding. For example, suppose we have a nominal (vector) channel parameter and find that an identical decoding results from a large ball of channels about this nominal setting. In such a case, we may conclude that the decoding is robust to possible deviations of the true parameter from the nominal parameter. On the other hand, if the nominal parameter is near the boundary of a partition cell then a slight perturbation of the true parameter could lead to a completely different decoding, indicating a non-robust condition.

To motivate our investigation further, consider the simple example of a binary symmetric channel (BSC) with crossover probability p. Suppose that we employ block coding as a means of transmitting data over this BSC. In this situation, it is obvious that the MAP decoding rule is the nearest-neighbor decoding (in Hamming space) for $p < 1/2$, and the farthest-neighbor decoding for $p > 1/2$. Thus, the parameter space of a BSC can be partitioned into the intervals $[0, 1/2)$ and $(1/2, 1]$. If our nominal setting for p is close to $1/2$, then we see that we are in a very non-robust situation. While this partitioning of the one-dimensional parameter space of a BSC is trivial, the determination of analogous partitions for discrete channel models with higher dimensional parameter spaces is highly non-trivial. In the sequel, we present a unified computational approach to solving this problem for discrete channels described by graphical models (e.g, finite state (Markov) channels [2]).

II. CHANNEL PARAMETER SPACE PARTITIONING

In this section, we formally define the channel parameter space partitioning problem under consideration. Consider a point-to-point digital communication system with finite input codebook $\mathcal{X}$ and finite output codebook $\mathcal{Y}$. An input codeword $x \in \mathcal{X}$ is transmitted through a discrete channel which produces an output codeword $y \in \mathcal{Y}$. It is further assumed that the discrete channel is represented by a directed acyclic graphical model and the joint probability of an input-output codeword pair can be written as a monomial in a d-dimensional parameter $\theta = (a_1, ..., a_d)$, that is,

$$P(x, y; \theta) = a_1^{\nu_1(x,y)} a_2^{\nu_2(x,y)} \cdots a_d^{\nu_d(x,y)} \ , \qquad (1)$$

where the powers $\nu_j(x, y)$, $j = 1, \dots, d$, are integer-valued functions of the pair (x, y), and the parameter θ is an element of the parameter space Θ that defines all possible channels. Finite state channels (FSCs) [2] are one class of channel models having this form and, for the sake of this exposition, we would focus exclusively on them. In particular, as a special case of the general FSC, we will carefully investigate a two-state (binary input, quaternary output) intersymbol interference (ISI) channel model in Section V. Note that in the case of an FSC, one also includes dependence on the channel state, s, which takes value in the state space

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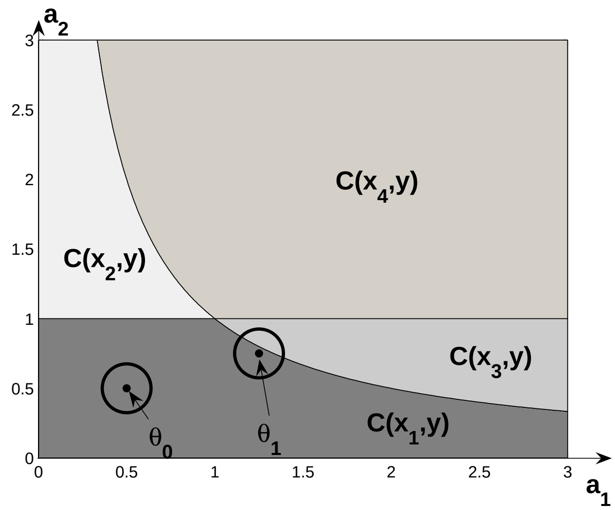

Fig. 1. Partitioning of the channel parameter space Θ for a given output codeword y.

$\mathcal{S}$ [2]. That is, the channel is characterized by the joint probability distribution of the input and output codewords, and the channel state as follows

$$P(x,y,s;\theta) = a_1^{\nu_1(x,y,s)} a_2^{\nu_2(x,y,s)} \cdots a_d^{\nu_d(x,y,s)} \,, \qquad (2)$$

where again the powers $\nu_j(x,y,s)$, $j = 1,\ldots,d$, are integer-valued functions. Finally, note that the MAP decoding rule for a fixed channel parameter θ_0 is given by

$$\widehat{x}(\theta_0) = \arg\max_{x\in\mathcal{X}} \left\{ \max_{s\in\mathcal{S}} P(x,y,s;\theta_0) \right\} \qquad (3)$$

or, alternatively, if the state of the channel is known to be $s^{(0)}$ through some side information then the MAP decoding is taken to be

$$\widehat{x}(\theta_0) = \arg\max_{x\in\mathcal{X}} P(x,y,s^{(0)};\theta_0). \qquad (4)$$

Our main interest here concerns the subset $\Theta_0 \subset \Theta$ such that $\widehat{x}(\theta) = \widehat{x}(\theta_0)$ for every $\theta \in \Theta_0$. Thus, Θ_0 defines the collection of channel parameters that are equivalent to θ_0 for the output codeword y. More generally, we are interested in a partition of Θ into subsets/cells corresponding to channel parameters that produce identical MAP decodings for the output codeword y. Partitions of this form allow us to assess the robustness of a given decoding and simultaneously recover a list decoding that is ordered relative to the proximity of cell boundaries to a nominal parameter setting.

To illustrate this idea further, consider the partition of a two-dimensional channel parameter space given by $\Theta = \{(a_1,a_2) \,:\, a_1,a_2 \geq 0\}$, as shown in Fig. 1. The partition of Θ is determined by the output codeword y (and the underlying graphical model) and the partition cells can be enumerated in terms of the input codewords. For example, $C(x_4,y) \subset \Theta$ is the set of all channel parameters that decode y to the input codeword x_4. To illustrate the notion of decoding robustness, consider the parameter θ_0. As can be seen from the figure, θ_0 is well contained within the interior of cell $C(x_1,y)$ and hence, all channel parameters within a sizeable ball of θ_0 will also decode y to x_1. Thus, we can say (loosely) that the decoding of y based on θ_0 is robust. On the other hand, parameter θ_1 is near the boundary of a cell and so the decoding of y based on θ_1 is not robust. It may be desirable in such cases to provide the decoder with a list decoding. In this specific case, the ordered list might be (x_1,x_3,x_4,x_2) based on the proximity of cell boundaries to θ_1, which is immediately available from the partition.

III. Newton Polytopes of Finite State Channels

In this section, we introduce concepts from tropical geometry that enable the polynomial-time construction of the partition of the channel parameter space. For channels with joint probability distributions of the form given in (2), we can write the (marginal) probability distribution of y as

$$\begin{aligned} f_y &:= P(y;\theta) \\ &= \sum_{x\in\mathcal{X}} \sum_{s\in\mathcal{S}} P(x,y,s;\theta) \\ &= \sum_{x\in\mathcal{X}} \sum_{s\in\mathcal{S}} a_1^{\nu_1(x,y,s)} a_2^{\nu_2(x,y,s)} \cdots a_d^{\nu_d(x,y,s)}. \end{aligned} \qquad (5)$$

Now consider the tropicalization of f_y, denoted by g_y, which is obtained by replacing the operators $(+,\times)$ with the operators $(\min,+)$ and a_i with $-\log(a_i)$ [1], [3]. This gives us

$$\begin{aligned} g_y &= \min_{x\in\mathcal{X}} \min_{s\in\mathcal{S}} \sum_{i=1}^{d} \left(-\nu_i(x,y,s)\log(a_i)\right) \\ &= \min_{x\in\mathcal{X}} \min_{s\in\mathcal{S}} \left\langle \nu(x,y,s), -\log(\theta)\right\rangle , \end{aligned} \qquad (6)$$

where $\langle\cdot,\cdot\rangle$ is used to denote an inner product, $\nu(x,y,s) = (\nu_1(x,y,s),\,\ldots,\,\nu_d(x,y,s))$ and $-\log(\theta) = (-\log(a_1),\,\ldots,\,-\log(a_d))$. The interesting thing to observe here is that g_y coincides with the negative-log joint probability evaluated at the MAP values of x and s.

The Newton polytope of f_y, denoted by $\mathrm{NP}(f_y)$, is simply the convex hull of $\nu(x,y,s)$ for all $x\in\mathcal{X}$ and $s\in\mathcal{S}$, namely,

$$\mathrm{NP}(f_y) = \mathrm{conv}\left\{\nu(x,y,s) : x\in\mathcal{X}, s\in\mathcal{S}\right\}. \qquad (7)$$

A nice consequence of tropicalizing f_y by g_y is that the function g_y is piecewise linear on the cones in the normal fan of $\mathrm{NP}(f_y)$. Note that the normal cone to a closed, convex set $K \subset \mathbb{R}^d$ at the point $v\in K$ is traditionally defined to be the set $\mathrm{NC}(v,K)\subset\mathbb{R}^d$ such that

$$\mathrm{NC}(v,K) = \{b\in\mathbb{R}^d : \langle v-u, b\rangle \geq 0 \,\forall\, u\in K\}. \qquad (8)$$

However, in this context, we define the normal cone of a vertex $v \in \mathrm{NP}(f_y)$ to be the set of (log) parameters $\mathrm{NC}(v,f_y)$ such that v **minimizes** $\langle v,b\rangle$ for all $b\in\mathrm{NC}(v,f_y)$ and all $u\in\mathrm{NP}(f_y)$, namely,

$$\mathrm{NC}(v,f_y) = \{b\in\mathbb{R}^d : \langle v-u,b\rangle \leq 0 \,\forall\, u\in\mathrm{NP}(f_y)\}. \qquad (9)$$

The usefulness of these Newton polytopes lies in the fact that, given the nominal channel parameter θ_0, they can be used to decode a given output codeword y to the optimal

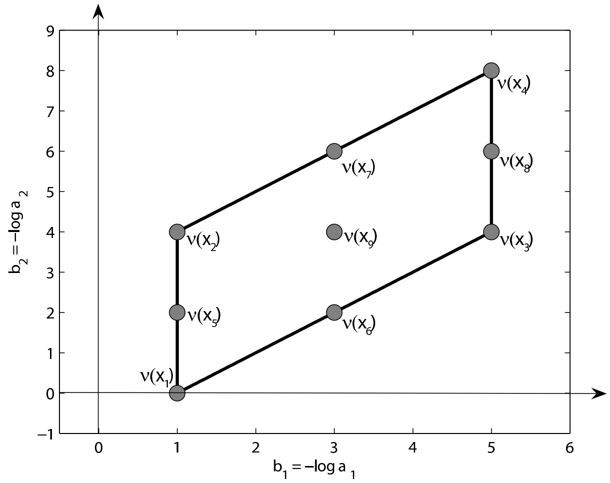

Fig. 2. An example of a Newton polytope.

input codeword $\widehat{x}$ and optimal state vector $\widehat{s}$. To observe this, note that

$$\begin{aligned}\widehat{x}(\theta_0), \widehat{s}(\theta_0) &= \arg \max_{x \in \mathcal{X}, s \in \mathcal{S}} P(x, y, s; \theta_0) \\ &= \arg \min_{x \in \mathcal{X}, s \in \mathcal{S}} -\log(P(x, y, s; \theta_0)) \\ &= \arg \min_{x \in \mathcal{X}, s \in \mathcal{S}} \sum_{i=1}^{d} (-\nu_i(x, y, s) \log(a_i)) \\ &= \arg \min_{x \in \mathcal{X}, s \in \mathcal{S}} \langle \nu(x, y, s), -\log(\theta_0) \rangle \qquad (10)\end{aligned}$$

and thus, by a simple convexity argument, the point $\nu(\widehat{x}(\theta_0), y, \widehat{s}(\theta_0))$ is a vertex of $\mathrm{NP}(f_y)$; hence, the point $b_0 = -\log(\theta_0)$ lies in $\mathrm{NC}(\nu(\widehat{x}(\theta_0), y, \widehat{s}(\theta_0)), f_y)$. If the optimal channel state vector is deemed to be unimportant, we may simply decode y to $\widehat{x}(\theta_0)$ as given in (3). Additionally, if the channel state vector is known, we may use (4) to decode the output codeword y.

Finally, note that typically one would expect the number of possible decodings to be on the order of $|\mathcal{X}||\mathcal{S}|$, that is, exponential in the input codeword length. A rather remarkable consequence of using Newton polytopes for MAP detection is that the number of possible decodings for a fixed output codeword y is, at worst, polynomial in the number of parameters in the graphical model [3], [4].

Example 1 (A Newton Polytope): Consider a fixed observation y arising from a graphical model with hidden variable X over nine possible values $x_1, ..., x_9$. Assume that the channel has only one state. Let the joint probabilities be given by

$$\begin{aligned}
P(X = x_1, Y = y) &= a_1 && \Rightarrow \nu(x_1) = (1, 0)\\
P(X = x_2, Y = y) &= a_1 a_2^4 && \Rightarrow \nu(x_2) = (1, 4)\\
P(X = x_3, Y = y) &= a_1^5 a_2^4 && \Rightarrow \nu(x_3) = (5, 4)\\
P(X = x_4, Y = y) &= a_1^5 a_2^8 && \Rightarrow \nu(x_4) = (5, 8)\\
P(X = x_5, Y = y) &= a_1 a_2^2 && \Rightarrow \nu(x_5) = (1, 2)\\
P(X = x_6, Y = y) &= a_1^3 a_2^2 && \Rightarrow \nu(x_6) = (3, 2)\\
P(X = x_7, Y = y) &= a_1^3 a_2^6 && \Rightarrow \nu(x_7) = (3, 6)\\
P(X = x_8, Y = y) &= a_1^5 a_2^6 && \Rightarrow \nu(x_8) = (5, 6)\\
P(X = x_9, Y = y) &= a_1^3 a_2^4 && \Rightarrow \nu(x_9) = (3, 4)
\end{aligned}$$

The Newton polytope $\mathrm{NP}(f_y)$ arising out of this example is plotted in Fig. 2. The vertices of $\mathrm{NP}(f_y)$ are given by $\nu(x_1)$, $\nu(x_2)$, $\nu(x_3)$, and $\nu(x_4)$. We will return to this example in Section IV.

IV. Robust Decoding Using Newton Polytopes

In this section, we rely on some of the recent advances in the field of tropical geometry employing Newton polytopes to partition the channel parameter space Θ into distinct decoding regions. Note that this is in stark contrast to the more traditional way of decoding using one parameter at a time, as in the sum-product algorithm [5].

A. Decoding regions in regular and tropical parameter space

After tropicalization of the regular channel parameter space, the regions of identical MAP decoding in the tropical parameter space are given by the normal cones of an output codeword y. Specifically, let v be a vertex of $\mathrm{NP}(f_y)$, which corresponds to some input codeword x and state vector s, that is, $v = \nu(x, y, s)$. Then, the *tropical decoding region* for x corresponding to the vertex v is simply the normal cone of $\mathrm{NP}(f_y)$ at v, and the tropical decoding region for x, which we will denote by $\tilde{C}(x, y)$, is simply the union of the tropical decoding regions for the input codeword x corresponding to all vertices v; denoting the collection of such vertices to be V_x, we get

$$\tilde{C}(x, y) := \bigcup_{v \in V_x} \mathrm{NC}(v, f_y). \qquad (11)$$

To identify the corresponding decoding regions in the regular parameter space, we first define the following quantity

$$\begin{aligned}a^v &:= \prod_{i=1}^{d} a_i^{v_i} \\ &= e^{\langle v, \log(a) \rangle}. \qquad (12)\end{aligned}$$

The decoding region $C(x, y)$ in the regular parameter space is simply the *de-tropicalized* tropical decoding region given by

$$\begin{aligned}C(x, y) &= \bigcup_{v \in V_x} \left\{ \theta : e^{\langle u - v, \log(\theta) \rangle} \geq 1 \; \forall \, u \in \mathrm{NP}(f_y) \right\} \\ &= \bigcup_{v \in V_x} \left\{ \theta : \theta^{u - v} \geq 1 \; \forall \, u \in \mathrm{NP}(f_y) \right\}. \qquad (13)\end{aligned}$$

B. Perturbation bounds in tropical parameter space

As discussed earlier in Section III, the vertices of the Newton polytope of an output codeword (or observation) correspond to the input codewords and channel states that are the solutions to the MAP criterion for a given parameter vector $\theta = (a_1, \ldots, a_d) \in \mathbb{R}^d$.

Consider a tropical decoding region $\tilde{C}(x, y)$ as given in (11). If a given tropical parameter vector $b = -\log(\theta)$ lies in this region, then we decode y to x. Furthermore, we can determine the radius δ of the largest possible ball centered about b and entirely contained in this decoding region, which is the distance to the nearest facet of the decoding region.

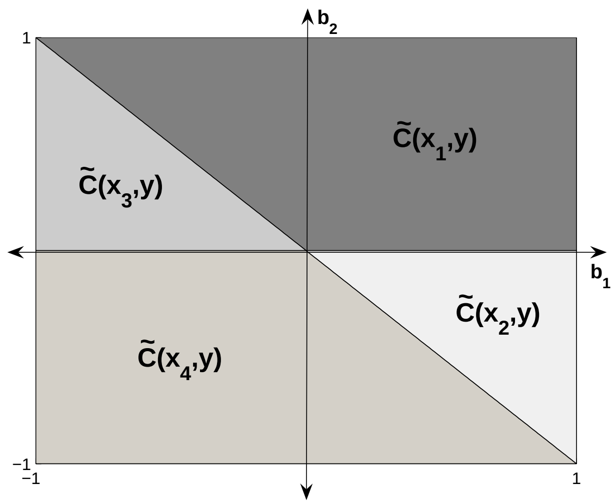

Fig. 3. Decoding regions for example 1 in the tropical channel parameter space.

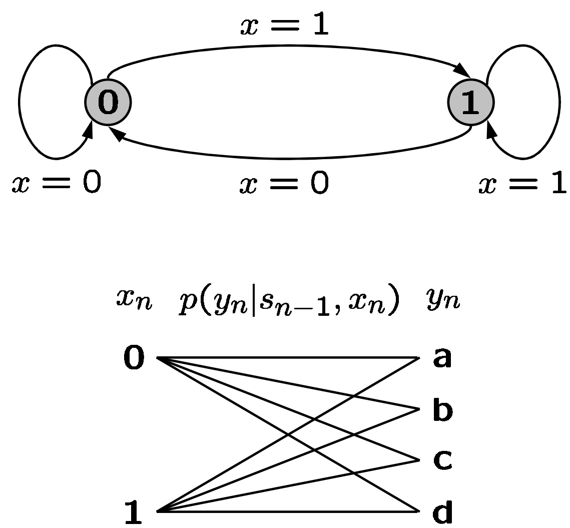

Fig. 4. A two-state binary input, quaternary output pure intersymbol interference channel model.

Specifically, let $\bar{V}_x$ be the vertices of $\mathrm{NP}(f_y)$ that do not correspond to x, then

$$\delta = \min_{u \in \bar{V}_x} \frac{|\langle v-u, b\rangle|}{||v-u||}, \quad (14)$$

where v is the vertex of $\mathrm{NP}(f_y)$ corresponding to the MAP solution.

Example 1 (Continued): We return to the example of Section III. Again, we assume that the channel has only one state. With Newton polytope $\mathrm{NP}(f_y)$ as given in Fig. 2, we can also determine the tropical decoding regions $\tilde{C}(x,y)$ and the regular decoding regions $C(x,y)$ for x corresponding to the vertices of the Newton polytope. Let $b_1 = -\log a_1$ and $b_2 = -\log a_2$. Then, the tropical decoding regions are given by the normal cones of the vertices of the Newton polytope and are described by

$$\begin{aligned} \tilde{C}(x_1,y) &= \{(b_1,b_2) : b_2 > 0, b_1+b_2 > 0\} \\ \tilde{C}(x_2,y) &= \{(b_1,b_2) : b_2 < 0, b_1+b_2 > 0\} \\ \tilde{C}(x_3,y) &= \{(b_1,b_2) : b_2 > 0, b_1+b_2 < 0\} \\ \tilde{C}(x_4,y) &= \{(b_1,b_2) : b_2 < 0, b_1+b_2 < 0\} \end{aligned} \quad (15)$$

A plot of these tropical decoding regions is given in Fig. 3. Similarly, the decoding regions $C(x,y)$ in the regular channel parameter space can be written as

$$\begin{aligned} C(x_1,y) &= \{(a_1,a_2) : a_2 < 1, a_1a_2 < 1\} \\ C(x_2,y) &= \{(a_1,a_2) : a_2 > 1, a_1a_2 < 1\} \\ C(x_3,y) &= \{(a_1,a_2) : a_2 < 1, a_1a_2 > 1\} \\ C(x_4,y) &= \{(a_1,a_2) : a_2 > 1, a_1a_2 > 1\} \end{aligned} \quad (16)$$

The reader should be reminded that a_1, a_2 are not probabilities and hence, they need not be contained within the unit square. These regular decoding regions are plotted in Fig. 1.

V. Application to a Symmetric ISI Channel

An intersymbol interference (ISI) channel is a special case of the general FSC in which the channel state vector s depends statistically on the input codeword x [2, pp. 97-100]. A *pure* ISI channel is the one in which the channel state vector depends *deterministically* on the input codeword. In this section, we apply the channel partitioning techniques developed so far to decode the output of a two-state binary input, quaternary output pure ISI channel.

The channel in Fig. 4 is a simple model of such an ISI channel. The channel has input codewords $x = (x_1, \ldots, x_N) \in \mathcal{X} \subset \{0,1\}^N$ and output codewords $y = (y_1, \ldots, y_N) \in \mathcal{Y} = \{a,b,c,d\}^N$. Further, as can be seen from the figure, the channel state at any time instant is the same as the channel input at the previous time instant. Without loss of generality, it is also assumed that the channel starts from rest in the sense that $s_0 = x_1$. Thus, the probability assignment on the current output y_n conditioned on the input codeword x is simply given by

$$p(y_n|x) = p(y_n|x_{n-1}x_n) = p_{y_n|x_{n-1}x_n} \quad (17)$$

for $n \in \{2, \ldots, N\}$, and $p(y_n|x) = p(y_n|x_n) = p_{y_n|x_nx_n}$ for the specific case of $n = 1$. Additionally, it is assumed that we have a uniform prior over the input codebook, that is, $p(x) = 1/|\mathcal{X}|$, and that the channel outputs are statistically independent conditioned on the input codeword, that is,

$$p(y|x) = p_{y_1|x_1x_1} \prod_{n=2}^{N} p_{y_n|x_{n-1}x_n}. \quad (18)$$

Finally, we assume that the ISI channel is symmetric in the sense that

$$\begin{aligned} p_{a|00} &= p_{d|11}, & p_{a|01} &= p_{d|10}, \\ p_{b|00} &= p_{c|11}, & p_{b|01} &= p_{c|10}, \\ p_{c|00} &= p_{b|11}, & p_{c|01} &= p_{b|10}, \\ p_{d|00} &= p_{a|11}, & p_{d|01} &= p_{a|10}. \end{aligned}$$

Note that in this specific case of a symmetric ISI channel, the channel parameter space is 6-dimensional – in the general case of a non-symmetric ISI channel, however, it can be as large as 14-dimensional.

TABLE I

COMPLEXITY OF MAP DECODINGS BASED ON NEWTON POLYTOPES

N	$\lvert\mathcal{X}\rvert = 2^{N-3}$	# of NP(f_y) Vertices
6	8	8
7	16	16
8	32	32
9	64	60
10	128	101
11	256	158
12	512	238
13	1024	352

A. Experiment 1: Complexity of Newton Polytopes

As noted earlier, a particularly exciting consequence of using Netwon polytopes for MAP detection is that the number of vertices of these Netwon polytopes grows only polynomially in the number of parameters of the graphical model [3]. In the context of this symmetric ISI channel, therefore, it means that the number of MAP decodings can be no larger than $O(N^6)$, since the Newton polytope of an output codeword y of this channel can have dimensions no larger than six. On the other hand, since any reasonable input codebook will have cardinality exponential in N, one would expect the number of *possible* MAP decodings to be exponential in the block length N.

In this experiment, we numerically demonstrate the polynomial (or subexponential) complexity of MAP decodings based on Newton polytopes. The input codebook $\mathcal{X}$ in this experiment is given by a random codebook of cardinality 2^{N-3}, where N is the block length. Note that given an output codeword y, the corresponding Newton polytope NP(f_y) can be calculated by applying (7) to (18). However, these Newton polytopes can also be more easily calculated using the polytope propagation algorithm described in [3].[1] The outcome of this experiment is summarized in Table 1.

B. Experiment 2: Perturbation bounds under parametric variation

In this experiment, we consider the effect of varying channel parameters on the perturbation bound δ as described in Section IV. The input codebook $\mathcal{X}$ in this experiment is also given by a random codebook of cardinality 2^{N-3}, where the block length is given by $N = 5$.

In order to visualize the outcome of this experiment, we reduce the dimensionality of the channel parameter space to two by assigning the conditional probabilities as follows

$$\begin{aligned}
p_{a|00} &= p_{d|11} = \frac{\alpha}{A}, & p_{a|01} &= p_{d|10} = \frac{\beta^4}{B},\\
p_{b|00} &= p_{c|11} = \frac{\alpha^2}{A}, & p_{b|01} &= p_{c|10} = \frac{\beta^3}{B},\\
p_{c|00} &= p_{b|11} = \frac{\alpha^3}{A}, & p_{c|01} &= p_{b|10} = \frac{\beta^2}{B},\\
p_{d|00} &= p_{a|11} = \frac{\alpha^4}{A}, & p_{d|01} &= p_{a|10} = \frac{\beta}{B},
\end{aligned}$$

where $0 \leq \alpha, \beta \leq 1$ are the two (variable) channel parameters, and A and B are the normalization constants given by $A = \alpha + \alpha^2 + \alpha^3 + \alpha^4$ and $B = \beta + \beta^2 + \beta^3 + \beta^4$, respectively. Note that these geometrically progressing probability assignments are rather natural in the sense that, for example, if '00' was transmitted then one is more likely to receive an 'a' at the receiver than a 'b', a 'c' or a 'd'.

For the sake of this experiment, we fix the output codeword to be $y = bcacd$ while the input codebook is given by $\mathcal{X} = \{10101, 00101, 11101, 01111\}$. We vary α and β from 10^{-10} to 1, and calculate the corresponding perturbation bound δ in the tropical channel parameter space as well as the optimal (MAP) input codeword. The results of this experiment are plotted in Fig. 5. The sudden drop in the perturbation bound δ in the figure is an indication of the fact that the channel parameter $\theta = (\alpha, \beta)$ is very close to the boundary separating two decoding regions – on one side of this boundary is the decoding region for $\widehat{x}(\theta) = 01111$, while on the other side is the decoding region for $\widehat{x}(\theta) = 10101$.

[1]Ch. 6 and 7 in [4] also contain a detailed description of this algorithm.

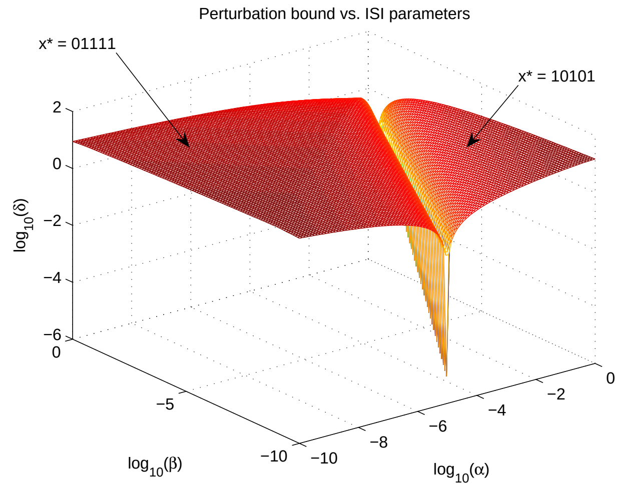

Fig. 5. Perturbation bound δ in the tropical channel parameter space as a function of the ISI channel parameters α and β.

VI. CONCLUSION

Based on recent advances in the field of tropical geometry, this work provides a new methodology for partitioning the parameter space of finite state channels into separate decoding regions. The approach is based on a polynomial-time polytope propagation algorithm [3], [4], and the partitioning process is highly efficient and scalable. Furthermore, we have provided a framework for characterizing the robustness of decoding under uncertainties in the channel parameters.

REFERENCES

[1] J. Richter-Gebert, B. Sturmfels, and T. Theobald, "First steps in tropical geometry," in *Proc. Idempotent Mathematics and Mathematical Physics*, ser. Contemporary Mathematics, vol. 377. Vienna, Austria: American Mathematical Society, Feb. 2003, pp. 289–317.

[2] R. G. Gallager, *Information Theory and Reliable Communication*. New York: John Wiley and Sons, Inc., 1968.

[3] L. Pachter and B. Sturmfels, "Tropical geometry of statistical models," *Proc. Natl. Acad. Sci.*, vol. 101, pp. 16 132–16 137, 2004.

[4] L. Pachter and B. Sturmfels, Eds., *Algebraic Statistics for Computational Biology*. Cambridge University Press, 2005.

[5] F. Kschischang, B. Frey, and H.-A. Loeliger, "Factor graphs and the sum-product algorithm," *IEEE Transactions on Information Theory*, vol. 47, pp. 498–519, 2001.