First and second cohomologies of grading-restricted vertex algebras

Yi-Zhi Huang

Abstract

Let V be a grading-restricted vertex algebra and W a V-module. We show that for any $m \in \mathbb{Z}_+$, the first cohomology $H^1_m(V, W)$ of V with coefficients in W introduced by the author is linearly isomorphic to the space of derivations from V to W. In particular, $H^1_m(V, W)$ for $m \in \mathbb{N}$ are equal (and can be denoted using the same notation $H^1(V, W)$). We also show that the second cohomology $H^2_{\frac{1}{2}}(V, W)$ of V with coefficients in W introduced by the author corresponds bijectively to the set of equivalence classes of square-zero extensions of V by W. In the case that $W = V$, we show that the second cohomology $H^2_{\frac{1}{2}}(V, V)$ corresponds bijectively to the set of equivalence classes of first order deformations of V.

1 Introduction

The present paper is a sequel to the paper [H]. We discuss the first and second cohomologies of grading-restricted vertex algebras introduced by the author in that paper.

Let V be a grading-restricted vertex algebra and W a V-module. Recall from [H] that for each $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, we have an n-th cohomology $H^n_m(V, W)$ of V with coefficients in W. For each $n \in \mathbb{N}$, We also have an n-th cohomology $H^n_\infty(V, W)$ of V with coefficients in W which is isomorphic to the inverse limit of the inverse system $\{H^n_m(V, W)\}_{m\in\mathbb{Z}_+}$. We also have an additional second cohomology $H^2_{\frac{1}{2}}(V, W)$ of V with coefficients in W. In the present paper, we disuss only $H^1_m(V, W)$ for $m \in \mathbb{Z}_+$ and $H^2_{\frac{1}{2}}(V, W)$.

Let V be a grading-restricted vertex algebra and W a V-module. A grading-preserving linear map $f: V \to W$ is called a *derivation* if

$$\begin{aligned} f(Y_V(u,z)v) &= Y_{WV}^W(f(u),z)v + Y_W(u,z)f(v) \\ &= e^{zL(-1)}Y_W(v,-z)f(u) + Y_W(u,z)f(v) \end{aligned}$$

for $u, v \in V$. We use Der (V, W) to denote the space of all such derivations. We have the following result for the first cohomologies of V with cofficients in W:

Theorem 1.1. *Let V be a grading-restricted vertex algebra and W a V-module. Then $H^1_m(V, W)$ is linearly isomorphic to the space of derivations from V to W for any $m \in \mathbb{Z}_+$, that is, $H^1_m(V, W)$ is linearly isomorphic to* Der (V, W) *for any $m \in \mathbb{Z}_+$.*

In particular, $H^1_m(V, W)$ for $m \in \mathbb{N}$ are isomorphic (and can be denoted using the same notation $H^1(V, W)$).

Definition 1.2. Let V be a grading-restricted vertex algebra. A *square-zero ideal of* V is an ideal W of V such that for any $u, v \in W$, $Y_V(u, x)v = 0$.

Definition 1.3. Let V be a grading-restricted vertex algebra and W a $\mathbb{Z}$-graded V-module. A *square-zero extension* (Λ, f, g) *of* V *by* W is a grading-restricted vertex algebra Λ together with a surjective homomorphism f : $\Lambda \to V$ of grading-restricted vertex algebras such that $\ker f$ is a square-zero ideal of Λ (and therefore a V-module) and an injective homomorphism g of V-modules from W to Λ such that $g(W) = \ker f$. Two square-zero extensions (Λ_1, f_1, g_1) and (Λ_2, f_2, g_2) of V by W are *equivalent* if there exists an isomorphism of grading-restricted vertex algebras $h: \Lambda_1 \to \Lambda_2$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \xrightarrow[g_1]{} & \Lambda_1 & \xrightarrow[f_1]{} & V & \longrightarrow & 0 \\ & & {\scriptstyle 1_W}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle 1_V} & & \\ 0 & \longrightarrow & W & \xrightarrow[g_2]{} & \Lambda_2 & \xrightarrow[f_2]{} & V & \longrightarrow & 0, \end{array}$$

is commutative.

The notion of square-zero extension of V by W is an analogue of the notion of square-zero extension of an associative algebra by a bimodule. (see, for example, Section 9.3 of [W]).

We have the following result for the second cohomology $H^2_{\frac{1}{2}}(V, W)$ of V with cofficients in W:

Theorem 1.4. *Let V be a grading-restricted vertex algebra and W a V-module. Then the set of the equivalence classes of square-zero extensions of V by W corresponds bijectively to $H^2_{\frac{1}{2}}(V, W)$.*

Definition 1.5. Let t be a complex variable. A *family of grading-restricted vertex algebras up to the first order in t* is a $\mathbb{Z}$-graded vector space V, a family $Y_t : V \otimes V \to V((x))$ for $t \in \mathbb{C}$ of linear maps of the form $Y_t = Y_0 + t\Psi$ where Y_0 and Ψ are linear maps from $V \otimes V$ to $V((x))$ independent of t, and an element $\mathbf{1} \in V$ such that $(V, Y_t, \mathbf{1})$ satisfies all the axioms for grading-restricted vertex algebras up to the first order in t.

Definition 1.6. Let $(V, Y_V, \mathbf{1})$ be a grading-restricted vertex algebra. A *first order deformation of* V is a family $Y_t : V \otimes V \to V((x))$ for $t \in \mathbb{C}$ of linear maps of the form $Y_t = Y_V + t\Psi$ where

$$\begin{aligned} \Psi : V \otimes V &\to V((x)) \\ v_1 \otimes v_2 &\to \Psi(v_1, x)v_2 \end{aligned}$$

is a linear map such that $(V, Y_t, \mathbf{1})$ for $t \in \mathbb{C}$ is a family of grading-restricted vertex algebras up to the first order in t. Two first order deformations $Y_t^{(1)}$ and $Y_t^{(2)}$, $t \in \mathbb{C}$, of $(V, Y_V, \mathbf{1})$ are *equivalent* if there exists a family $f_t : V \to V$, $t \in \mathbb{C}$ of linear maps of the form $f_t = 1_V + tg$ where $g : V \to V$ is a linear map preserving the gradings of V such that

$$f_t(Y_t^{(1)}(v_1, x)v_2) - Y_t^{(1)}(f_t(v_1), x)f_t(v_2) \in t^2 V((x)) \tag{1.1}$$

for $v_1, v_2 \in V$.

We have:

Theorem 1.7. *The set of equivalence classes of first order deformations of a grading-restricted vertex algebra is in bijection with the set of equivalence classes of square-zero extensions of V by V.*

From Theorems 1.4 and 1.8, we obtain immediately the following result for the second cohomology $H^2_{\frac{1}{2}}(V, V)$ of V with cofficients in V:

Theorem 1.8. *Let V be a grading-restricted vertex algebra. Then the set of the equivalence classes of first order deformations of V correspond bijectively to $H^2_{\frac{1}{2}}(V, V)$.*

We prove Theorems 1.1, 1.4 and 1.7 in Sections 2, 3 and 4, respectively.

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2 First cohomologies and spaces of derivations

We prove Theorem 1.1 in the present section. First, we need the following:

Lemma 2.1. *Let $f : V \to W$ be a derivation. Then $f(\mathbf{1}) = 0$.*

Proof. By defintion,

$$\begin{aligned} f(\mathbf{1}) &= f(Y_V(\mathbf{1}, z)\mathbf{1}) \\ &= \lim_{z\to 0} f(Y_V(\mathbf{1}, z)\mathbf{1}) \\ &= \lim_{z\to 0} e^{zL(-1)}Y_W(\mathbf{1}, -z)f(\mathbf{1}) + \lim_{z\to 0} Y_W(\mathbf{1}, z)f(\mathbf{1}) \\ &= 2f(\mathbf{1}). \end{aligned}$$

So $f(\mathbf{1}) = 0$. ∎

Let $\Phi : V \to \widetilde{W}_{z_1}$ be an element of $C^1_m(V, W)$ satisfying $\delta^1_m\Phi = 0$. Since Φ satisfies the $L(0)$-conjugation property, for $v \in V_{(n)}$ and $z \in \mathbb{C}^\times$,

$$\begin{aligned} z^{L(0)}(\Phi(v))(0) &= (\Phi(z^{L(0)}v))(0) \\ &= z^n(\Phi(v))(0). \end{aligned}$$

Thus $(\Phi(v))(0) \in W_{(n)}$. So $(\Phi(v))(0)$ is a grading-preserving linear map from V to W.

Since $\delta^1_m\Phi = 0$,

$$\begin{aligned} &R(\langle w', Y_W(v_1, z_1)(\Phi(v_2))(z_2)\rangle) - R(\langle w', (\Phi(Y_V(v_1, z_1 - z_2)v_2))(z_2)\rangle) \\ &\quad + R(\langle w', Y_W(v_2, z_2)(\Phi(v_1))(z_1)\rangle) \\ &= 0 \end{aligned}$$

for $v_1, v_2 \in V$ and $w' \in W'$. By $L(-1)$-derivative property for Φ and the vertex operator map Y_W,

$$R(\langle w', Y_W(v_2, z_2)(\Phi(v_1))(z_1)\rangle) = R(\langle w', e^{z_1L(-1)}Y_W(v_2, -z_1 + z_2)(\Phi(v_1))(0)\rangle).$$

Thus we have

$$\begin{aligned}
&R(\langle w', Y_W(v_1,z_1)(\Phi(v_2))(z_2)\rangle) - R(\langle w', (\Phi(Y_V(v_1,z_1-z_2)v_2))(z_2)\rangle)\\
&\quad +R(\langle w', e^{z_1L(-1)}Y_W(v_2,-z_1+z_2)(\Phi(v_1))(0)\rangle)\\
&= 0.
\end{aligned}$$

Let $z_2=0$. We obtain

$$\begin{aligned}
&R(\langle w', Y_W(v_1,z_1)(\Phi(v_2))(0)\rangle) - R(\langle w', (\Phi(Y_V(v_1,z_1)v_2))(0)\rangle)\\
&\quad +R(\langle w', e^{z_1L(-1)}Y_W(v_2,-z_1)(\Phi(v_1))(0)\rangle)\\
&= 0.
\end{aligned}$$

Since w' is arbitrary, we obtain

$$\begin{aligned}
&(\Phi(Y_V(v_1,z_1)v_2))(0)\\
&\quad = e^{z_1L(-1)}Y_W(v_2,-z_1)(\Phi(v_1))(0) + Y_W(v_1,z_1)(\Phi(v_2))(0)\\
&\quad = Y_{WV}^W((\Phi(v_1))(0),z_1)(\Phi(v_2))(0) + Y_W(v_1,z_1)(\Phi(v_2))(0)
\end{aligned}$$

for $v_1,v_2\in V$. This means that $(\Phi(\cdot))(0): V\to W$ is a derivation from V to W. Note that $\delta_m^0(C_m^0(V,W))=0$. So we obtain a linear map from $H^1(V,W)$ to the space of derivations from V to W.

Conversely, given any derivation f from V to W, let $\Phi_f: V\to \widetilde{W}_{z_1}$ be given by

$$(\Phi_f(v))(z_1) = f(Y_V(v,z_1)\mathbf{1}) = Y_{WV}^W(f(v),z_1)\mathbf{1}$$

for $v\in V$, where we have used Lemma 2.1. By Theorem 5.6.2 in [FHL], the map from V to $\widetilde{W}_{z_1}$ given by $v\mapsto Y_{WV}^W((\Phi(v))(0),z_1)\mathbf{1}$ is composable with m vertex operators for any $m\in\mathbb{N}$. Thus $\Phi_f\in C_m^1(V,W)$ for any $m\in\mathbb{N}$. For $v_1,v_2\in V$ and $w'\in W'$,

$$\begin{aligned}
&((\delta_m^1\Phi_f)(v_1\otimes v_2))(z_1,z_2)\\
&\quad = R(\langle w', Y_W(v_1,z_1)Y_{WV}^W(f(v_2),z_2)\mathbf{1}\rangle)\\
&\qquad -R(\langle w', Y_{WV}^W(f(Y_V(v_1,z_1-z_2)v_2),z_2)\mathbf{1})\rangle)\\
&\qquad +R(\langle w', Y_W(v_2,z_2)Y_{WV}^W(f(v_1),z_1)\mathbf{1}\rangle)\\
&\quad = R(\langle w', Y_W(v_1,z_1)Y_{WV}^W(f(v_2),z_2)\mathbf{1}\rangle)\\
&\qquad -R(\langle w', Y_{WV}^W(Y_{WV}^W(f(v_1),z_1-z_2)v_2),z_2)\mathbf{1})\rangle)\\
&\qquad -R(\langle w', Y_{WV}^W(Y_W(v_1,z_1-z_2)f(v_2),z_2)\mathbf{1})\rangle)
\end{aligned}$$

$$
\begin{aligned}
&+R(\langle w', Y_W(v_2,z_2)Y_{WV}^W(f(v_1),z_1)\mathbf{1}\rangle)\\
&= R(\langle w', Y_W(v_1,z_1)Y_{WV}^W(f(v_2),z_2)\mathbf{1}\rangle)\\
&\quad -R(\langle w', e^{z_2L_W(-1)}Y_{WV}^W(f(v_1),z_1-z_2)v_2\rangle)\\
&\quad -R(\langle w', e^{z_2L_W(-1)}Y_W(v_1,z_1-z_2)f(v_2)\rangle)\\
&\quad +R(\langle w', Y_W(v_2,z_2)Y_{WV}^W(f(v_1),z_1)\mathbf{1}\rangle)\\
&= R(\langle w', Y_W(v_1,z_1)Y_{WV}^W(f(v_2),z_2)\mathbf{1}\rangle)\\
&\quad -R(\langle w', Y_{WV}^W(f(v_1),z_1)e^{z_2L_V(-1)}v_2\rangle)\\
&\quad -R(\langle w', Y_W(v_1,z_1)e^{z_2L_W(-1)}f(v_2)\rangle)\\
&\quad +R(\langle w', Y_W(v_2,z_2)Y_{WV}^W(f(v_1),z_1)\mathbf{1}\rangle)\\
&= R(\langle w', Y_W(v_1,z_1)Y_{WV}^W(f(v_2),z_2)\mathbf{1}\rangle)\\
&\quad -R(\langle w', Y_{WV}^W(f(v_1),z_1)Y_W(v_2,z_2)\mathbf{1}\rangle)\\
&\quad -R(\langle w', Y_W(v_1,z_1)Y_{WV}^W(f(v_2),z_2)\mathbf{1}\rangle)\\
&\quad +R(\langle w', Y_W(v_2,z_2)Y_{WV}^W(f(v_1),z_1)\mathbf{1}\rangle)\\
&= -R(\langle w', Y_{WV}^W(f(v_1),z_1)Y_V(v_2,z_2)\mathbf{1}\rangle)\\
&\quad +R(\langle w', Y_W(v_2,z_2)Y_{WV}^W(f(v_1),z_1)\mathbf{1}\rangle).
\end{aligned}
\tag{2.1}
$$

From Theorem 5.6.2 in [FHL], we know that the right-hand side of (2.1) is 0. So we obtain a linear map $f \mapsto \Phi_f$ from the space Der (V, W) to $H^1_m(V, W) = C^1_m(V, W)$.

Clearly these two maps are inverse to each other and thus Der (V, W) and $H^1_m(V, W)$ are isomorphic. ∎

3 Second cohomologies and square-zero extensions

In this section, we prove Theorem 1.4.

Let (Λ, f, g) be a square-zero extension of V by W. Then there is an injective linear map $\Gamma : V \to \Lambda$ such that the linear map $h : V \oplus W \to \Lambda$ given by $h(v, w) = \Gamma(v) + g(w)$ is a linear isomorphism. By definition, the restriction of h to W is the isomorphism g from W to $\ker f$. Then the grading-restricted vertex algebra structure and the V-module structure on Λ give a grading-restricted vertex algebra structure and a V-module structure on $V \oplus W$ such that the embedding $i_2 : W \to V \oplus W$ and the projection $p_1 : V \oplus W \to V$ are homormorphisms of grading-restricted vertex algebras. Moreover, $\ker p_1$ is a square-zero ideal of $V \oplus W$, i_2 is an injective homorphism such that $i_2(W) = \ker p_1$ and the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \xrightarrow{i_2} & V \oplus W & \xrightarrow{p_1} & V & \longrightarrow & 0 \\ & & {\scriptstyle 1_W}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle 1_V} & & \\ 0 & \longrightarrow & W & \xrightarrow[g]{} & \Lambda & \xrightarrow[f]{} & V & \longrightarrow & 0 \end{array} \tag{3.1}$$

of V-moduels is commutative. So we obtain a square-zero extension $(V \oplus W, p_1, i_2)$ equivalent to (Λ, f, g). We need only consider square-zero extension of V by W of the particular form $(V \oplus W, p_1, i_2)$. Note that the difference between two such square-zero extensions are in the vertex operator maps. So we use $(V \oplus W, Y_{V\oplus W}, p_1, i_2)$ to denote such a square-zero extension.

We now write down the vertex operator map for $V \oplus W$ explicitly. Since $(V \oplus W, Y_{V\oplus W}, p_1, i_2)$ is a square-zero extension of V, there exists $\Psi(u, x)v \in W((x))$ for $u, v \in V$ such that

$$\begin{array}{rcl} Y_{V\oplus W}((v_1, 0), x)(v_2, 0) & = & (Y_V(v_1, x)v_2, \Psi(v_1, x)v_2), \\ Y_{V\oplus W}((v_1, 0), x)(0, w) & = & (0, Y_V(v_1, x)w_2), \\ Y_{V\oplus W}((0, w_1), x)(v_2, 0) & = & (0, Y_{WV}^W(w, x)v_2), \\ Y_{V\oplus W}((0, w_1), x)(0, w_2) & = & 0 \end{array}$$

for $v_1, v_2 \in V$ and $w_1, w_2 \in W$. Thus we have

$$\begin{aligned} & Y_{V\oplus W}((v_1, w_1), x)(v_2, w_2) \\ & \quad = (Y_V(v_1, x)v_2, Y_W(v_1, x)w_2 + Y_{WV}^W(w_1, x)v_2 + \Psi(v_1, x)v_2) \end{aligned} \tag{3.2}$$

for $v_1, v_2 \in V$ and $w_1, w_2 \in W$.

The vacuum of $V \oplus W$ is $(\mathbf{1}, 0)$. Since

$$\begin{array}{rcl} Y_{V\oplus W}((v, w), x)(\mathbf{1}, 0) & = & e^{xL_{V\oplus W}(-1)}(v, w) \\ & = & (e^{xL_V(-1)}v, e^{xL_W(-1)}w) \\ & = & (Y_V(v, x)\mathbf{1}, Y_{WV}^W(w, x)\mathbf{1}) \end{array}$$

for $v \in W$ and $w \in W$, we have

$$\Psi(v, x)\mathbf{1} = 0 \tag{3.3}$$

for $v\in V$.

We identify $(V\oplus W)'$ with $V'\oplus W'$. For $v_1, v_2\in V$ and $w'\in W'$,

$$\begin{aligned}
&\langle (0, w'), Y_{V\oplus W}((v_1, 0), z_1)Y_{V\oplus W}((v_2, 0), z_2)(\mathbf{1}, 0)\rangle\\
&\quad = \langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1} + Y_W(v_1, z_1)\Psi(v_2, z_2)\mathbf{1}\rangle\\
&\quad = \langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}\rangle,\\
&\langle (0, w'), Y_{V\oplus W}((v_2, 0), z_2)Y_{V\oplus W}((v_1, 0), z_1)(\mathbf{1}, 0)\rangle\\
&\quad = \langle w', \Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1} + Y_W(v_2, z_2)\Psi(v_1, z_1)\mathbf{1}\rangle\\
&\quad = \langle w', \Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1}\rangle,\\
&\langle (0, w'), Y_{V\oplus W}(Y_{V\oplus W}((v_1, 0), z_1-z_2)(v_2, 0), z_2)(\mathbf{1}, 0)\rangle\\
&\quad = \langle w', Y_{WV}^{W}(\Psi(v_1, z_1-z_2)v_2, z_2)\mathbf{1} + \Psi(Y_V(v_1, z_1-z_2)v_2, z_2)\mathbf{1}\rangle\\
&\quad = \langle w', Y_{WV}^{W}(\Psi(v_1, z_1-z_2)v_2, z_2)\mathbf{1}\rangle
\end{aligned}$$

are absolutely convergent in the region $|z_1|>|z_2|>0$, $|z_2|>|z_1|>0$, $|z_2|>|z_1-z_2|>0$, respectively, to one rational function in z_1 and z_2 with the only possible poles at $z_1, z_2=0$ and $z_1=z_2$. Using our notation in [H], we denote this rational function by

$$R(\langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}\rangle)$$

or

$$R(\langle w', \Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1}\rangle)$$

or

$$R(\langle w', Y_{WV}^{W}(\Psi(v_1, z_1-z_2)v_2, z_2)\mathbf{1}\rangle).$$

Then we obtain an element, denoted by

$$E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1})$$

or

$$E(\Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1})$$

or

$$E(Y_{WV}^{W}(\Psi(v_1, z_1-z_2)v_2, z_2)\mathbf{1}),$$

of $\widetilde{W}_{z_1, z_2}$ given by

$$\langle w', E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1})\rangle = R(\langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}\rangle)$$

or

$$\langle w', E(\Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1})\rangle = R(\langle w', \Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1}\rangle)$$

or

$$\langle w', E(Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)\mathbf{1})\rangle = R(\langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)\mathbf{1}\rangle).$$

By definition, we have

$$\begin{aligned} E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}) &= E(\Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1}) \\ &= E(Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)\mathbf{1}) \end{aligned}$$

for $v_1, v_2 \in V$.

Let

$$\Phi : V \otimes V \to \widetilde{W}_{z_1, z_2}$$

be the linear map given by

$$\begin{aligned} (\Phi(v_1 \otimes v_2))(z_1, z_2) &= E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}) \\ &= E(\Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1}) \\ &= E(Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)\mathbf{1}) \end{aligned} \tag{3.4}$$

for $v_1, v_2 \in V$ and $(z_1, z_2) \in F_2\mathbb{C}$. We first prove that $\Phi \in \widehat{C}^2_{\frac{1}{2}}(V, W)$.

By the $L(-1)$-derivative property and the $L(0)$-bracket formula for $V \oplus W$, we have

$$\frac{d}{dx}Y_{V\oplus W}((v, 0), x) = Y_{V\oplus W}((L_V(-1)v, 0), x) \tag{3.5}$$

$$= [L_{V+W}(-1), Y_{V\oplus W}((v, 0), x)], \tag{3.6}$$

$$[L_{V+W}(0), Y_{V\oplus W}((v, 0), x)] = Y_{V\oplus W}((L_V(0)v, 0), x) + x\frac{d}{dx}Y_{V\oplus W}((v, 0), x) \tag{3.7}$$

for $v \in V$. By (3.5), (3.6), (3.7), (3.2) and the $L(-1)$-derivative property and the $L(0)$-bracket formula for V, we obtain

$$\frac{d}{dx}\Psi(v, x) = \Psi(L_V(-1)v, x) \tag{3.8}$$

$$= L_W(-1)\Psi(v, x) - \Psi(v, x)L_V(-1), \tag{3.9}$$

$$L_W(0)\Psi(v, x) - \Psi(v, x)L_V(0) = \Psi(L_V(0)v, x) + x\frac{d}{dx}\Psi(v, x) \tag{3.10}$$

for $v \in V$. From (3.10), we obtain

$$z^{L_W(0)}\Psi(v, x) = \Psi(z^{L_V(0)}v, zx)z^{L_V(0)} \tag{3.11}$$

for $v \in V$.

For $v_1, v_2 \in V$ and $w' \in W'$, by (3.8) and the $L(-1)$-derivative property for V, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial z_1}\langle w', (\Phi(v_1 \otimes v_2))(z_1, z_2)\rangle \\
&\quad = \frac{\partial}{\partial z_1}\langle w', E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1})\rangle \\
&\quad = \frac{\partial}{\partial z_1}R(\langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}\rangle) \\
&\quad = R\left(\left\langle w', \frac{\partial}{\partial z_1}\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}\right\rangle\right) \\
&\quad = R(\langle w', \Psi(L_V(-1)v_1, z_1)Y_V(v_2, z_2)\mathbf{1}\rangle) \\
&\quad = \langle w', E(\Psi(L_V(-1)v_1, z_1)Y_V(v_2, z_2)\mathbf{1})\rangle \\
&\quad = \langle w', (\Phi(L_V(-1)v_1 \otimes v_2))(z_1, z_2)\rangle
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial z_2}\langle w', (\Phi(v_1 \otimes v_2))(z_1, z_2)\rangle \\
&\quad = \frac{\partial}{\partial z_2}\langle w', E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1})\rangle \\
&\quad = \frac{\partial}{\partial z_2}R(\langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1}\rangle) \\
&\quad = R\left(\left\langle w', \Psi(v_1, z_1)\frac{\partial}{\partial z_2}Y_V(v_2, z_2)\mathbf{1}\right\rangle\right) \\
&\quad = R(\langle w', \Psi(v_1, z_1)Y_V(L_V(-1)v_2, z_2)\mathbf{1}\rangle) \\
&\quad = \langle w', E(\Psi(v_1, z_1)Y_V(L_V(-1)v_2, z_2)\mathbf{1})\rangle \\
&\quad = \langle w', (\Phi(v_1 \otimes L_V(-1)v_2))(z_1, z_2)\rangle.
\end{aligned} \tag{3.13}$$

Using (3.8), (3.9) and the $L(-1)$-derivative property for V, we obtain

$$\left(\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_2}\right)\langle w', (\Phi(v_1 \otimes v_2))(z_1, z_2)\rangle$$

$$
\begin{aligned}
&= \left(\frac{\partial}{\partial z_2}+\frac{\partial}{\partial z_2}\right)\langle w', E(\Psi(v_1,z_1)Y_V(v_2,z_2)\mathbf{1})\rangle\\
&= \left(\frac{\partial}{\partial z_2}+\frac{\partial}{\partial z_2}\right)R(\langle w', \Psi(v_1,z_1)Y_V(v_2,z_2)\mathbf{1}\rangle)\\
&= R\left(\left\langle w', \frac{\partial}{\partial z_1}\Psi(v_1,z_1)Y_V(v_2,z_2)\mathbf{1}\right\rangle\right)\\
&\quad +R\left(\left\langle w', \Psi(v_1,z_1)\frac{\partial}{\partial z_2}Y_V(v_2,z_2)\mathbf{1}\right\rangle\right)\\
&= R(\langle w', \Psi(L_V(-1)v_1,z_1)Y_V(v_2,z_2)\mathbf{1}\rangle)\\
&\quad +R(\langle w', \Psi(v_1,z_1)Y_V(L_V(-1)v_2,z_2)\mathbf{1}\rangle)\\
&= R(\langle w', L_W(-1)\Psi(v_1,z_1)Y_V(v_2,z_2)\mathbf{1}\rangle)\\
&= R(\langle L_{W'}(1)w', \Psi(v_1,z_1)Y_V(v_2,z_2)\mathbf{1}\rangle)\\
&= \langle L_{W'}(1)w', E(\Psi(v_1,z_1)Y_V(v_2,z_2)\mathbf{1})\rangle\\
&= \langle w', L_W(-1)E(\Psi(v_1,z_1)Y_V(v_2,z_2)\mathbf{1})\rangle\\
&= \langle w', L_W(-1)(\Phi(v_1\otimes v_2))(z_1,z_2)\rangle
\end{aligned}
\tag{3.14}
$$

for $v_1, v_2 \in V$ and $w' \in W'$, From (3.12), (3.13) and (3.14), we see that Φ satisfies the $L(-1)$-derivative property.

Also for $v_1, v_2 \in V$ and $w' \in W'$, by (3.11) and the $L(0)$-bracket formula for V, we have

$$
\begin{aligned}
&\langle w', z^{L_W(0)}(\Phi(v_1\otimes v_2))(z_1,z_2)\rangle\\
&\quad= \langle w', z^{L_W(0)}E(\Psi(v_1,z_1)Y_V(v_2,z_2)\mathbf{1})\rangle\\
&\quad= \langle z^{L_{W'}(0)}w', E(\Psi(v_1,z_1)Y_V(v_2,z_2)\mathbf{1})\rangle\\
&\quad= R(\langle z^{L_{W'}(0)}w', \Psi(v_1,z_1)Y_V(v_2,z_2)\mathbf{1}\rangle)\\
&\quad= R(\langle w', z^{L_W(0)}\Psi(v_1,z_1)Y_V(v_2,z_2)\mathbf{1}\rangle)\\
&\quad= R(\langle w', \Psi(z^{L_V(0)}v_1,zz_1)Y_V(z^{L_V(0)}v_2,zz_2)\mathbf{1}\rangle)\\
&\quad= \langle w', E(\Psi(z^{L_V(0)}v_1,zz_1)Y_V(z^{L_V(0)}v_2,zz_2)\mathbf{1})\rangle\\
&\quad= \langle w', (\Phi(z^{L_V(0)}v_1\otimes z^{L_V(0)}v_2))(zz_1,zz_2)\rangle,
\end{aligned}
$$

that is, Φ satisfies the $L(0)$-conjugation property.

Since $V\oplus W$ is a grading-restricted vertex algebra, for $v_1, v_2, v_3 \in V$ and $w' \in W'$, the series

$$
\langle (0,w'), Y_{V\oplus W}((v_1,0),z_1)Y_{V\oplus W}((v_2,0),z_2)Y_{V\oplus W}((v_3,0),z_3)(\mathbf{1},0)\rangle
$$

and

$$\langle (0, w'), Y_{V\oplus W}(Y_{V\oplus W}((v_1, 0), z_1 - z_2)(v_2, 0), z_2)Y_{V\oplus W}((v_3, 0), z_3)(\mathbf{1}, 0)\rangle$$

are absolutely convergent in the regions given by $|z_1| > |z_2| > |z_3| > 0$ and by $|z_2| > |z_1 - z_2|, |z_3| > 0$ and $|z_2 - z_3| > |z_1 - z_2|$, respectively, to a same rational function with the only possible poles at $z_1 = z_2$, $z_1 = z_3$, $z_2 = z_3$. But by (3.2) and (3.3), these series are equal to

$$\langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle + \langle w', Y_W(v_1, z_1)\Psi(v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle$$

and

$$\begin{aligned}\langle w', \Psi(Y_V(v_1, z_1 - z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle\\ +\langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle,\end{aligned}$$

respectively, and are absolutely convergent to a same rational function which in our convention is equal to

$$R(\langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle + \langle w', Y_W(v_1, z_1)\Psi(v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle)$$

and

$$\begin{aligned}R(\langle w', \Psi(Y_V(v_1, z_1 - z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle\\ +\langle w', Y_{WV}^W(\Psi(v_1, z_1 - z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle).\end{aligned}$$

In particular, we have proved that $\Phi \in \widehat{C}^2_{\frac{1}{2}}(V, W)$.

Since by (3.4),

$$\begin{aligned}(\Phi(v_1 \otimes v_2))(z_1, z_2) &= E(\Psi(v_1, z_1)Y_V(v_2, z_2)\mathbf{1})\\ &= E(\Psi(v_2, z_2)Y_V(v_1, z_1)\mathbf{1})\\ &= (\Phi(v_2 \otimes v_1))(z_2, z_1)\\ &= (\sigma_{12}(\Phi(v_2 \otimes v_1)))(z_1, z_2)\end{aligned}$$

for $v_1, v_2 \in V$ and $(z_1, z_2) \in F_2\mathbb{C}$, that is,

$$\Phi(v_1 \otimes v_2) - \sigma_{12}(\Phi(v_2 \otimes v_1)) = 0$$

for $v_1, v_2 \in V$, We obtain

$$\sum_{\sigma\in J_{2;1}} (-1)^{|\sigma|}\sigma(\Phi(v_1\otimes v_2))$$
$$= \Phi(v_1\otimes v_2) - \sigma_{12}(\Phi(v_2\otimes v_1))$$
$$= 0$$

for $v_1, v_2 \in V$. So $\Phi \in C^2(V, W)$.

Next we show that $\delta^2_{\frac{1}{2}}(\Phi) = 0$. For $v_1, v_2, v_3 \in V$, $w' \in W'$,

$$\begin{aligned}
&\langle w', ((\delta^2_{\frac{1}{2}}(\Phi))(v_1\otimes v_2\otimes v_3))(z_1, z_2, z_3)\rangle\\
&\quad = R(\langle w', (E^{(1)}_W(v_1; \Phi(v_2\otimes v_3)))(z_1, z_2, z_3)\rangle\\
&\qquad\qquad +\langle w', (\Phi(v_1\otimes E^{(2)}(v_2\otimes v_3; \mathbf{1})))(z_1, z_2, z_3)\rangle)\\
&\qquad -R(\langle w', (\Phi(E^{(2)}(v_1\otimes v_2; \mathbf{1})\otimes v_3))(z_1, z_2, z_3)\rangle\\
&\qquad\qquad +\langle w', (E^{W;(1)}_{WV}(\Phi(v_1\otimes v_2); v_3))(z_1, z_2, z_3)\rangle)\\
&\quad = R(\langle w', Y_W(v_1, z_1)\Psi(v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle\\
&\qquad\qquad +\langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle)\\
&\qquad -R(\langle w', \Psi(Y_V(v_1, z_1-z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle\\
&\qquad\qquad +\langle w', Y^W_{WV}(\Psi(v_1, z_1-z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle).
\end{aligned}\tag{3.15}$$

Since $V\oplus W$ is a grading-restricted vertex algebra, we have the associativity property

$$\begin{aligned}
&R(\langle (0, w'), Y_{V\oplus W}((v_1, 0), z_1)Y_{V\oplus W}((v_2, 0), z_2)Y_{V\oplus W}((v_3, 0), z_3)(\mathbf{1}, 0)\rangle)\\
&\quad = R(\langle (0, w'), Y_{V\oplus W}(Y_{V\oplus W}((v_1, 0), z_1-z_2)(v_2, 0), z_2)\cdot\\
&\qquad\qquad \cdot Y_{V\oplus W}((v_3, 0), z_3)(\mathbf{1}, 0)\rangle),
\end{aligned}$$

which, by (3.2) and (3.3), is equivalent to

$$\begin{aligned}
&R(\langle w', \Psi(v_1, z_1)Y_V(v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle\\
&\qquad\qquad +\langle w', Y_W(v_1, z_1)\Psi(v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle)\\
&\quad = R(\langle w', \Psi(Y_V(v_1, z_1-z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle\\
&\qquad\qquad +\langle w', Y^W_{WV}(\Psi(v_1, z_1-z_2)v_2, z_2)Y_V(v_3, z_3)\mathbf{1}\rangle),
\end{aligned}$$

as we have noticed above. So the right-hand side of (3.15) is 0. Thus $\Phi + \delta^1_2 C^1_2(V, W)$ is an element of $H^2_{\frac{1}{2}}(V, W)$.

Conversely, given any element of $H^2_{\frac{1}{2}}(V,W)$, let $\Phi\in C^2_{\frac{1}{2}}(V,W)$ be a representative of this element. Then for any $v_1,v_2\in V$, there exists N such that for $w'\in W'$, $\langle w',(\Phi(v_1\otimes v_2))(z,0)\rangle$ is a rational function of z with the only possible pole at $z=0$ of order less than or equal to N. For $v_1,v_2\in V$, let $\Psi(v_1,x)v_2\in W((x))$ be given by

$$\langle w',\Psi(v_1,x)v_2\rangle|_{x=z}=\langle w',(\Phi(v_1\otimes v_2))(z,0)\rangle.$$

for $z\in\mathbb{C}^\times$. For $v_1,v_2\in V$, define $Y_{V\oplus W}(v_1,x)v_2$ using (3.2). So we obtain a vertex operator map $Y_{V\oplus W}$. Reversing the proof above, we see that $V\oplus W$ equipped with the vertex operator map Y_{V+W} and the vacuum $(\mathbf{1},0)$ is a grading-restricted vertex algebra and together with the projection $p_1: V\oplus W\to V$ and the embedding $i_2: W\to V\oplus W$, $V\oplus W$ is a square-zero extension of V by W.

Next we prove that two elements of $\ker\delta^2_{\frac{1}{2}}$ obtained this way are differed by an element of $\delta_1 C^1(V,W)$ if and only if the corresponding square-zero extensions of V by W are equivalent.

Let $\Phi_1,\Phi_2\in\ker\delta^2_{\frac{1}{2}}$ be two such elements obtained from square-zero extensions $(V\oplus W, Y^{(1)}_{V\oplus W},p_1,i_2)$ and $(V\oplus W, Y^{(2)}_{V\oplus W},p_1,i_2)$. Assume that $\Phi_1=\Phi_2+\delta_1(\Gamma)$ where $\Gamma\in C^1(V,W)$. Since

$$\begin{aligned}&\langle w',((\delta_1(\Gamma))(v_1\otimes v_2))(z_1,z_2)\rangle\\ &\quad=R(\langle w',Y_W(v_1,z_1)(\Gamma(v_2))(z_2)\rangle)\\ &\quad\quad-R(\langle w',(\Gamma(Y_V(v_1,z_1-z_2)v_2))(z_2)\rangle)\\ &\quad\quad+R(\langle w',Y_W(v_2,z_2)(\Gamma(v_1))(z_1)\rangle),\end{aligned}$$

we have

$$\begin{aligned}&R(\langle w',\Psi_1(v_1,z_1)Y_V(v_2,z_2)\mathbf{1}\rangle)\\ &\quad=\langle w',(\Phi_1(v_1\otimes v_2))(z_1,z_2)\rangle\\ &\quad=\langle w',(\Phi_2(v_1\otimes v_2))(z_1,z_2)\rangle\\ &\quad\quad+\langle w',(\delta_1(\Gamma))(z_1,z_2)\rangle\\ &\quad=R(\langle w',\Psi_2(v_1,z_1)Y_V(v_2,z_2)\mathbf{1}\rangle\\ &\quad\quad+R(\langle w',Y_W(v_1,z_1)(\Gamma(v_2))(z_2)\rangle)\\ &\quad\quad-R(\langle w',(\Gamma(Y_V(v_1,z_1-z_2)v_2))(z_2)\rangle)\\ &\quad\quad+R(\langle w',Y_W(v_2,z_2)(\Gamma(v_1))(z_1)\rangle)\end{aligned}$$

$$
\begin{aligned}
&= R(\langle w', \Psi_2(v_1, z_1) Y_V(v_2, z_2)\mathbf{1}\rangle \\
&\quad + R(\langle w', Y_W(v_1, z_1)(\Gamma(v_2))(z_2)\rangle) \\
&\quad - R(\langle w', (\Gamma(Y_V(v_1, z_1 - z_2)v_2))(z_2)\rangle) \\
&\quad + R(\langle w', e^{(z_1 - z_2)L_W(-1)} Y_W(v_2, -z_1)(\Gamma(v_1))(z_2)\rangle). \qquad (3.16)
\end{aligned}
$$

Let z_2 go to zero on both sides of (3.16). We obtain

$$
\begin{aligned}
\langle w', \Psi_1(v_1, z_1)v_2\rangle &= \langle w', \Psi_2(v_1, z_1)v_2\rangle \\
&\quad + \langle w', Y_W(v_1, z_1)(\Gamma(v_2))(0)\rangle \\
&\quad - \langle w', (\Gamma(Y_V(v_1, z_1)v_2))(0)\rangle \\
&\quad + \langle w', e^{z_1 L_W(-1)} Y_W(v_2, -z_1)(\Gamma(v_1))(0)\rangle \\
&= \langle w', \Psi_2(v_1, z_1)v_2\rangle \\
&\quad + \langle w', Y_W(v_1, z_1)(\Gamma(v_2))(0)\rangle \\
&\quad - \langle w', (\Gamma(Y_V(v_1, z_1)v_2))(0)\rangle \\
&\quad + \langle w', Y_{WV}^W((\Gamma(v_1))(0), z_1)v_2\rangle.
\end{aligned}
$$

Then

$$
\begin{aligned}
\Psi_1(v_1, x)v_2 &= \Psi_2(v_1, x)v_2 + Y_W(v_1, x)(\Gamma(v_2))(0) \\
&\quad - (\Gamma(Y_V(v_1, x)v_2))(0) + Y_{WV}^W((\Gamma(v_1))(0), x)v_2. \qquad (3.17)
\end{aligned}
$$

For $v_1, v_2 \in V$ and $w_1, w_2 \in W$, by (3.2) and (3.17), we have

$$
\begin{aligned}
&Y_{V\oplus W}^{(1)}((v_1, w_1), x)(v_2, w_2) \\
&\quad = (Y_V(v_1, x)v_2, Y_W(v_1, x)w_2 + Y_{WV}^W(w_1, x)v_2 + \Psi_1(v_1, x)v_2) \\
&\quad = (Y_V(v_1, x)v_2, Y_W(v_1, x)w_2 + Y_{WV}^W(w_1, x)v_2 + \Psi_2(v_1, x)v_2) \\
&\qquad + (Y_V(v_1, x)v_2, Y_W(v_1, x)(\Gamma(v_2))(0)) \\
&\qquad - (Y_V(v_1, x)v_2, (\Gamma(Y_V(v_1, x)v_2))(0)) \\
&\qquad + (Y_V(v_1, x)v_2, Y_{WV}^W((\Gamma(v_1))(0), x)v_2) \\
&\quad = Y_{V\oplus W}^{(2)}((v_1, w_1 + (\Gamma(v_1))(0)), x)(v_2, w_2 + (\Gamma(v_2))(0)) \\
&\qquad - (Y_V(v_1, x)v_2, (\Gamma(Y_V(v_1, x)v_2))(0)). \qquad (3.18)
\end{aligned}
$$

We now define a linear map $h : V \oplus W \to V \oplus W$ by

$$
e(v, w) = (v, w + (\Gamma(v))(0))
$$

for $v \in V$ and $w \in W$. Then h is a linear isomorphism and (3.18) can be rewritten as

$$h(Y^{(1)}_{V\oplus W}((v_1, w_1), x)(v_2, w_2)) = Y^{(2)}_{V\oplus W}(h(v_1, w_1), x)h(v_2, w_2). \tag{3.19}$$

for $v_1, v_2 \in V$ and $w_1, w_2 \in W$. Thus h is an isomorphism of grading-restricted vertex algebras from $(V \oplus W, Y^{(1)}_{V\oplus W}, (\mathbf{1}, 0))$ to $(V \oplus W, Y^{(2)}_{V\oplus W}, (\mathbf{1}, 0))$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W & \xrightarrow{i_2} & V \oplus W & \xrightarrow{p_1} & V & \longrightarrow & 0 \\ & & {\scriptstyle 1_W}\big\downarrow & & {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle 1_V} & & \\ 0 & \longrightarrow & W & \xrightarrow{i_2} & V \oplus W & \xrightarrow{p_1} & V & \longrightarrow & 0 \end{array} \tag{3.20}$$

is commutative. Thus the two square-zero extensions of V by W are equivalent.

Conversely, let $(V \oplus W, Y^{(1)}_{V+W}, p_1, i_2)$ and $(V \oplus W, Y^{(2)}_{V+W}, p_1, i_2)$ be two equivalent square-zero extensions of V by W. So there exists an isomorphism $h : V \oplus W \to V \oplus W$ of grading-restricted vertex algebras such that (3.20) is commutative. We have the following lemma which is also needed in the next section:

Lemma 3.1. *There exists a linear map $g : V \to V$ such that*

$$h(v, w) = (v, w + g(v))$$

for $v \in V$ and $w \in W$.

Proof. Let $h(v, w) = (f(v, w), g(v, w))$ for $v \in V$ and $w \in W$. Then by (3.20), we have $f(v, w) = v$ and $g(0, w) = w$. Since h is linear, we have $g(v, w) = g(v, 0) + g(0, w) = w + g(v, 0)$. So $h(v, w) = (v, w + g(v, 0))$. Taking $g(v)$ to be $g(v, 0)$, we see that the conclusion holds. ∎

Let $(\Gamma(v))(z_1) = e^{z_1 L_W(-1)}g(v) \in \overline{W}$. Then $\Gamma : V \to \widetilde{W}_{z_1}$ is an element of $C^1_{\frac{1}{2}}(V, W)$. By defintion, we have $g(v) = (\Gamma(v))(0)$ and $h(v, w) = (v, w + (\Gamma(v))(0))$ for $v \in V$ and $w \in W$. Let Φ_1 and Φ_2 be elements of $\ker \delta^2_{\frac{1}{2}}$ obtained from $(V \oplus W, Y^{(1)}_{V\oplus W}, p_1, i_2)$ and $(V \oplus W, Y^{(2)}_{V+W}, p_1, i_2)$, respectively, and Ψ_1 and Ψ_2 the corresponding maps from $V \otimes V$ to $W((x))$. Then since h is a homomorphism of grading-restricted vertex algebras, (3.19) holds for

$v_1, v_2 \in V$ and $w_1, w_2 \in W$. Thus the two sides of (3.18) are equal for $v_1, v_2 \in V$ and $w_1, w_2 \in W$. So the two expressions in the middle of (3.18) are equal for $v_1, v_2 \in V$ and $w_1, w_2 \in W$. Thus we have (3.17) for $v_1, v_2 \in V$. Formula (3.17) implies that the two sides of (3.16) are equal for $v_1, v_2 \in V$. Thus the middle expressions in (3.16) are all equal for $v_1, v_2 \in V$. In particular, we obtain $\Phi_1 = \Phi_2 + \Gamma$. So Φ_1 and Φ_2 are differed by an element of $\delta_1 C^1(V, W)$. ∎

4 Square-zero extensions and first order deformations

In this section, we prove Theorem 1.7.

Let $Y_t : V \otimes V \to V((x))$, $t \in U$, be a first order deformation of V. By definition, there exists

$$\begin{aligned} \Psi : V \otimes V &\to V((x)) \\ v_1 \otimes v_2 &\to \Psi(v_1, x)v_2 \end{aligned}$$

such that

$$Y_t(v_1, x)v_2 = Y_V(v_1, x)v_2 + t\Psi(v_1, x)v_2$$

for $v_1, v_2 \in V$. By definition, $(V, Y_t, \mathbf{1})$ satisfies all the axioms for grading-restricted vertex algebras up to the first order in t and consequently have all properties of grading-restricted vertex algebras up to the first order in t.

The identity property for $(V, Y_t, \mathbf{1})$ up to the first order in t gives

$$Y_V(\mathbf{1}, x)v + t\Psi(\mathbf{1}, x)v = v + O(t^2)$$

for $v \in V$. So we obtain

$$\Psi(\mathbf{1}, x)v = 0 \tag{4.1}$$

for $v \in V$. The creation property for $(V, Y_t, \mathbf{1})$ up to the first order in t gives

$$\lim_{t\to 0}(Y_V(v, x) + t\Psi(v, x))\mathbf{1} = v + O(t^2)$$

for $v \in V$. Then we have

$$\lim_{t\to 0} \Psi(v, x)\mathbf{1} = 0 \tag{4.2}$$

for $v \in V$.

We have the following duality property up to the first order in t: For $v_1, v_2, v_3 \in V$ and $v' \in V'$, the coefficients of t^0 and t^1 terms of

$$\langle v', Y_t(v_1, z_1) Y_t(v_2, z_2) v_3 \rangle$$
$$\langle v', Y_t(v_2, z_2) Y_t(v_1, z_1) v_3 \rangle$$
$$\langle v', Y_t(Y_t(v_1, z_1 - z_2) v_2, z_2) v_3 \rangle$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to common rational functions in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$, or equivalently,

$$\langle v', (Y_V(v_1, z_1)\Psi(v_2, z_2) + \Psi(v_1, z_1) Y_V(v_2, z_2)) v_3 \rangle \tag{4.3}$$
$$\langle v', (Y_V(v_2, z_2)\Psi(v_1, z_1) + \Psi(v_2, z_2) Y_V(v_1, z_1)) v_3 \rangle \tag{4.4}$$
$$\langle v', (Y_V(\Psi(v_1, z_1 - z_2) v_2, z_2) + \Psi(Y_V(v_1, z_1 - z_2) v_2, z_2)) v_3 \rangle \tag{4.5}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

Let

$$\begin{aligned} Y_{V\oplus V} : (V \oplus V) \otimes (V \oplus V) &\to (V \oplus V)[[x, x^{-1}]] \\ (u_1, v_1) \otimes (u_2, v_2) &\mapsto Y_{V\oplus V}((u_1, v_1), x)(u_2, v_2) \end{aligned}$$

be given by

$$\begin{aligned} &Y_{V\oplus V}((u_1, v_1), x)(u_2, v_2) \\ &\quad = (Y_V(u_1, x)u_2, Y_V(u_1, x)v_2 + Y_V(v_1, x)u_2 + \Psi(u_1, x)u_2) \end{aligned} \tag{4.6}$$

for $u_1, u_2, v_1, v_2 \in V$. By (4.6) and (4.1),

$$\begin{aligned} Y_{V\oplus V}((\mathbf{1}, 0), x)(u, v) &= (Y_V(\mathbf{1}, x)u, Y_V(\mathbf{1}, x)v + Y_V(0, x)u + \Psi(\mathbf{1}, x)u) \\ &= (u, v) \end{aligned}$$

for $u, v \in V$, that is, $(V \oplus V, Y_{V\oplus V}, (\mathbf{1}, 0))$ has the identity property. By (4.6) and (4.2),

$$\begin{aligned} &\lim_{x\to 0} Y_{V\oplus V}((u, v), x)(\mathbf{1}, 0) \\ &\quad = (\lim_{x\to 0} Y_V(u, x)\mathbf{1}, \lim_{x\to 0} Y_V(u, x)0 + \lim_{x\to 0} Y_V(v, x)\mathbf{1} + \Psi(u, x)\mathbf{1}) \\ &\quad = (u, v) \end{aligned}$$

for $u, v \in V$, that is, $(V \oplus V, Y_{V\oplus V}, (\mathbf{1}, 0))$ has the creation property.

By (4.6), we have

$$\begin{aligned}
&\langle (u', v'), Y_{V\oplus V}((u_1, v_1), z_1) Y_{V\oplus V}((u_2, v_2), z_2)(u_3, v_3)\rangle \\
&\quad = \langle (u', v'), Y_{V\oplus V}((u_1, v_1), z_1) \cdot \\
&\qquad\qquad \cdot (Y_V(u_2, z_2)u_3, Y_V(u_2, z_2)v_3 + Y_V(v_2, z_2)u_3 + \Psi(u_2, z_2)u_3)\rangle \\
&\quad = \langle u', Y_V(u_1, z_1)Y_V(u_2, z_2)u_3\rangle + \langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v_3\rangle \\
&\qquad + \langle v', Y_V(u_1, z_1)Y_V(v_2, z_2)u_3\rangle + \langle v', Y_V(u_1, z_1)\Psi(u_2, z_2)u_3\rangle \\
&\qquad + \langle v', Y_V(v_1, z_1)Y_V(u_2, z_2)u_3\rangle + \langle v', \Psi(u_1, z_1)Y_V(u_2, z_2)u_3\rangle. \qquad (4.7)
\end{aligned}$$

By the properties of V and the absolute convergence of (4.3), we see that the left-hand side of (4.7) is absolutely convergent when $|z_1| > |z_2| > 0$. Similarly, by (4.6), we have

$$\begin{aligned}
&\langle (u', v'), Y_{V\oplus V}((u_2, v_2), z_2) Y_{V\oplus V}((u_1, v_1), z_1)(u_3, v_3)\rangle \\
&\quad = \langle u', Y_V(u_2, z_2)Y_V(u_1, z_1)u_3\rangle + \langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v_3\rangle \\
&\qquad + \langle v', Y_V(u_2, z_2)Y_V(v_1, z_1)u_3\rangle + \langle v', Y_V(u_2, z_2)\Psi(u_1, z_1)u_3\rangle \\
&\qquad + \langle v', Y_V(v_2, z_2)Y_V(u_1, z_1)u_3\rangle + \langle v', \Psi(u_2, z_2)Y_V(u_1, z_1)u_3\rangle \qquad (4.8)
\end{aligned}$$

and the left-hand side of (4.8) is absolutely convergent when $|z_2| > |z_1| > 0$. Moreover, since (4.3) and (4.4) converges absolutely when $|z_1| > |z_2| > 0$ and when $|z_2| > |z_1| > 0$, respectively, to a common rational function with the only possible poles at $z_1, z_2, z_1 - z_2 = 0$, the left-hand side of (4.7) and left-hand side of (4.8) also converges absolutely when $|z_1| > |z_2| > 0$ and when $|z_2| > |z_1| > 0$, respectively, to a common rational function with the only possible pole at $z_1 - z_2 = 0$. By (4.6) again, we have

$$\begin{aligned}
&\langle (u', v'), Y_{V\oplus V}(Y_{V\oplus V}((u_1, v_1), z_1 - z_2)(u_2, v_2), z_2)(u_3, v_3)\rangle \\
&\quad = \langle (u', v'), Y_{V\oplus V}((Y_V(u_1, z_1 - z_2)u_2, \\
&\qquad\qquad Y_V(u_1, z_1 - z_2)v_2 + Y_V(v_1, z_1 - z_2)u_2 \\
&\qquad\qquad\qquad + \Psi(u_1, z_1 - z_2)u_2), z_2)(u_3, v_3)\rangle \\
&\quad = \langle u', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)u_3\rangle + \langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v_3\rangle \\
&\qquad + \langle v', Y_V(Y_V(u_1, z_1 - z_2)v_2, z_2)u_3\rangle + \langle v', Y_V(Y_V(v_1, z_1 - z_2)u_2, z_2)u_3\rangle \\
&\qquad + \langle v', Y_V(\Psi(u_1, z_1 - z_2)u_2, z_2)u_3\rangle + \langle v', \Psi(Y_V(u_1, z_1 - z_2)u_2, z_2)u_3\rangle. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.9)
\end{aligned}$$

By the properties of V and the absolute convergence of (4.5) and (4.9), we see that the left-hand side of (4.9) is absolutely convergent when $|z_2| > |z_1 - z_2| > 0$. Moreover, since (4.3) and (4.5) converges absolutely when $|z_1| > |z_2| > 0$ and when $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function with the only possible poles at $z_1, z_2, z_1 - z_2 = 0$, the left-hand side of (4.7) and left-hand side of (4.9) also converges absolutely when $|z_1| > |z_2| > 0$ and when $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function with the only possible poles at $z_1, z_2, z_1 - z_2 = 0$. So $(V \oplus V, Y_{V\oplus V}, (\mathbf{1}, 0))$ has the duality property.

Note that the $L(-1)$-derivative propery and the $L(-1)$-bracket property are consequences of the other axioms. Thus $(V \oplus V, Y_{V\oplus V}, (\mathbf{1}, 0))$ is a grading-restricted vertex algebra.

By definition,

$$\begin{aligned}
&p_1(Y_{V\oplus V}((u_1, v_1), x)(u_2, v_2)) \\
&\quad = p_1(Y_V(u_1, x)u_2, Y_V(u_1, x)v_2 + Y_V(v_1, x)u_2 + \Psi(u_1, x)u_2) \\
&\quad = Y_V(u_1, x)u_2 \\
&\quad = Y_V(p_1(u_1, v_1), x)p_1(u_2, v_2)
\end{aligned}$$

for $u_1, u_2, v_1, v_2 \in V$. Also

$$\ker p_1 = 0 \oplus V$$

and

$$Y_{V\oplus V}((0, v_1), x)(0, v_2) = (0, 0)$$

for $v_1.v_2 \in V$. So p_1 is a surjective homomorphism of grading-restricted vertex algebras and $\ker p_1$ is a square-zero ideal of $V \oplus V$.

We use $Y^V_{V\oplus V}$ to denote the vertex operator map for $V \oplus V$ when $V \oplus V$ is viewed as a V-module. Then by definition,

$$\begin{aligned}
i_2(Y_V(v_1, x)v_2) &= (0, Y_V(v_1, x)v_2) \\
&= Y^V_{V\oplus V}(v_1, x)(0, v_2) \\
&= Y^V_{V\oplus V}(v_1, x)i_2(v_2)
\end{aligned}$$

for $v_1, v_2 \in V$. So i_2 is an injective homomorphism of V-modules. Clearly, we have $i_2(V) = \ker p_1$. Thus $(V \oplus V, Y_{V\oplus V}, p_1, i_2)$ is a square-zero extension of V by V.

Conversely, let $(V\oplus V, Y_{V\oplus V}, p_1, i_2)$ be a square-zero extension of V by V. Then there exists

$$\begin{aligned}\Psi: V\otimes V &\to V((x))\\ v_1\otimes v_2 &\to \Psi(v_1, x)v_2\end{aligned}$$

such that

$$Y_{V\oplus V}((u_1, 0), x)(u_2, 0) = (Y_V(u_1, x)u_2, \Psi(u_1, x)u_2)$$

for $u_1, u_2\in V$. The identity property and the creation property of the grading-restricted vertex algebra $(V\oplus V, Y_{V\oplus V}, (\mathbf{1}, 0))$ give (4.1) and (4.2). The duality property for $(V\oplus V, Y_{V\oplus V}, (\mathbf{1}, 0))$ gives (4.3), (4.4) and (4.5).

For $t\in\mathbb{C}$, define

$$Y_t(v_1, x)v_2 = Y_V(v_1, x)v_2 + t\Psi(v_1, x)v_2$$

for $v_1, v_2\in V$. Then (4.1) and (4.2) imply that Y_t satisfies the identity property and the creation property up to the first order in t and (4.3), (4.4) and (4.5) imply that Y_t satisfies the duality property up to the first order in t. Thus $(V, Y_t, \mathbf{1})$ is a grading-restricted vertex algebras up to the first order in t, that is, Y_t is a first-order deformation of $(V, Y_V, \mathbf{1})$.

Now we prove that two first-order deformations of V are equivalent if and only if the corresponding square-zero extensions of V by V are equivalent.

Consider two equivalent first-order deformations of V given by $Y_t^{(1)}$: $V\otimes V\to V((x))$ and $Y_t^{(2)}: V\otimes V\to V((x))$ for $t\in\mathbb{C}$. Then there exist a family $f_t: V\to V$, $t\in\mathbb{C}$, of linear maps of the form $f_t = 1_V + tg$ where $g: V\to V$ is a linear map preserving the grading of V such that (1.1) holds for $v_1, v_2\in V$. By definition, there exist linear maps

$$\begin{aligned}\Psi_1: V\otimes V &\to V((x))\\ v_1\otimes v_2 &\to \Psi_1(v_1, x)v_2\end{aligned}$$

and

$$\begin{aligned}\Psi_2: V\otimes V &\to V((x))\\ v_1\otimes v_2 &\to \Psi_2(v_1, x)v_2\end{aligned}$$

such that $Y_t^{(1)} = Y_V + t\Psi_1$ and $Y_t^{(2)} = Y_V + t\Psi_2$. By (1.1), we have

$$\begin{aligned}&\Psi_1(v_1, x)v_2 - \Psi_2(v_1, x)v_2\\ &\quad = -g(Y_V(v_1, x)v_2) + Y_V(g(v_1), x)v_2 + Y_V(v_1, x)g(v_2)\end{aligned} \tag{4.10}$$

for $v_1, v_2 \in V$.

Let $(V \oplus V, Y^{(1)}_{V\oplus V}, p_1, i_2)$ and $(V \oplus V, Y^{(2)}_{V\oplus V}, p_1, i_2)$ be the suqare-zero extensions of V by V constructed from $Y_t^{(1)}$ and $Y_t^{(2)}$. Let $h : V \oplus V \to V \oplus V$ be defined by

$$h(v_1, v_2) = (v_1, v_2 + g(v_1))$$

for $v_1, v_2 \in V$. Clearly, h is a linear isomorphism. For $u_1, u_2, v_1, v_2 \in V$, by definition and (4.10),

$$\begin{aligned}
& h(Y^{(1)}_{V\oplus V}((u_1, v_1), x)(u_2, v_2)) \\
&\quad = h(Y_V(u_1, x)u_2, Y_V(u_1, x)v_2 + Y_V(v_1, x)u_2 + \Psi_1(u_1, x)u_2) \\
&\quad = (Y_V(u_1, x)u_2, Y_V(u_1, x)v_2 + Y_V(v_1, x)u_2 \\
&\qquad\qquad + \Psi_1(u_1, x)u_2 + g(Y_V(u_1, x)u_2)) \\
&\quad = (Y_V(u_1, x)u_2, Y_V(u_1, x)v_2 + Y_V(v_1, x)u_2 \\
&\qquad\qquad + \Psi_2(u_1, x)u_2 + Y_V(g(u_1), x)u_2 + Y_V(u_1, x)g(u_2))) \\
&\quad = (Y_V(u_1, x)u_2, Y_V(u_1, x)(v_2 + g(u_2)) \\
&\qquad\qquad + Y_V((v_1 + g(u_1)), x)u_2 + \Psi_2(u_1, x)u_2) \\
&\quad = Y^{(2)}_{V\oplus V}(h(u_1, v_1), x)h(u_2, v_2).
\end{aligned}$$

So h is in fact an isomorphism from the algebra $(V \oplus V, Y^{(1)}_{V\oplus V}, (\mathbf{1}, 0))$ to the algebra $(V \oplus V, Y^{(2)}_{V\oplus V}, (\mathbf{1}, 0))$. Now it is clear that the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V & \underset{i_2}{\longrightarrow} & V \oplus V & \underset{p_1}{\longrightarrow} & V & \longrightarrow & 0 \\
 & & {\scriptstyle 1_W}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle 1_V} & & \\
0 & \longrightarrow & V & \underset{i_2}{\longrightarrow} & V \oplus V & \underset{p_1}{\longrightarrow} & V & \longrightarrow & 0,
\end{array}$$

So these two first order deformations are equivalent.

Conversely, let $(V \oplus V, Y^{(1)}_{V\oplus V}, p_1, i_2)$ and $(V \oplus V, Y^{(2)}_{V\oplus V}, p_1, i_2)$ be two equivalent suqare-zero extensions of V by V. Let $\Psi_1, \Psi_2 : V \otimes V \to V((x))$ be given by

$$\begin{aligned}
Y^{(1)}_{V\oplus V}((u_1, 0), x)(u_2, 0)) &= (Y_V(u_1, x)u_2, \Psi_1(u_1, x)u_2), \\
Y^{(2)}_{V\oplus V}((u_1, 0), x)(u_2, 0)) &= (Y_V(u_1, x)u_2, \Psi_2(u_1, x)u_2)
\end{aligned}$$

for $u_1, u_2 \in V$. Then $Y_t^{(1)}, Y_t^{(2)} : V \otimes V \to V((x))$ given by

$$Y_t^{(1)}(v_1, x)v_2 = Y_V(v_1, x)v_2 + t\Psi_1(v_1, x)v_2,$$
$$Y_t^{(2)}(v_1, x)v_2 = Y_V(v_1, x)v_2 + t\Psi_2(v_1, x)v_2$$

for $v_1, v_2 \in V$ are first-order deformations of $(V, Y_V, \mathbf{1})$ by the proof above.

Let $h : V \oplus V \to V \oplus V$ be an equivalence from $(V \oplus V, Y^{(1)}_{V\oplus V}, p_1, i_2)$ to $(V \oplus V, Y^{(2)}_{V\oplus V}, p_1, i_2)$. Then by Lemma 3.1, there exists a linear map $g : V \to V$ such that

$$h(v_1, v_2) = (v_1, v_2 + g(v_1))$$

for $v_1, v_2 \in V$. Using the fact that h is an isomorphism of grading-restricted vertex algebras from $(V \oplus V, Y^{(1)}_{V\oplus V}, (\mathbf{1}, 0))$ to $(V \oplus V, Y^{(2)}_{V\oplus V}, (\mathbf{1}, 0))$, we obtain (4.10) which implies (1.1). Thus the two first-order deformations $Y_t^{(1)}$ and $Y_t^{(2)}$ are equivalent. ∎

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Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China,

Kavali Institute For Theoretical Physics China, CAS, Beijing 100190, China

and

Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019 (permanent address)

E-mail address: yzhuang@math.rutgers.edu