Answers to review problems for Exam #2, Math 151, Sections 13, 14, 15

1. i) \( \frac{5 \cos(5x) + 2e^{2x}}{\sin(5x) + e^{2x}} \). **Hint:** Use the chain rule.

   ii) \( \frac{6x - 5}{1 + (3x^2 - 5x + 2)^2} \). **Hint:** Use the chain rule.

   iii) \(-e^x \cos(x) \sin(x) + e^x \cos(x)(e^x)\). **Hint:** Use the product rule and the chain rule.

2. \( y = 8x - 2 \). **Hint:** Use the implicit differentiation to find the slope.

3. \( f(4) \geq 16 \). **Hint:** Use the mean value theorem: \( f(4) - f(1) = f'(c)(4 - 1) \) for some \( c \) satisfying \( 1 \leq c \leq 4 \). Since \( f'(c) \geq 2 \), \( f'(c)(4 - 1) \geq 6 \). So \( f(4) \geq 16 \).

4. (a) \( h'(x) = 10x\sqrt{44 - 35x^2} \) and \( h'(1) = 30 \). **Hint:** Use the chain rule.

   (b) \( h(0.95) \simeq 1.5 \). **Hint:** Use \( h(0.95) \simeq h(1) + h'(1)(0.95 - 1) \), \( h(1) = f(2) = 3 \) and \( h'(1) = 30 \).

   (c) Likely to be greater than the true value of \( h(0.95) \). Reason: Since \( h''(1) = -\frac{260}{3} \) is negative, the graph of \( h(x) \) is concave downward near \( x = 1 \). So the tangent line is above the graph. Thus the linear approximation is greater than the true value.

5. (a) \( 1 - \frac{x}{2} \). **Hint:** Use the linear approximation formula: \( f(x) \simeq f(a) + f'(a)(x - a) \).

   (b) 0.995. **Hint:** Take \( x \) to be 0.01 in the linear approximation above.

6. i) 0. **Hint:** Use L'Hôpital's rule.

   ii) \( \frac{2}{\pi^2} \). **Hint:** Use L'Hôpital's rule.

   iii) 0. **Hint:** Use L'Hôpital's rule.

   iv) \( \frac{3}{2} \). **Hint:** Use \( x - \sqrt{x^2 - 3x} = \frac{3x}{x + \sqrt{x^2 - 3x}} \) and then use either the method of compute limits of algebraic functions or L'Hôpital's rule.

   v) \( e^{2} \). **Hint:** Take the logarithm of \( (1 + 2x)^{\frac{1}{2}} \) and then use L'Hôpital's rule.

7. (a) We have \( f(0) = 1 \) and \( f(1) = -\sin 3 \). Since \( 0 < 3 < \pi \), we know that \( \sin 3 > 0 \). So \( f(1) = -\sin 3 < 0 \). Thus 0 is between \( f(0) \) and \( f(1) \). By the intermediate value theorem, there is \( c \) between 0 and 1 such that \( f(c) = 0 \), that is, \( f(x) \) has a root in \([0, 1]\).

   (b) \( g(x) = x + \frac{1 - x - \sin(3x)}{1 + 3\cos(3x)} \). **Hint:** Use the formula \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \).
\( x_1 = 0.5 + \frac{0.5 - \sin(1.5)}{1 + 3\cos(1.5)} \) and
\[ x_2 = 0.5 + \frac{0.5 - \sin(1.5)}{1 + 3\cos(1.5)} + \frac{0.5 - \frac{0.5 - \sin(1.5)}{1 + 3\cos(1.5)} - \sin(1.5 + 3\frac{0.5 - \sin(1.5)}{1 + 3\cos(1.5)})}{1 + 3\cos(1.5 + 3\frac{0.5 - \sin(1.5)}{1 + 3\cos(1.5)})}. \]

8. (a) \( \lim_{x \to \infty} f(x) = \infty \) and \( \lim_{x \to -\infty} f(x) = 0 \). \textbf{Hint:} Since \( \lim_{x \to \infty}(x^2 - 1) = \infty \) and \( \lim_{x \to \infty} e^x = \infty \), \( \lim_{x \to \infty} f(x) = \infty \). For \( \lim_{x \to -\infty} f(x) \), use L'Hôpital's rule.

(b) \( f'(x) = (x^2 + 2x - 1)e^x \). The exact solutions of \( f'(x) = 0 \) are \( -1 \pm \sqrt{2} \). When \( x \leq -1 - \sqrt{2} \) or \( x > -1 + \sqrt{2} \), \( f'(x) > 0 \). When \( -1 - \sqrt{2} < x < -1 + \sqrt{2} \), \( f'(x) < 0 \).

(c) \( f''(x) = (x^2 + 4x + 1)e^x \). The exact solutions of \( f''(x) = 0 \) are \( -2 \pm \sqrt{3} \). When \( x \leq -2 - \sqrt{3} \) or \( x > -2 + \sqrt{3} \), \( f''(x) > 0 \). When \( -2 - \sqrt{3} < x < -2 + \sqrt{3} \), \( f''(x) < 0 \).

(d) See the picture.

9. (a) Local minimum: 0. Local maximum: 2. \textbf{Hint:} Use the information on \( f'(x) \).

(b) Inflection points: \( -1, 1, 3 \). \textbf{Hint:} Use the information on \( f''(x) \).

(c) See the picture.

(d) No. For \( x > 4 \), \( f''(x) > 0 \). So \( f'(x) \) is increasing when \( x > 4 \). But \( f'(4) = 0 \). So \( f'(x) \geq 0 \) for \( x > 4 \).

10. (a) \( \theta = \tan^{-1}\frac{a}{b} \). \textbf{Hint:} Draw a picture.

(b) \( \theta = \tan^{-1} 2 \) and \( \frac{d\theta}{dt} = -\frac{1}{90} \). \textbf{Hint:} \( \theta \) can be obtained by just substitute the values of \( a \) and \( b \). \( \frac{d\theta}{dt} \) is obtained by using chain rules (both \( a \) and \( b \) are functions of \( t \)) and then using \( a = 10, b = 5, a' = 0.3 \) and \( b' = 0.4 \).

11. 5062.5. \textbf{Hint:} Let \( y \) be the length of the sides with parallel fencing inside and \( x \) the length of the other two sides. Then the area \( A = xy \) and we have the constraint \( 2x + 5y = 450 \). So \( A = 90x - \frac{2}{5}x^2 \). The domain is \([0, 225]\). \( A \) reaches its maximum at \( x = 112.5 \). Then calculate the area at this \( x \).

12. \( f\left(\frac{\pi}{6}\right) = -\frac{325\pi^3}{108} + \frac{5\pi}{6} - \frac{\sqrt{3}}{2} + 1 \). \textbf{Hint:} Find \( f'(x) = -6x^2 + \sin x + 6\pi^2 - 1 \) first. Then find \( f(x) = -2x^3 - \cos x + (6\pi^2 - 1)x + 1 - 4\pi^3 + \pi \).