Differential equations, duality and modular invariance

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Abstract

We solve the problem of constructing all chiral genus-one correlation functions from chiral genus-zero correlation functions associated to a vertex operator algebra satisfying the following conditions: (i) $V_{(n)} = 0$ for $n < 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$, (ii) every $\mathbb{N}$-gradable weak V-module is completely reducible and (iii) V is C_2-cofinite. We establish the fundamental properties of these functions, including suitably formulated commutativity, associativity and modular invariance. The method we develop and use here is completely different from the one previously used by Zhu and other people. In particular, we show that the q-traces of products of certain geometrically-modified intertwining operators satisfy modular invariant systems of differential equations which, for any fixed modular parameter, reduce to doubly-periodic systems with only regular singular points. Together with the results obtained by the author in the genus-zero case, the results of the present paper solves essentially the problem of constructing chiral genus-one weakly conformal field theories from the representations of a vertex operator algebra satisfying the conditions above.

Contents

0 Introduction

The present paper is one of the main steps in the author's program of constructing conformal field theories in the sense of Segal [S1] [S2] [S3] and Kontsevich from representations of suitable vertex operator algebras. We construct all the chiral genus-one correlation functions from chiral genus-zero correlation functions (or equivalently, from intertwining operators) for a vertex operator algebra satisfying suitable natural conditions, and we establish their fundamental properties, including suitably formulated commutativity, associativity and modular invariance. Together with the results previously obtained by the author in the genus-zero case, the results of the present paper solve essentially the problem of constructing chiral genus-one weakly conformal field theories. The results of the present paper imply the rigidity of the ribbon tensor category of modules for a suitable vertex operator algebra. Together with the results in the genus-zero case and the results obtained by Moore and Seiberg in [MS1] and [MS2], the results of the present paper also imply the Verlinde conjecture that the fusion rules are diagonalized by the action of the modular transfermation given by $\tau \mapsto -1/\tau$. These applications will be discussed in [H10] and [H11]. We believe that the results of the present paper will be interesting and useful not only to people working on vertex operator algebras and conformal field theories, but also to people in other related fields.

In [H3], [H4], [H5], [H6], [H8] and [H9], the author has constructed a chiral genus-zero weakly conformal field theory in the sense of Segal [S2] [S3] from representations of a vertex operator algebra satisfying the C_1-cofiniteness condition and certain finite reductivity conditions. The genus-zero part of the program is thus more or less finished. The next natural step is the construction of all the chiral genus-one correlation functions and the proof of their basic properties. In [Z], Zhu constructed chiral genus-one correlation functions associated to elements of a vertex operator algebra V from V-modules when V satisfies the C_2-cofiniteness condition and certain (stronger) finite reductivity conditions, and he proved their basic properties, including modular invariance. Zhu's work was generalized by Dong-Li-Mason [DLM] to the case of twisted representations of vertex operator algebras and by Miyamoto [M1] to the case of chiral genus-one correlation functions associated to elements of modules among which at most one is not isomorphic to the algebra. Miyamoto [M2] further constructed certain one-point chiral genus-one correlation functions involving logarithms when the vertex operator algebra satisfies only the C_2-cofiniteness condition.

The method used by Zhu, Dong-Li-Mason and Miyamoto depends on the commutator formula for (untwisted or twisted) vertex operators or the commutator formula between vertex operators and intertwining operators. These formulas were needed to find recurrence relations such that the study of the formal q-traces of products of more than one operator can be reduced to the study of the formal q-traces of vertex or intertwining operators, not products of such operators.

However, this method cannot be generalized to give a construction of general chiral genus-one correlation functions because there is no commutator formula for general intertwining operators, instead see [H1]. Note that chiral genus-one correlation functions are the the main objects to construct in the genus-one case. Also, most of the conjectures in chiral genus-one weakly conformal field theories can be formulated and proved directly using these correlation functions. For example, based on the assumptions that all the chiral genus-zero and genus-one correlation functions have been constructed and the desired properties (including the duality properties and the modular invariance) hold, Moore and Seiberg in [MS1] and [MS2] derived the genus-one coherence relations and proved the Verlinde conjecture [V] (the diagonalization of the fusion rules by the action of the modular transformation $\tau \mapsto -1/\tau$). It turns out that the proof of the rigidity of the ribbon tensor category of modules for a suitable vertex operator algebra also needs such correlation functions and their properties [H11]. In the present paper, we develop a method completely different from the one used in [Z], [DLM], [M1] and [M2] and completely solve the problem of constructing all the chiral genus-zero correlation functions and establishing all the desired properties.

Here is a brief description of our main results. For simplicity, here we shall discuss the results under certain strong conditions, though, as we show in this paper, many of these results in fact hold under weaker conditions. Let V be a vertex operator algebra satisfying the following conditions: (i) $V_{(n)} = 0$ for

$n < 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$, (ii) every $\mathbb{N}$-gradable weak V-module is completely reducible (see Section 6 for a definition of $\mathbb{N}$-gradable weak V-module) and (iii) V is C_2-cofinite (that is, $\dim V/C_2(V) < \infty$, where $C_2(V)$ is the subspace of V spanned by the elements of the form $u_{-2}v$ for $u, v \in V$). We prove that the q-traces of products of certain geometrically-modified intertwining operators satisfy modular invariant systems of different equations which reduce to doubly-periodic systems with only regular singular points when the modular parameter τ is fixed. Using these systems of differential equations, we prove the convergence of the q-traces. The analytic extensions of these convergent q-traces give chiral genus-one correlation functions. We prove that these chiral genus-one correlation functions satisfy suitable "genus-one commutativity" and "genus-one associativity" properties. We also introduce a new product in a vertex operator algebra and, using this new product, we construct and study an associative algebra which looks very different from, but is in fact isomorphic to, Zhu's algebra (see [Z]). Using these results, we prove the modular invariance of the space of chiral genus-one correlation functions. Geometrically, this modular invariance and the double-periodicity of the reduced systems mentioned above give vector bundles with flat connections over the moduli spaces of genus-one Riemann surfaces with punctures and standard local coordinates vanishing at the punctures.

Together with the results obtained by the author in the genus-zero case, the results of the present paper solves essentially the problem of constructing chiral genus-one weakly conformal field theories. The results of the present paper also imply the rigidity of the ribbon tensor category of modules for a vertex operator algebra. The results of the present paper are the genus-one parts of the assumptions on rational conformal field theories in Moore-Seiberg's important work [MS1] [MS2]. Thus, together with the author's results in [H1], [H2] and [H9] in the genus-zero case, the results of the present paper imply all the results of Moore and Seiberg in [MS1] and [MS2] obtained using only the genus-zero and genus-one parts of their fundamental assumptions on rational conformal field theories. In particular, the the results of the present paper together with the author's results in [H1], [H2] and [H9] in the genus-zero case imply the Verlinde conjecture for modules for a vertex operator algebra satisfying the conditions mentioned above. The details of the proof of the rigidity of the ribbon tensor category and the proof of the Verlinde conjecture will be given in the papers [H10] and [H11], repectively.

Our construction of the chiral genus-one correlation functions actually verifies the sewing property which states that chiral genus-one correlation functions can be obtained from chiral genus-zero correlation functions by taking q-traces. In particular, we construct the spaces of chiral genus-one correlation functions and the factorization property for these spaces is an easy consequence of the sewing property. We would like to emphasize that the converse is not true, that is, even if the spaces of chiral genus-one correlation functions are identified abstractly and the factorization property is proved, additional structures such as those constructed in the present paper are still needed to construct the chiral genus-one correlation functions and to prove the sewing property.

Our modular invariance result gives, in particular, new proofs of the modular invariance result of Zhu in [Z] and its direct, straightforward generalization by Miyamoto in [M1] since these results are (very) special cases. The differential equations and the duality properties obtained in the present paper also have straightforward generalizations to the case of twisted modules and intertwining operators among them. Together with a straightforward generalization of some of the results on the associative algebras discussed in the present paper, they also give a modular invariance result in the twisted case. In particular, as a special case, this modular invariance gives a new proof of the modular invariance result of Dong, Li and Mason in [DLM]. We shall discuss the twisted case elsewhere.

We assume that the reader is familiar with basic notions and results in the algebraic and geometric theory of vertex operator algebras as presented in [FLM], [FHL] and [H3]. We also assume that the reader is familiar with the theory of intertwining operator algebras as developed and presented by the author in [H1], [H2], [H4], [H5], [H7], [H9], based on the tensor product theory developed by Lepowsky and the author in [HL1]–[HL4] and [H1]. We do not, however, assume that the reader is familiar with the modular invariance result proved by Zhu in [Z] and the method used there and in other papers, since our method is completely different.

The present paper is organized as follows: In Section 1, we discuss geometrically-modified intertwining operators, which play an important role in our construction. Then we prove some fundamental identities for the q-traces of products of these geometrically-modified intertwining operators in Section 2. We establish the existence of the systems of differential equations in Section 3. In Section 4, we prove the convergence of the q-traces of products of geometrically-modified intertwining operators and construct chiral genus-one correlation functions. In the same section, we also establish genus-one commutativity and associativity for these chiral genus-one correlation functions. In Section 5, we prove that for fixed modular parameter τ, the systems consisting of those equations in the systems mentioned above not involving derivatives of τ have only regular singular points. We introduce our new product on a vertex operator algebra and construct and study the corresponding associative algebra in Section 6. The modular invariance of the space of chiral genus-one correlation functions is proved in Section 7.

Notations and conventions In this paper, i is either $\sqrt{-1}$ or an index, and it should be easy to tell which is which. The symbols $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ denote the nonnegative integers, positive integers, integers, rational numbers, real numbers, complex numbers and the upper half plane, respectively. We shall use $x, , y, ...$ to denote commuting formal variables and $z, z_1, z_2, ...$ to denote complex numbers or complex variables. For any nonzero $z \in \mathbb{C}$ and any $n \in \mathbb{Z}$, $\log z = \log |z| + i \arg z$ with $0 \leq \arg z < 2\pi$ and $z^n = e^{n \log z}$. For any $z \in \mathbb{C}$, $q_z = e^{2\pi i z}$. As in [FLM] and [FHL], for a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$ or a module (W, Y), we use the u_n for $u \in V$ and $n \in \mathbb{Z}$ to denote the coefficients of the formal series $Y(u, x)$. We shall use $w_1, w_2, ...$ to denote elements of V-modules. Note that when the V-module is the adjoint module, the notations u_n and $w_1, w_2, ...$ clearly have different meanings.

Acknowledgment This research is supported in part by NSF grant DMS-0070800. I am grateful to Jim Lepowsky and Antun Milas for helpful comments.

1 Geometrically-modified intertwining operators

We need the theory of composition-invertible power series and their actions on modules for the Virasoro algebra developed in [H3]. Let A_j, $j \in \mathbb{Z}_+$, be the complex numbers defined by

$$\frac{1}{2\pi i} \log(1 + 2\pi i y) = \left(\exp \left(\sum_{j \in \mathbb{Z}_+} A_j y^{j+1} \frac{\partial}{\partial y} \right) \right) y.$$

In particular, a simple calculation gives

$$A_1 = -\pi i, \quad (1.1)$$

$$A_2 = -\frac{\pi^2}{3}. \quad (1.2)$$

Note that the composition inverse of $\frac{1}{2\pi i}\log(1+2\pi i y)$ is $\frac{1}{2\pi i}(e^{2\pi i y}-1)$ and thus we have

$$\frac{1}{2\pi i}(e^{2\pi i y}-1) = \left(\exp\left(-\sum_{j\in\mathbb{Z}_+} A_j y^{j+1}\frac{\partial}{\partial y}\right)\right) y.$$

In this paper, we fix a vertex operator algebra V. We shall use Y to denote the vertex operator maps for the algebra V and for V-modules. For any V-module W, we shall denote the operator $\sum_{j\in\mathbb{Z}_+} A_j L(j)$ on W by $L_+(A)$. Then

$$e^{-\sum_{j\in\mathbb{Z}_+} A_j L(j)} = e^{-L_+(A)}.$$

For convenience, we introduce the operator

$$\mathcal{U}(x) = (2\pi i x)^{L(0)} e^{-L^+(A)} \in (\mathrm{End}\, W)\{x\} \quad (1.3)$$

where $(2\pi i)^{L(0)} = e^{(\log 2\pi + i\frac{\pi}{2})L(0)}$ (and we shall use this convention throughout this paper). In fact we can substitute for x in the operator $\mathcal{U}(x)$ a number or any formal expression which makes sense. For example,

$$\mathcal{U}(1) = (2\pi i)^{L(0)} e^{-L_+(A)}$$

and we have

$$\mathcal{U}(xy) = x^{L(0)}\mathcal{U}(y). \quad (1.4)$$

For any $z \in \mathbb{C}$, we shall use q_z to denote $e^{2\pi i z}$.

Lemma 1.1 *For the conformal element $\omega = L(-2)\mathbf{1} \in V$,*

$$\mathcal{U}(1)\omega = (2\pi i)^2\left(\omega - \frac{c}{24}\mathbf{1}\right),$$

where c is the central charge of V.

Proof. By definition, we have

$$\begin{aligned}
\mathcal{U}(1)\omega &= (2\pi i)^{L(0)}e^{-L_+(A)}L(-2)\mathbf{1}\\
&= (2\pi i)^{L(0)}L(-2)\mathbf{1} - (2\pi i)^{L(0)}A_2L(2)L(-2)\mathbf{1}\\
&= (2\pi i)^2L(-2)\mathbf{1} + \frac{\pi^2 c}{6}\mathbf{1}\\
&= (2\pi i)^2\left(\omega - \frac{c}{24}\mathbf{1}\right),
\end{aligned}$$

where we have used (1.1) and (1.2). ∎

We have the following conjugation and commutator formulas:

Proposition 1.2 *Let W_1, W_2 and W_3 be V-modules and $\mathcal{Y}$ an intertwining operator of type $\binom{W_3}{W_1 W_2}$. Then for $u \in V$ and $w \in W_1$,*

$$\mathcal{U}(1)\mathcal{Y}(w,x)(\mathcal{U}(1))^{-1} = \mathcal{Y}(\mathcal{U}(e^{2\pi i x})w, e^{2\pi i x} - 1) \tag{1.5}$$

and

$$\begin{aligned}
&[Y(\mathcal{U}(x_1)u, x_1), \mathcal{Y}(\mathcal{U}(x_2)w, x_2)]\\
&\quad = 2\pi i \mathrm{Res}_y \delta\left(\frac{x_1}{e^{2\pi i y}x_2}\right)\mathcal{Y}(\mathcal{U}(x_2)Y(u,y)w, x_2).
\end{aligned} \tag{1.6}$$

Proof. Let B_j, $j \in \mathbb{Z}_+$, be rational numbers defined by

$$\log(1+y) = \left(\exp\left(\sum_{j\in\mathbb{Z}_+} B_j y^{j+1}\frac{\partial}{\partial y}\right)\right) y.$$

Since the composition inverse of $\log(1+y)$ is $e^y - 1$, we have

$$e^y - 1 = \left(\exp\left(-\sum_{j\in\mathbb{Z}_+} B_j y^{j+1}\frac{\partial}{\partial y}\right)\right) y.$$

By the change-of-variable formula in [H3], we have

$$Y(e^{yL(0)}u, e^y - 1) = e^{-L_+(B)}Y(e^{L_+(B)}u, y)e^{L_+(B)}. \tag{1.7}$$

By the definition of the sequences $\{A_j\}_{j\in\mathbb{Z}_+}$ and $\{B_j\}_{j\in\mathbb{Z}_+}$, we have

$$\left(\exp\left(\sum_{j\in\mathbb{Z}_+} B_j y^{j+1}\frac{\partial}{\partial y}\right)\right) y$$
$$= (2\pi i)^{-y\frac{\partial}{\partial y}}\left(\exp\left(\sum_{j\in\mathbb{Z}_+} A_j y^{j+1}\frac{\partial}{\partial y}\right)\right)(2\pi i)^{y\frac{\partial}{\partial y}} y.$$

Thus

$$\begin{aligned} e^{-L_+(B)} &= (2\pi i)^{L(0)} e^{-L_+(A)} (2\pi i)^{-L(0)} \\ &= \mathcal{U}(1)(2\pi i)^{-L(0)}, \quad (1.8)\\ e^{L_+(B)} &= (2\pi i)^{L(0)} e^{L_+(A)} (2\pi i)^{-L(0)} \\ &= (2\pi i)^{L(0)} (\mathcal{U}(1))^{-1}. \quad (1.9)\end{aligned}$$

Substituting $\mathcal{U}(1)w$ for w in (1.7) and using (1.4), (1.8), (1.9) and the $L(0)$-conjugation formula

$$(2\pi i)^{-L(0)}\mathcal{Y}((2\pi i)^{L(0)}w, x)(2\pi i)^{L(0)} = \mathcal{Y}\left(w, \frac{x}{2\pi i}\right), \quad (1.10)$$

we obtain

$$\mathcal{U}(1)\mathcal{Y}\left(w, \frac{x}{2\pi i}\right)(\mathcal{U}(1))^{-1} = \mathcal{Y}(\mathcal{U}(e^x)w, e^x - 1),$$

which is equivalent to (1.5).

As a special case of the Jacobi identity [FHL] defining intertwining operators, we have the commutator formula between vertex operators and intertwining operators

$$[Y(u, x_1), \mathcal{Y}(w, x_2)] = \mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}(Y(u, x_0)w, x_2).$$

Thus

$$\begin{aligned} &[Y(x_1^{L(0)}u, x_1), \mathcal{Y}(x_2^{L(0)}w, x_2)] \\ &\quad = \mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}(Y(x_1^{L(0)}u, x_0)x_2^{L(0)}w, x_2) \\ &\quad = \mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)\mathcal{Y}\left(x_2^{L(0)}Y\left(\left(\frac{x_1}{x_2}\right)^{L(0)}u, \frac{x_0}{x_2}\right)w, x_2\right). \end{aligned} \quad (1.11)$$

Now we change the variable x_0 to y as follows:

$$x_0 = \sum_{k\geq 1} \frac{x_2(2\pi i y)^k}{k!} = (e^{2\pi i y} - 1)x_2.$$

(Note that by definition, for any $r \in \mathbb{C}$,

$$(x_2 + (e^{2\pi i y} - 1)x_2)^r = e^{2\pi i y r} x_2^r.)$$

Then the right-hand side of (1.11) becomes

$$\begin{aligned}
&\mathrm{Res}_y \frac{\partial x_0}{\partial y} x_2^{-1} \delta\left(\frac{x_1 - (e^{2\pi i y} - 1)x_2}{x_2}\right) \mathcal{Y}(x_2^{L(0)} Y((x_1/x_2)^{L(0)} u, e^{2\pi i y} - 1)w, x_2)\\
&\quad = 2\pi i \mathrm{Res}_y e^{2\pi i y} x_2 (x_2 + (e^{2\pi i y} - 1)x_2)^{-1} \delta\left(\frac{x_1}{x_2 + (e^{2\pi i y} - 1)x_2}\right) \cdot\\
&\qquad\qquad \cdot \mathcal{Y}(x_2^{L(0)} Y((x_1/x_2)^{L(0)} u, e^{2\pi i y} - 1)w, x_2)\\
&\quad = 2\pi i \mathrm{Res}_y \delta\left(\frac{x_1}{e^{2\pi i y} x_2}\right) \mathcal{Y}(x_2^{L(0)} Y((x_1/x_2)^{L(0)} u, e^{2\pi i y} - 1)w, x_2)\\
&\quad = 2\pi i \mathrm{Res}_y \delta\left(\frac{x_1}{e^{2\pi i y} x_2}\right) \mathcal{Y}(x_2^{L(0)} Y(e^{2\pi i y L(0)} u, e^{2\pi i y} - 1)w, x_2)\\
&\quad = 2\pi i \mathrm{Res}_y \delta\left(\frac{x_1}{e^{2\pi i y} x_2}\right) \mathcal{Y}(x_2^{L(0)} \mathcal{U}(1) Y((\mathcal{U}(1))^{-1} u, y)(\mathcal{U}(1))^{-1} w, x_2),
\end{aligned} \tag{1.12}$$

where in the last step, we have used (1.5). From (1.11) and (1.12), we obtain

$$\begin{aligned}
&[Y(x_1^{L(0)} u, x_1), \mathcal{Y}(x_2^{L(0)} w, x_2)]\\
&\quad = 2\pi i \mathrm{Res}_y \delta\left(\frac{x_1}{e^{2\pi i y} x_2}\right) \mathcal{Y}\left(\mathcal{U}(x_2) Y((\mathcal{U}(1))^{-1} u, y)(\mathcal{U}(1))^{-1} w, x_2\right).
\end{aligned} \tag{1.13}$$

Substituting $\mathcal{U}(1)u$ for u and $\mathcal{U}(1)w$ for w in (1.13) and using (1.4) and (1.10), we obtain (1.6). ∎

The operators $Y(\mathcal{U}(x_1)u, x_1)$ and $\mathcal{Y}(\mathcal{U}(x_2)w, x_2)$ are called *geometrically-modified vertex operator* and *geometrically-modified intertwining operator*, respectively. These operators play an important role in this paper. See Remark 3.5 in Section 3 for the reason why we need these modified operators.

Proposition 1.3 *Let W_1, W_2 and W_3 be V-modules and $\mathcal{Y}$ an intertwining operator of type $\binom{W_3}{W_1 W_2}$. Then for any $w_1 \in W_1$,*

$$2\pi i x \frac{d}{dx}\mathcal{Y}(\mathcal{U}(x)w_1, x) = \mathcal{Y}(\mathcal{U}(x)L(-1)w_1, x).$$

Proof. Using the $L(-1)$-derivative property for $\mathcal{Y}$, we obtain

$$\begin{aligned}
&2\pi i x \frac{d}{dx}\mathcal{Y}(\mathcal{U}(x)w_1, x)\\
&\quad = 2\pi i x \frac{d}{dx}\mathcal{Y}((2\pi i x)^{L(0)}e^{-L_+(A)}w_1, x)\\
&\quad = \mathcal{Y}((2\pi i x L(-1) + 2\pi i L(0))(2\pi i x)^{L(0)}e^{-L_+(A)}w_1, x)\\
&\quad = \mathcal{Y}(x^{L(0)}(2\pi i L(-1) + 2\pi i L(0))\mathcal{U}(1)w_1, x).
\end{aligned} \tag{1.14}$$

Using (1.5) for the vertex operator Y defining the module W_1 and the conformal element ω, we obtain

$$\mathcal{U}(1)Y(\omega, x) = Y(\mathcal{U}(e^{2\pi i x})\omega, e^{2\pi i x} - 1)\mathcal{U}(1).$$

This formula together with Lemma 1.1, the identity property and by changing the variable from x to $y = e^{2\pi i x} - 1$ gives

$$\begin{aligned}
\mathcal{U}(1)L(-1) &= \mathrm{Res}_x Y(\mathcal{U}(e^{2\pi i x})\omega, e^{2\pi i x} - 1)\mathcal{U}(1)\\
&= \mathrm{Res}_x Y(e^{2\pi i x L(0)}\mathcal{U}(1)\omega, e^{2\pi i x} - 1)\mathcal{U}(1)\\
&= \mathrm{Res}_x (2\pi i)^2 Y\left(e^{2\pi i x L(0)}\left(\omega - \frac{c}{24}\mathbf{1}\right), e^{2\pi i x} - 1\right)\mathcal{U}(1)\\
&= \mathrm{Res}_x (2\pi i)^2 Y(e^{2\pi i x L(0)}\omega, e^{2\pi i x} - 1)\mathcal{U}(1)\\
&= \mathrm{Res}_x (2\pi i)^2 e^{4\pi i x} Y(\omega, e^{2\pi i x} - 1)\mathcal{U}(1)\\
&= \mathrm{Res}_y (2\pi i + 2\pi i y) Y(\omega, y)\mathcal{U}(1)\\
&= (2\pi i L(-1) + 2\pi i L(0))\mathcal{U}(1).
\end{aligned} \tag{1.15}$$

Using (1.15), we see that the right-hand side of (1.14) is equal to

$$\mathcal{Y}(\mathcal{U}(x)L(-1)w_1, x),$$

proving the proposition. ∎

Proposition 1.4 *Let W_i, $i = 0, \dots, 5$, be V-modules and $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_3$ and $\mathcal{Y}_4$ intertwining operators of types $\binom{W_0}{W_1 W_4}$, $\binom{W_4}{W_2 W_3}$, $\binom{W_5}{W_1 W_2}$ and $\binom{W_0}{W_5 W_3}$, respectively, such that when $|z_1| > |z_2| > |z_1 - z_2| > 0$, we have the associativity*

$$\langle w_0', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle = \langle w_0', \mathcal{Y}_4(\mathcal{Y}_3(w_1, z_1)w_2, z_2)w_3\rangle$$

for $w_i \in W_i$, $i = 1, 2, 3$ and $w_0' \in W_0'$. Then when $|q_{(z_1 - z_2)}| > 1 > |q_{(z_1 - z_2)} - 1| > 0$,

$$\begin{aligned}&\langle w_0', \mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\mathcal{Y}_2(\mathcal{U}(q_{z_2})w_2, q_{z_2})w_3\rangle\\ &\quad = \langle w_0', \mathcal{Y}_4(\mathcal{U}(q_{z_2})\mathcal{Y}_3(w_1, z_1 - z_2)w_2, q_{z_2})w_3\rangle.\end{aligned}$$

for $w_i \in W_i$, $i = 1, 2, 3$ and $w_0' \in W_0'$.

Proof. By associativity, (1.4), the $L(0)$-conjugation property and (1.5),

$$\begin{aligned}&\langle w_0', \mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\mathcal{Y}_2(\mathcal{U}(q_{z_2})w_2, q_{z_2})w_3\rangle\\ &\quad = \langle w_0', \mathcal{Y}_4(\mathcal{Y}_3(\mathcal{U}(q_{z_1})w_1, q_{z_1} - q_{z_2})\mathcal{U}(q_{z_2})w_2, q_{z_2})w_3\rangle\\ &\quad = \langle w_0', \mathcal{Y}_4(\mathcal{Y}_3(\mathcal{U}(q_{z_1})w_1, q_{z_1} - q_{z_2})q_{z_2}^{L(0)}\mathcal{U}(1)w_2, q_{z_2})w_3\rangle\\ &\quad = \langle w_0', \mathcal{Y}_4(q_{z_2}^{L(0)}\mathcal{Y}_3(\mathcal{U}(q_{z_1 - z_2})w_1, q_{z_1 - z_2} - 1)\mathcal{U}(1)w_2, q_{z_2})w_3\rangle\\ &\quad = \langle w_0', \mathcal{Y}_4(q_{z_2}^{L(0)}\mathcal{U}(1)\mathcal{Y}_3(w_1, z_1 - z_2)w_2, q_{z_2})w_3\rangle\\ &\quad = \langle w_0', \mathcal{Y}_4(\mathcal{U}(q_{z_2})\mathcal{Y}_3(w_1, z_1 - z_2)w_2, q_{z_2})w_3\rangle. \qquad (1.16)\end{aligned}$$

Since the associativity used above holds in the region $|q_{z_1}| > |q_{z_2}| > |q_{z_1} - q_{z_2}| > 0$ or equivalently $|q_{(z_1 - z_2)}| > 1 > |q_{(z_1 - z_2)} - 1| > 0$, we see that (1.16) also holds in the region $|q_{(z_1 - z_2)}| > 1 > |q_{(z_1 - z_2)} - 1| > 0$. ∎

2 Identities for formal q-traces

In this section, we prove several identities on which the main results in the present paper are based. We shall use both the formal variable and the complex variable approaches. The formal variable approach is needed because the q-traces are still formal in this section and the complex variable approach is needed because we need to use analytic extensions to obtained our results.

For $m \geq 0$, let

$$P_{m+1}(x; q) = (2\pi i)^{m+1} \sum_{l > 0} \left(\frac{l^m}{m!} \frac{x^l}{1 - q^l} - \frac{(-1)^m l^m}{m!} \frac{q^l x^{-l}}{1 - q^l} \right)$$

where $(1-q^l)^{-1}$ for $l>0$ is understood as the series $\sum_{k\geq 0} q^{kl}$. For $\tau, z\in\mathbb{C}$ satisfying $|q_\tau|<|q_z|<1$, the series $P_{m+1}(q_z;q_\tau)$ for $m\geq 0$ are absolutely convergent and, for $z\in\mathbb{C}$ satisfying $|q_z|<1$, the q-coefficients of $P_{m+1}(q_z;q)$ is absolutely convergent.

As we mentioned in Section 1, we shall fix a vertex operator algebra V throughout this paper. Let W be a V-module. and let $o(u): W\to W$ be the linear map defined by $o(u)=u_{\mathrm{wt}\ u-1}$ for homogeneous $u\in V$. We have:

Proposition 2.1 *Let W_i and $\tilde{W}_i$, $i=1,\dots,n$, V-modules, and $\mathcal{Y}_i$, $i=1,\dots,n$, intertwining operators of types $\binom{\tilde{W}_{i-1}}{W_i\tilde{W}_i}$, respectively, where we use the convention $\tilde{W}_0=\tilde{W}_n$. Then for any $u\in V$, $w_i\in W_i$, $i=1,\dots,n$, we have*

$$\begin{aligned}
&\mathrm{Tr}_{\tilde{W}_n} Y(\mathcal{U}(x)u,x)\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}\\
&\quad=\sum_{i=1}^n\sum_{m\geq 0}P_{m+1}\left(\frac{x_i}{x};q\right)\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdot\\
&\qquad\qquad\cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1},x_{i-1})\mathcal{Y}_i(\mathcal{U}(x_i)u_mw_i,x_i)\cdot\\
&\qquad\qquad\cdot\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1},x_{i+1})\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}\\
&\quad\quad+\mathrm{Tr}_{\tilde{W}_n}o(\mathcal{U}(1)u)\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}
\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}
&\sum_{i=1}^n\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1},x_{i-1})\cdot\\
&\quad\cdot\mathcal{Y}_i(\mathcal{U}(x_i)u_0w_i,x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1},x_{i+1})\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}\\
&\quad=0.
\end{aligned}\tag{2.2}$$

Proof. The left-hand side of (2.1) is equal to

$$\begin{aligned}
&\sum_{i=1}^n\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1},x_{i-1})\cdot\\
&\qquad\cdot[Y(\mathcal{U}(x)u,x),\mathcal{Y}_i(\mathcal{U}(x_i)w_i,x_i)]\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1},x_{i+1})\cdot\\
&\qquad\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}\\
&\quad+\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1w_1,x_1)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)Y(\mathcal{U}(x)u,x)q^{L(0)}\\
&=\sum_{i=1}^n 2\pi i\mathrm{Res}_y\delta\left(\frac{x}{e^{2\pi iy}x_i}\right)\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdot
\end{aligned}$$

$$
\begin{aligned}
&\cdot\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1})\mathcal{Y}_i(\mathcal{U}(x_i)Y(u,y)w_i, x_i)\cdot\\
&\cdot\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1})\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}\\
&+\mathrm{Tr}_{\tilde{W}_n}Y\left(\mathcal{U}\left(\frac{x}{q}\right)u, \frac{x}{q}\right)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}\\
&= \mathrm{Res}_y\sum_{i=1}^n\sum_{m\geq 0}\frac{(2\pi i)^{m+1}y^m}{m!}\left(\left(x_i\frac{\partial}{\partial x_i}\right)^m\delta\left(\frac{x}{x_i}\right)\right)\cdot\\
&\quad\cdot\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1})\cdot\\
&\quad\cdot\mathcal{Y}_i(\mathcal{U}(x_i)Y(u,y)w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1})\cdot\\
&\quad\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}\\
&+q^{-x\frac{\partial}{\partial x}}\mathrm{Tr}_{\tilde{W}_n}Y(\mathcal{U}(x)u, x)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)},
\end{aligned}
\tag{2.3}
$$

where we have used (1.6), the $L(0)$-conjugation property for vertex operators and a property of traces ($\mathrm{Tr}AB = \mathrm{Tr}BA$).

From (2.3), we obtain

$$
\begin{aligned}
&(1-q^{-x\frac{\partial}{\partial x}})\mathrm{Tr}_{\tilde{W}_n}Y(\mathcal{U}(x)u, x)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}\\
&\quad= \mathrm{Res}_y\sum_{i=1}^n\sum_{m\geq 0}\frac{(2\pi i)^{m+1}y^m}{m!}\left(\left(x_i\frac{\partial}{\partial x_i}\right)^m\delta\left(\frac{x}{x_i}\right)\right)\cdot\\
&\qquad\cdot\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1})\cdot\\
&\qquad\cdot\mathcal{Y}_i(\mathcal{U}(x_i)Y(u,y)w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1})\cdot\\
&\qquad\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}\\
&\quad= \sum_{i=1}^n\sum_{m\geq 0}\sum_{l>0}\frac{(2\pi i)^{m+1}}{m!}\left(\left(x_i\frac{\partial}{\partial x_i}\right)^m\left(\frac{x^l}{x_i^l}\right)\right)\cdot\\
&\qquad\cdot\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1})\cdot\\
&\qquad\cdot\mathcal{Y}_i(\mathcal{U}(x_i)u_mw_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1})\cdot\\
&\qquad\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}\\
&\qquad+\sum_{i=1}^n\sum_{m\geq 0}\sum_{l>0}\frac{(2\pi i)^{m+1}}{m!}\left(\left(x_i\frac{\partial}{\partial x_i}\right)^m\left(\frac{x^{-l}}{x_i^{-l}}\right)\right)\cdot\\
&\qquad\cdot\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1})\cdot\\
&\qquad\cdot\mathcal{Y}_i(\mathcal{U}(x_i)u_mw_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1})\cdot\\
&\qquad\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}
\end{aligned}
$$

$$
\begin{aligned}
&+2\pi i\sum_{i=1}^{n}\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1},x_{i-1})\cdot\\
&\quad\cdot\mathcal{Y}_i(\mathcal{U}(x_i)u_0w_i,x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1},x_{i+1})\cdot\\
&\quad\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}.
\end{aligned}\tag{2.4}
$$

Note that the left-hand side of (2.4) has no constant term as a series in x. Thus the constant term (the last term) of the right-hand side as a series in x must be 0. So we obtain (2.2) and consequently the left-hand side of (2.4) is equal to the sum of the first two terms in the right-hand side of (2.4). Since $1-q^{-x\frac{\partial}{\partial x}}$ acting on any term independent of x is 0, we obtain

$$
\begin{aligned}
&(1-q^{-x\frac{\partial}{\partial x}})\big(\mathrm{Tr}_{\tilde{W}_n}Y(\mathcal{U}(x)u,x)\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}\\
&\qquad\qquad -\mathrm{Tr}_{\tilde{W}_n}o(\mathcal{U}(1)u)\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}\big)\\
&\quad=\sum_{i=1}^{n}\sum_{m\geq 0}\sum_{l>0}\frac{(2\pi i)^{m+1}}{m!}\left(\left(x_i\frac{\partial}{\partial x_i}\right)^m\left(\frac{x^l}{x_i^l}\right)\right)\cdot\\
&\qquad\cdot\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1},x_{i-1})\cdot\\
&\qquad\cdot\mathcal{Y}_i(\mathcal{U}(x_i)u_mw_i,x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1},x_{i+1})\cdot\\
&\qquad\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}\\
&\qquad+\sum_{i=1}^{n}\sum_{m\geq 0}\sum_{l>0}\frac{(2\pi i)^{m+1}}{m!}\left(\left(x_i\frac{\partial}{\partial x_i}\right)^m\left(\frac{x^{-l}}{x_i^{-l}}\right)\right)\cdot\\
&\qquad\cdot\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1},x_{i-1})\cdot\\
&\qquad\cdot\mathcal{Y}_i(\mathcal{U}(x_i)u_mw_i,x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1},x_{i+1})\cdot\\
&\qquad\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}\\
&\quad=\sum_{i=1}^{n}\sum_{m\geq 0}\sum_{l>0}\left(\frac{(2\pi i)^{m+1}}{m!}\left(\left(x_i\frac{\partial}{\partial x_i}\right)^m\left(\frac{x^l}{x_i^l}\right)\right)\right.\\
&\qquad\qquad\left.+\frac{(2\pi i)^{m+1}}{m!}\left(\left(x_i\frac{\partial}{\partial x_i}\right)^m\left(\frac{x^{-l}}{x_i^{-l}}\right)\right)\right)\cdot\\
&\qquad\cdot\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1},x_{i-1})\cdot\\
&\qquad\cdot\mathcal{Y}_i(\mathcal{U}(x_i)u_mw_i,x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1},x_{i+1})\cdot\\
&\qquad\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}.
\end{aligned}\tag{2.5}
$$

Note that when acting on nonzero power of x, $1-q^{-x\frac{\partial}{\partial x}}$ is invertible. But on different rings, it has different inverses. Since all the terms in our formulas as series in q are lower truncated, $(1-q^{-x\frac{\partial}{\partial x}})^{-1}$ acting on any nonzero power of x should also be lower truncated as a series in q. Thus we have

$$(1-q^{-x\frac{\partial}{\partial x}})^{-1}x^l = \begin{cases} \dfrac{-q^l x^l}{1-q^l} & l>0 \\ \dfrac{x^l}{1-q^l} & l<0. \end{cases} \tag{2.6}$$

By (2.5) and (2.6), we obtain

$$\begin{aligned}
&\mathrm{Tr}_{\tilde{W}_n} Y(\mathcal{U}(x)u,x)\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}\\
&\quad -\mathrm{Tr}_{\tilde{W}_n} o(\mathcal{U}(1)u)\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}\\
&= (1-q^{-x\frac{\partial}{\partial x}})^{-1}\sum_{i=1}^n\sum_{m\geq 0}\sum_{l>0}\left(\frac{(2\pi i)^{m+1}}{m!}\left(\left(x_i\frac{\partial}{\partial x_i}\right)^m\left(\frac{x^l}{x_i^l}\right)\right)\right.\\
&\qquad\qquad\left.+\frac{(2\pi i)^{m+1}}{m!}\left(\left(x_i\frac{\partial}{\partial x_i}\right)^m\left(\frac{x^{-l}}{x_i^{-l}}\right)\right)\right)\cdot\\
&\qquad\cdot\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1},x_{i-1})\cdot\\
&\qquad\cdot\mathcal{Y}_i(\mathcal{U}(x_i)u_mw_i,x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1},x_{i+1})\cdot\\
&\qquad\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}\\
&= \sum_{i=1}^n\sum_{m\geq 0}\sum_{l>0}\left(\frac{(2\pi i)^{m+1}}{m!}\left(\left(x_i\frac{\partial}{\partial x_i}\right)^m\left(\frac{-q^lx^l}{(1-q^l)x_i^l}\right)\right)\right.\\
&\qquad\qquad\left.+\frac{(2\pi i)^{m+1}}{m!}\left(\left(x_i\frac{\partial}{\partial x_i}\right)^m\left(\frac{x^{-l}}{(1-q^l)x_i^{-l}}\right)\right)\right)\cdot\\
&\qquad\cdot\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1},x_{i-1})\cdot\\
&\qquad\cdot\mathcal{Y}_i(\mathcal{U}(x_i)u_mw_i,x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1},x_{i+1})\cdot\\
&\qquad\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}\\
&= \sum_{i=1}^n\sum_{m\geq 0}\sum_{l>0}\left(-\frac{(-1)^m(2\pi i)^{m+1}l^m}{m!}\left(\frac{q^l\left(\frac{x_i}{x}\right)^{-l}}{1-q^l}\right)\right.\\
&\qquad\qquad\left.+\frac{(2\pi i)^{m+1}l^m}{m!}\left(\frac{\left(\frac{x_i}{x}\right)^l}{1-q^l}\right)\right)\cdot\\
&\qquad\cdot\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1},x_{i-1})\cdot\\
&\qquad\cdot\mathcal{Y}_i(\mathcal{U}(x_i)u_mw_i,x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1},x_{i+1})\cdot\\
&\qquad\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}
\end{aligned}$$

$$= \sum_{i=1}^{n} \sum_{m\geq 0} P_{m+1}\left(\frac{x_i}{x}; q\right) \mathrm{Tr}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdot$$
$$\cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \mathcal{Y}_i(\mathcal{U}(x_i)u_m w_i, x_i) \cdot$$
$$\cdot \mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1}) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) q^{L(0)}. \tag{2.7}$$

The identity (2.7) is equivalent to (2.1). ∎

The identities above are proved using purely formal variable approach. But starting from the next identity, we need to use both formal variables and complex variables.

Let

$$\begin{aligned}
\wp_1(z;\tau) &= \frac{1}{z} + \sum_{(k,l)\neq(0,0)} \left(\frac{1}{z-(k\tau+l)} + \frac{1}{k\tau+l} + \frac{z}{(k\tau+l)^2} \right), \\
\wp_2(z;\tau) &= \frac{1}{z^2} + \sum_{(k,l)\neq(0,0)} \left(\frac{1}{(z-(k\tau+l))^2} - \frac{1}{(k\tau+l)^2} \right) \\
&= -\frac{\partial}{\partial z}\wp_1(z;\tau)
\end{aligned}$$

be the Weierstrass zeta function and the Weierstrass $\wp$-function, respectively, and let $\wp_m(z;\tau)$ for $m > 2$ be the elliptic functions defined recursively by

$$\wp_{m+1}(z,\tau) = -\frac{1}{m}\frac{\partial}{\partial z}\wp_m(z;\tau). \tag{2.8}$$

These functions have the following Laurent expansions in the region $0 < |z| < \min(1, |\tau|)$: For $m \geq 1$

$$\wp_m(z;\tau) = \frac{1}{z^m} + (-1)^m \sum_{k\geq 1} \binom{2k+1}{m-1} G_{2k+2}(\tau) z^{2k+2-m} \tag{2.9}$$

where

$$G_{2k+2}(\tau) = \sum_{(m,l)\neq(0,0)} \frac{1}{(m\tau+l)^{2k+2}}$$

for $k \geq 1$ are the Eisenstein series. We also have another Eisenstein series

$$G_2(\tau) = \frac{\pi^2}{3} + \sum_{m\in\mathbb{Z}\setminus\{0\}} \sum_{l\in\mathbb{Z}} \frac{1}{(m\tau+l)^2}.$$

The Eisenstein series have the following q-expansions:

$$G_{2k+2}(\tau) = 2\zeta(2k+2) + \frac{2(2\pi i)^{2k+2}}{(2k+1)!} \sum_{l\geq 1} \frac{l^{2k+1} q_\tau^l}{1 - q_\tau^l}$$

where

$$\zeta(2k+2) = \sum_{l\geq 1} \frac{1}{l^{2k+2}}$$

and $(1-q_\tau^l)^{-1}$ is understood as $\sum_{j\geq 0} q_\tau^{jl}$. For $k \geq 1$, let

$$\tilde{G}_{2k+2}(q) = 2\zeta(2k+2) + \frac{2(2\pi i)^{2k+2}}{(2k+1)!} \sum_{l\geq 1} \frac{l^{2k+1} q^l}{1 - q^l}$$

where $(1-q^l)^{-1}$ is the formal power series $\sum_{j\geq 0} q^{jl}$. Then $G_{2k+2}(\tau) = \tilde{G}_{2k+2}(q_\tau)$.

The functions $\wp_m(z;\tau)$ for $m \geq 1$ also have the following q-expansions: For $z \in \mathbb{C}$ satisfying $0 < |z| < \min(1, |\tau|)$ and $|q_\tau| < |q_z| < \frac{1}{|q_\tau|}$ and for $m \geq 1$,

$$\wp_m(z;\tau) = (-1)^m \left(P_m(q_z; q_\tau) - \frac{\partial^{m-1}}{\partial z^{m-1}} (\tilde{G}_2(q_\tau) z - \pi i) \right). \tag{2.10}$$

For $m \geq 1$, let

$$\tilde{\wp}_m(y;q) = \frac{1}{y^m} + (-1)^m \sum_{k\geq 1} \binom{2k+1}{m-1} \tilde{G}_{2k+2}(q) y^{2k+2-m} \in y^{-m}\mathbb{C}[[y,q]]. \tag{2.11}$$

Then for $z \in \mathbb{C}$ satisfying $0 < |z| < 1$ and $m \geq 1$,

$$\tilde{\wp}_m(z;q) = (-1)^m \left(P_m(q_z; q) - \frac{\partial^{m-1}}{\partial z^{m-1}} (\tilde{G}_2(q) z - \pi i) \right). \tag{2.12}$$

We also have

$$\tilde{\wp}_m(z; q_\tau) = \wp_m(z;\tau)$$

when $m \geq 1$ and $0 < |z| < \min(1, |\tau|)$ and $|q_\tau| < |q_z| < \frac{1}{|q_\tau|}$. For more details on the Weierstrass zeta function, the Weierstrass $\wp$-function and the Eisenstein series, see [La] and [K].

Now we assume that for any W_i and $\tilde{W}_i$, $\mathcal{Y}_i$, $i = 1, \dots, n$, as in Proposition 2.1, any $w_i \in W_i$, $i = 1, \dots, n$, $\tilde{w}_n \in \tilde{W}_n$ and $\tilde{w}'_n \in \tilde{W}'_n$,

$$\langle \tilde{w}'_n, \mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_n(w_n, z_n)\, \tilde{w}_n \rangle$$

is absolutely convergent when $|z_1| > \cdots > |z_n| > 0$. We also assume that commutativity and associativity for intertwining operators hold. (For more details on convergence, commutativity and associativity for intertwining operators, see [H1], [H2], [H4] and [H7].)

Theorem 2.2 *Let W_i and $\tilde{W}_i$, $\mathcal{Y}_i$, $i = 1, \dots, n$, be as in Proposition 2.1. Then for any $u \in V$, $w_i \in W_i$, $i = 1, \dots, n$ and any integer j satisfying $1 \le j \le n$, we have*

$$\begin{aligned}
&\mathrm{Tr}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{j-1}(\mathcal{U}(x_{j-1})w_{j-1}, x_{j-1}) \cdot \mathcal{Y}_j(\mathcal{U}(x_j)Y(u,y)w_j, x_j)\\
&\qquad\qquad \cdot \mathcal{Y}_{j+1}(\mathcal{U}(x_{j+1})w_{j+1}, x_{j+1}) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) q^{L(0)}\\
&\quad = \sum_{m \ge 0} (-1)^{m+1} \left(\tilde{\wp}_{m+1}(-y; q) + \frac{\partial^m}{\partial y^m}(\tilde{G}_2(q)y + \pi i) \right) \cdot\\
&\qquad\qquad \cdot \mathrm{Tr}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{j-1}(\mathcal{U}(x_{j-1})w_{j-1}, x_{j-1}) \cdot\\
&\qquad\qquad \cdot \mathcal{Y}_j(\mathcal{U}(x_j)u_m w_j, x_j) \mathcal{Y}_{j+1}(\mathcal{U}(x_{j+1})w_{j+1}, x_{j+1}) \cdot\\
&\qquad\qquad \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) q^{L(0)}\\
&\qquad + \sum_{i \ne j} \sum_{m \ge 0} P_{m+1}\left(\frac{x_i}{x_j e^{2\pi i y}}; q \right) \cdot\\
&\qquad\qquad \cdot \mathrm{Tr}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1}) \cdot\\
&\qquad\qquad \cdot \mathcal{Y}_i(\mathcal{U}(x_i)u_m w_i, x_i) \mathcal{Y}_{i+1}(w_{i+1}, x_{i+1}) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) q^{L(0)}\\
&\qquad + \mathrm{Tr}_{\tilde{W}_n} o(\mathcal{U}(1)u) \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1) \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n) q^{L(0)}.
\end{aligned} \tag{2.13}$$

Proof. We first prove the case $j = 1$. Note that by (1.5),

$$\begin{aligned}
\mathcal{U}(x_1)Y(u, y) &= x_1^{L(0)} \mathcal{U}(1) Y(u, y) (\mathcal{U}(1))^{-1} \mathcal{U}(1)\\
&= x_1^{L(0)} Y(\mathcal{U}(e^{2\pi i y})u, e^{2\pi i y} - 1) \mathcal{U}(1)\\
&= Y(x_1^{L(0)} \mathcal{U}(e^{2\pi i y})u, x_1 e^{2\pi i y} - 1) x_1^{L(0)} \mathcal{U}(1)\\
&= Y(\mathcal{U}(x_1 e^{2\pi i y})u, x_1(e^{2\pi i y} - 1)) \mathcal{U}(x_1)
\end{aligned} \tag{2.14}$$

(recall (1.4)).

Using (2.14), we see that in the case $j = 1$, the left-hand side of (2.13) is equal to

$$\begin{aligned}
&\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)Y(u,y)w_1,x_1)\mathcal{Y}_2(\mathcal{U}(x_2)w_2,x_2)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}\\
&\quad=\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(Y(\mathcal{U}(x_1e^{2\pi iy})u,x_1(e^{2\pi iy}-1))\mathcal{U}(x_1)w_1,x_1)\cdot\\
&\qquad\qquad\cdot\mathcal{Y}_2(\mathcal{U}(x_2)w_2,x_2)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}.
\end{aligned} \tag{2.15}$$

By the properties of intertwining operators, we know that the following associativity holds: For any $\tilde{w}_1\in\tilde{W}_1$, $\tilde{w}_n'\in\tilde{W}_n'$,

$$\begin{aligned}
&\langle\tilde{w}_n',\mathcal{Y}_1(Y(\mathcal{U}(z_1q_z)u,z_1(q_z-1))\mathcal{U}(z_1)w_1,z_1)\tilde{w}_1\rangle\\
&\quad=\langle\tilde{w}_n',Y(\mathcal{U}(z_1q_z)u,z_1q_z)\mathcal{Y}_1(\mathcal{U}(z_1)w_1,z_1)\tilde{w}_1\rangle
\end{aligned}$$

for any complex numbers z and z_1 satisfying $|z_1q_z|>|z_1|>|z_1(q_z-1)|>0$ (or equivalently, $|q_z|>1>|q_z-1|>0$ and $z_1\neq 0$), where we choose the values of z_1^n for $n\in\mathbb{C}$ to be $e^{n\log z_1}$, $0\leq\arg z_1<2\pi$ (and we shall choose this value throughout this paper). Thus for any complex numbers z and z_1 satisfying $|q_z|>1>|q_z-1|>0$ and $z_1\neq 0$ and for the values $e^{n\log z_1}$ of z_1^n for $n\in\mathbb{C}$,

$$\begin{aligned}
&\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(Y(\mathcal{U}(z_1q_z)u,z_1(q_z-1))\mathcal{U}(z_1)w_1,z_1)\cdot\\
&\qquad\qquad\cdot\mathcal{Y}_2(\mathcal{U}(x_2)w_2,x_2)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}\\
&\quad=\mathrm{Tr}_{\tilde{W}_n}Y(\mathcal{U}(z_1q_z)u,z_1q_z)\mathcal{Y}_1(\mathcal{U}(z_1)w_1,z_1)\cdot\\
&\qquad\qquad\cdot\mathcal{Y}_2(\mathcal{U}(x_2)w_2,x_2)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}
\end{aligned} \tag{2.16}$$

as formal series in $x_2,\dots,x_n$ and q. Since (2.16) holds for any complex number z satisfying $|q_z|>1>|q_z-1|>0$ and any nonzero complex number z_1, the expansions of both sides of (2.16) as Laurent series in z_1 are also equal for any complex number z satisfying $|q_z|>1>|q_z-1|>0$. Thus we can replace z_1 by the formal variable x_1 in (2.16) so that for any complex number z satisfying $|q_z|>1>|q_z-1|>0$,

$$\begin{aligned}
&\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(Y(\mathcal{U}(x_1q_z)u,x_1(q_z-1))\mathcal{U}(x_1)w_1,x_1)\cdot\\
&\qquad\qquad\cdot\mathcal{Y}_2(\mathcal{U}(x_2)w_2,x_2)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}\\
&\quad=\mathrm{Tr}_{\tilde{W}_n}Y(\mathcal{U}(x_1q_z)u,x_1q_z)\mathcal{Y}_1(\mathcal{U}(x_1)w_1,x_1)\cdot\\
&\qquad\qquad\cdot\mathcal{Y}_2(\mathcal{U}(x_2)w_2,x_2)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n,x_n)q^{L(0)}
\end{aligned} \tag{2.17}$$

as formal series in $x_1, \dots, x_n$ and q. Note that the left- and right-hand sides of (2.17) are well-defined formal series in $x_1, \dots, x_n$ and q for $z \in \mathbb{C}$ satisfying $1 > |q_z - 1| > 0$ and $|q_z| > 1$, respectively.

By (1.4) and (2.1), for any complex number z, we have

$$\begin{aligned}
&\mathrm{Tr}_{\tilde{W}_n} Y(\mathcal{U}(x_1q_z)u, x)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdot \\
&\qquad\qquad \cdot\mathcal{Y}_2(\mathcal{U}(x_2)w_2, x_2)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad= \mathrm{Tr}_{\tilde{W}_n} Y(\mathcal{U}(x)(\mathcal{U}(x))^{-1}\mathcal{U}(x_1q_z)u, x)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdot \\
&\qquad\qquad \cdot\mathcal{Y}_2(\mathcal{U}(x_2)w_2, x_2)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad= \sum_{i=1}^{n}\sum_{m\geq 0} P_{m+1}\left(\frac{x_i}{x}; q\right)\cdot \\
&\qquad\qquad \cdot\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1})\cdot \\
&\qquad\qquad \cdot\mathcal{Y}_i(\mathcal{U}(x_i)(2\pi i)^{L(0)}((\mathcal{U}(x))^{-1}\mathcal{U}(x_1q_z)u)_m w_i, x_i)\cdot \\
&\qquad\qquad \cdot\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1})\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad\quad +\mathrm{Tr}_{\tilde{W}_n} o(\mathcal{U}(1)(\mathcal{U}(x))^{-1}\mathcal{U}(x_1q_z)u)\cdot \\
&\qquad\qquad \cdot\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}
\end{aligned} \tag{2.18}$$

as formal series in $x, x_1, \dots, x_n$ and q.

Now for any complex number z satisfying $|q_z| > 1$ and $0 < |z| < 1$, substituting x_1q_z for x in (2.18) and using (2.12), we obtain

$$\begin{aligned}
&\mathrm{Tr}_{\tilde{W}_n} Y(\mathcal{U}(x_1q_z)u, x_1q_z)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdot \\
&\qquad\qquad \cdot\mathcal{Y}_2(\mathcal{U}(x_2)w_2, x_2)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad= \sum_{m\geq 0} P_{m+1}\left(\frac{1}{q_z}; q\right)\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)u_m w_1, x_1)\cdot \\
&\qquad\qquad \cdot\mathcal{Y}_2(\mathcal{U}(x_2)w_2, x_2)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad\quad +\sum_{i=2}^{n}\sum_{m\geq 0} P_{m+1}\left(\frac{x_i}{x_1q_z}; q\right)\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdot \\
&\qquad\qquad \cdots\mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1})\mathcal{Y}_i(\mathcal{U}(x_i)u_m w_i, x_i)\cdot \\
&\qquad\qquad \cdot\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1})\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad\quad +\mathrm{Tr}_{\tilde{W}_n} o(\mathcal{U}(1)u)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots\mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad= \sum_{m\geq 0}(-1)^{m+1}\left(\tilde{\wp}_{m+1}(-z; q) + \frac{\partial^m}{\partial z^m}(\tilde{G}_2(q)z + \pi i)\right)\cdot
\end{aligned}$$

$$
\begin{aligned}
&\cdot \mathrm{Tr}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)u_m w_1, x_1)\cdot \\
&\cdot \mathcal{Y}_2(\mathcal{U}(x_2)w_2, x_2)\cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&+\sum_{i=2}^{n}\sum_{m\geq 0} P_{m+1}\left(\frac{x_i}{x_1 q_z}; q\right)\cdot \\
&\cdot \mathrm{Tr}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1})\cdot \\
&\cdot \mathcal{Y}_i(\mathcal{U}(x_i)u_m w_i, x_i)\cdot \\
&\cdot \mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1})\cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&+\mathrm{Tr}_{\tilde{W}_n} o(\mathcal{U}(1)u)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \qquad (2.19)
\end{aligned}
$$

as a formal series in $x_1, \dots, x_n$ and q.

Combining (2.15), (2.17) and (2.19), we see that for any complex number z satisfying $|q_z| > 1$ and $0 < |z| < 1$,

$$
\begin{aligned}
&\mathrm{Tr}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)Y(u,z)w_1, x_1)\mathcal{Y}_2(\mathcal{U}(x_2)w_2, x_2)\cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\quad = \sum_{m\geq 0}(-1)^{m+1}\left(\tilde{\wp}_{m+1}(-z;q) + \frac{\partial^m}{\partial z^m}(\tilde{G}_2(q)z + \pi i)\right)\cdot \\
&\qquad\qquad \cdot \mathrm{Tr}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots \mathcal{Y}_{j-1}(\mathcal{U}(x_{j-1})w_{j-1}, x_{j-1})\cdot \\
&\qquad\qquad \cdot \mathcal{Y}_j(\mathcal{U}(x_j)u_m w_j, x_j)\mathcal{Y}_{j+1}(\mathcal{U}(x_{j+1})w_{j+1}, x_{j+1})\cdot \\
&\qquad\qquad \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\qquad +\sum_{i=2}^{n}\sum_{m\geq 0} P_{m+1}\left(\frac{x_i}{x_1 q_z}; q\right)\cdot \\
&\qquad\qquad \cdot \mathrm{Tr}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots \mathcal{Y}_{i-1}(\mathcal{U}(x_{i-1})w_{i-1}, x_{i-1})\cdot \\
&\qquad\qquad \cdot \mathcal{Y}_i(\mathcal{U}(x_i)u_m w_i, x_i)\mathcal{Y}_{i+1}(\mathcal{U}(x_{i+1})w_{i+1}, x_{i+1})\cdot \\
&\qquad\qquad \cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)} \\
&\qquad +\mathrm{Tr}_{\tilde{W}_n} o(\mathcal{U}(1)u)\mathcal{Y}_1(\mathcal{U}(x_1)w_1, x_1)\cdots \mathcal{Y}_n(\mathcal{U}(x_n)w_n, x_n)q^{L(0)}. \qquad (2.20)
\end{aligned}
$$

as a formal series in $x_1, \dots, x_n$ and q. Since both sides of (2.20) are formal series in $x_1, \dots, x_n$ and q whose coefficients are convergent Laurent series in z near $z = 0$, the truth of (2.20) in the region given by $|q_z| > 1$ and $0 < |z| < 1$ implies that (2.20) holds near $z = 0$. Thus (2.13) holds in the case of $j = 1$.

We now prove the theorem in the general case. We use induction on j. Assume that for $j = k < n$, (2.13) holds. We want to prove that (2.13) holds when $j = k + 1$. Fix $z_i^0 \in \mathbb{C}$, $i = 1, \dots, n$, satisfying $|z_1^0| > \cdots > |z_n^0| > 0$, let

γ_k be the path in

$$M^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq 0,\ z_i \neq z_l, i \neq l\}$$

from $(z_1^0, \dots, z_k^0, z_{k+1}^0, \dots, z_n^0)$ to $(z_1^0, \dots, z_{k+1}^0, z_k^0, \dots, z_n^0)$ given by

$$\gamma_k(t) = (z_1^0, \dots, e^{(1-t)\log z_k^0 + t\log z_{k+1}^0}, e^{t\log z_k^0 + (1-t)\log z_{k+1}^0}, \dots, z_n^0).$$

(Actually just as the choices of $z_i^0 \in \mathbb{C}$, $i = 1, \dots, n$, the choice of γ_k can also be arbitrary. Here to be more specific, we explicitly give one particular choice.) Given any branch of a multivalued analytic function of $z_1, \dots, z_n$ in the simply connected region $|z_1| > \cdots > |z_n| > 0$, $0 \leq \arg z_i < 2\pi$, $i = 1, \dots, n$, we have a unique analytic extension of this branch to the region

$$\begin{aligned} |z_1| > \cdots > |z_{k-1}| > |z_{k+1}| > |z_k| > |z_{k+2}| > \cdots > |z_n| > 0,\\ 0 \leq \arg z_i < 2\pi, \quad i = 1, \dots, n, \end{aligned} \tag{2.21}$$

determined by the path γ_k. By commutativity for intertwining operators, there exist V-modules $\hat{W}_k$ and intertwining operators $\hat{\mathcal{Y}}_k$ and $\hat{\mathcal{Y}}_{k+1}$ of types $\binom{\hat{W}_k}{W_k \tilde{W}_{k+1}}$ and $\binom{\tilde{W}_{k-1}}{W_{k+1} \hat{W}_k}$, respectively, such that for any $w_i \in W_i$, $i = 1, \dots, n$, $\tilde{w}_n \in \tilde{W}_n$ and $\tilde{w}'_n \in \tilde{W}'_n$, the branch of the analytic extension along γ_k of

$$\langle \tilde{w}'_n, \mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_k(w_k, z_k)\mathcal{Y}_{k+1}(w_{k+1}, z_{k+1}) \cdots \mathcal{Y}_n(w_n, z_n)\tilde{w}_n\rangle$$

to the region (2.21), is equal to

$$\begin{aligned} \langle \tilde{w}'_n, \mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_{k-1}(w_{k-1}, z_{k-1})\hat{\mathcal{Y}}_{k+1}(w_{k+1}, z_{k+1})\cdot\\ \cdot\hat{\mathcal{Y}}_k(w_k, z_k)\mathcal{Y}_{k+2}(w_{k+2}, z_{k+2}) \cdots \mathcal{Y}_n(w_n, z_n)\tilde{w}_n\rangle \end{aligned}$$

in the same region.

By induction assumption, (2.13) holds when $j = k$. So (2.13) holds when $w_j, w_{j+1}, x_j, x_{j+1}$, $\mathcal{Y}_j$ and $\mathcal{Y}_{j+1}$ in (2.13) are replaced by $w_{k+1}, w_k, x_{k+1}, x_k$, $\hat{\mathcal{Y}}_{k+1}$ and $\hat{\mathcal{Y}}_k$, respectively. Then when we substitute $z_1, \dots, z_n \in \mathbb{C}$ in the region (2.21) for $x_1, \dots, x_n$, respectively, (2.13) becomes an identity for formal Laurent series in q whose coefficients are single-valued analytic branches in the same region. By the commutativity stated above, we know that all these coefficients can be analytically extended back to the region $|z_1| > \cdots > |z_n| > 0$, $0 \leq \arg z_i < 2\pi$, $i = 1, \dots, n$, along the path γ_k^{-1} to the single-valued analytic branches which are nothing but the coefficients of the formal Laurent series in q obtained by substituting in (2.13) (with $j = k+1$) $z_1, \cdots, z_n \in \mathbb{C}$ for $x_1, \ldots, x_n$, respectively, when $z_1, \cdots, z_n$ are in the region $|z_1| > \cdots > |z_n| > 0$, $0 \leq \arg z_i < 2\pi$, $i = 1, \ldots, n$. Since these analytic extensions are unique, we see that (2.13) with $j = k+1$ holds. ∎

Remark 2.3 In the proof of the theorem above, the use of the complex variable approach is essential. If we use formal variable approach, in the case of $j = 1$, the proof seems to be more complicated and in the case of $j > 1$, certain identities can still be obtained but they would involve $P(\frac{x_i q}{x_j e^{2\pi i y}}; q)$ (note the extra q in the first argument) for $i < j$. (After an early version of the present paper was finished and circulated, Milas informed the author that he also obtained independently some identities similar to (2.1), (2.2) and the $j = 1$ case of (2.13).) In fact, eventually we will have multivalued analytic functions corresponding to flat sections of certain vector bundles with flat connections over the moduli spaces of genus-one Riemann surfaces with punctures and standard local coordinates vanishing at the punctures. So if the convergence is proved, those identities with extra q's and the identities in the theorem above are related by analytic extensions. But if we have only the identities involving extra q's, we would not even be able to prove the convergence of the series and therefore cannot construct the multivalued analytic functions.

Let $\mathbb{G}_{|z_1|>\cdots>|z_n|>0}$ be the space of all multivalued analytic functions in $z_1, \ldots, z_n$ defined on the region $|z_1| > \cdots > |z_n| > 0$ with preferred branches in the simply-connected region $|z_1| > \cdots > |z_n| > 0$, $0 \leq \arg z_i < 2\pi$, $i = 1, \ldots, n$. For any $f(z_1, \ldots, z_n) \in \mathbb{G}_{|z_1|>\cdots>|z_n|>0}$, we have a multivalued analytic functions $f(q_{z_1}, \ldots, q_{z_n})$ in $z_1, \ldots, z_n$ defined on the region $|q_{z_1}| > \cdots > |q_{z_n}| > 0$. All such functions form a space $\mathbb{G}_{|q_{z_1}|>\cdots>|q_{z_n}|>0}$. We see that

$$\mathrm{Tr}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1}) \cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n}) q^{L(0)}$$

and

$$\tilde{G}_{2k+2}(q) \mathrm{Tr}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1}) \cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n}) q^{L(0)}$$

for $k \geq 1$ are elements of $\mathbb{G}_{|q_{z_1}|>\cdots>|q_{z_n}|>0}((q))$.

Theorem 2.4 *Let W_i and $\tilde{W}_i$, $\mathcal{Y}_i$, $i = 1, \dots, n$, be as in Proposition 2.1. Then for any $u \in V$, $w_i \in W_i$, $i = 1, \dots, n$, any integer j satisfying $1 \le j \le n$ and any $l \in \mathbb{Z}_+$, in $\mathbb{G}_{|q_{z_1}|>\cdots>|q_{z_n}|>0}((q))$, we have*

$$\begin{aligned}
&\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdots\mathcal{Y}_{j-1}(\mathcal{U}(q_{z_{j-1}})w_{j-1}, q_{z_{j-1}})\mathcal{Y}_j(\mathcal{U}(q_{z_j})u_{-l}w_j, q_{z_j})\cdot\\
&\qquad\cdot\mathcal{Y}_{j+1}(\mathcal{U}(q_{z_{j+1}})w_{j+1}, q_{z_{j+1}})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}\\
&\quad=\sum_{k\ge 1}(-1)^{l+1}\binom{2k+1}{l-1}\tilde{G}_{2k+2}(q)\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdot\\
&\qquad\cdots\mathcal{Y}_{j-1}(\mathcal{U}(q_{z_{j-1}})w_{j-1}, q_{z_{j-1}})\mathcal{Y}_j(\mathcal{U}(q_{z_j})u_{2k+2-l}w_j, q_{z_j})\cdot\\
&\qquad\cdot\mathcal{Y}_{j+1}(\mathcal{U}(q_{z_{j+1}})w_{j+1}, q_{z_{j+1}})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}\\
&\qquad+\sum_{i\ne j}\sum_{m\ge 0}(-1)^{m+l}\binom{-m-1}{l-1}\tilde{\wp}_{m+l}(z_i-z_j;q)\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdot\\
&\qquad\cdots\mathcal{Y}_{i-1}(\mathcal{U}(q_{z_{i-1}})w_{i-1}, q_{z_{i-1}})\mathcal{Y}_i(\mathcal{U}(q_{z_i})u_m w_i, q_{z_i})\cdot\\
&\qquad\cdot\mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1}, q_{z_{i+1}})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}\\
&\qquad+\delta_{l,1}\tilde{G}_2(q)\sum_{i=1}^{n}\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdots\mathcal{Y}_{i-1}(\mathcal{U}(q_{z_{i-1}})w_{i-1}, q_{z_{i-1}})\cdot\\
&\qquad\cdot\mathcal{Y}_i(\mathcal{U}(q_{z_i})(u_1+u_0z_i)w_i, q_{z_i})\mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1}, q_{z_{i+1}})\cdot\\
&\qquad\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}\\
&\qquad+\delta_{l,1}\mathrm{Tr}_{\tilde{W}_n}o(\mathcal{U}(1)u)\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}. \qquad (2.22)
\end{aligned}$$

Proof. Let $z_1, \dots, z_n \in \mathbb{C}$ satisfying $|q_{z_1}| > \cdots > |q_{z_n}| > 0$. Then from (2.14) and the domain of convergence of the q-coefficients of $P_m(\frac{q_{z_i}}{q_{z_j}q_z}, q)$ for $m \ge 1$, we see that for z in a sufficiently small neighborhood of 0, we can substitute $z, q_{z_1}, \dots, q_{z_n}$ for $y, x_1, \dots, x_n$ in (2.13) so that both sides of (2.13) become formal series in q whose coefficients are analytic functions of z. Using (2.12) after these substitutions, we obtain

$$\begin{aligned}
&\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdots\mathcal{Y}_{j-1}(\mathcal{U}(q_{z_{j-1}})w_{j-1}, q_{z_{j-1}})\mathcal{Y}_j(\mathcal{U}(q_{z_j})Y(u,z)w_j, q_{z_j})\cdot\\
&\qquad\cdot\mathcal{Y}_{j+1}(\mathcal{U}(q_{z_{j+1}})w_{j+1}, q_{z_{j+1}})\cdots\mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}\\
&\quad=\sum_{m\ge 0}(-1)^{m+1}\left(\tilde{\wp}_{m+1}(-z;q)+\frac{\partial^m}{\partial z^m}(\tilde{G}_2(q)z+\pi i)\right)\cdot\\
&\qquad\cdot\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdots\mathcal{Y}_{j-1}(\mathcal{U}(q_{z_{j-1}})w_{j-1}, q_{z_{j-1}})\cdot\\
&\qquad\cdot\mathcal{Y}_j(\mathcal{U}(q_{z_j})u_m w_j, q_{z_j})\mathcal{Y}_{j+1}(\mathcal{U}(q_{z_{j+1}})w_{j+1}, q_{z_{j+1}})\cdot
\end{aligned}$$

$$
\begin{aligned}
&\cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}\\
&+\sum_{i\neq j}\sum_{m\geq 0}(-1)^{m+1}\biggl(\tilde{\wp}_{m+1}(z_i-z_j-z;q)\\
&\qquad\qquad +(-1)^{m+1}\frac{\partial^m}{\partial z_i^m}(\tilde{G}_2(q)(z_i-z_j-z)-\pi i)\biggr)\cdot\\
&\quad\cdot\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdots \mathcal{Y}_{i-1}(\mathcal{U}(q_{z_{i-1}})w_{i-1}, q_{z_{i-1}})\cdot\\
&\quad\cdot\mathcal{Y}_i(\mathcal{U}(q_{z_i})u_m w_i, q_{z_i})\mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1}, q_{z_{i+1}})\cdot\\
&\quad\cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}\\
&+\mathrm{Tr}_{\tilde{W}_n}o(\mathcal{U}(1)u)\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}. \qquad (2.23)
\end{aligned}
$$

For $l>2$, taking the z^{l-1}-coefficients of both sides of (2.23) and using (2.8), (2.11) and the Taylor expansion formulas for $\tilde{\wp}_{m+1}(z_i-z_j-z;q)$ as functions of z, we obtain (2.22) in this case. For $l=2$, the right-hand side contains an extra term

$$
\begin{aligned}
&-\tilde{G}_2(q)\sum_{i=1}^{n}\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdots \mathcal{Y}_{i-1}(\mathcal{U}(q_{z_{i-1}})w_{i-1}, q_{z_{i-1}})\cdot\\
&\qquad\cdot\mathcal{Y}_i(\mathcal{U}(q_{z_i})u_0 w_i, q_{z_i})\mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1}, q_{z_{i+1}})\cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}
\end{aligned}
$$

which is equal to 0 by (2.2). For $l=1$, the right-hand side contains the two extra terms in the right-hand side of (2.22) and also contains terms

$$
\begin{aligned}
&-\pi\sum_{i=1}^{n}\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdots \mathcal{Y}_{i-1}(\mathcal{U}(q_{z_{i-1}})w_{i-1}, q_{z_{i-1}})\cdot\\
&\qquad\cdot\mathcal{Y}_i(\mathcal{U}(q_{z_i})u_0 w_i, q_{z_i})\mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1}, q_{z_{i+1}})\cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}
\end{aligned}
$$

and

$$
\begin{aligned}
&-z_j\sum_{i=1}^{n}\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1})\cdots \mathcal{Y}_{i-1}(\mathcal{U}(q_{z_{i-1}})w_{i-1}, q_{z_{i-1}})\cdot\\
&\qquad\cdot\mathcal{Y}_i(\mathcal{U}(q_{z_i})u_0 w_i, q_{z_i})\mathcal{Y}_{i+1}(\mathcal{U}(q_{z_{i+1}})w_{i+1}, q_{z_{i+1}})\cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)}
\end{aligned}
$$

which are again equal to 0 by (2.2).

Since both sides of (2.22) are in $\mathbb{G}_{|q_{z_1}|>\cdots>|q_{z_n}|>0}((q))$, (2.22) hold in this space. ∎

3 Differential equations

We shall fix a positive integer n in this section. Let

$$R = \mathbb{C}[\tilde{G}_4(q), \tilde{G}_6(q), \tilde{\wp}_2(z_i - z_j; q), \tilde{\wp}_3(z_i - z_j; q)]_{i,j=1,\dots,n,\, i<j},$$

that is, the commutative associative algebra over $\mathbb{C}$ generated by the series $\tilde{G}_4(q)$, $\tilde{G}_6(q)$, $\tilde{\wp}_2(z_i - z_j; q)$ and $\tilde{\wp}_3(z_i - z_j; q)$ for $i, j = 1, \dots, n$ satisfying $i < j$. (Note that by definition, when $n = 1$, R is the commutative associative algebra over $\mathbb{C}$ generated by the series $\tilde{G}_4(q)$ and $\tilde{G}_6(q)$.) By definition, R is finitely generated over the field $\mathbb{C}$ and thus is Noetherian. We have:

Lemma 3.1 *For $k \geq 2$, $m \geq 2$, $i, j = 1, \dots, n$, $i \neq j$,*

$$\tilde{G}_{2k}(q), \tilde{\wp}_m(z_i - z_j; q) \in R.$$

Proof. It is known (see, for example, [K]) that any modular form can be written as a polynomial of $G_4(\tau)$ and $G_6(\tau)$. In particular, $G_{2k}(\tau)$ for $k \geq 2$ can be written as polynomials of $G_4(\tau)$ and $G_6(\tau)$. Since $\tilde{G}_{2k}(q)$ for $k \geq 2$ are nothing but the q-expansions of $G_{2k}(\tau)$, they can be written as polynomials of $\tilde{G}_4(q)$ and $\tilde{G}_6(q)$ and thus are elements of R. Also using the relation

$$\frac{\partial}{\partial z}\wp_2(z;\tau)^2 = 4\wp_2(z;\tau)^3 - 60G_4(\tau)\wp_2(z;\tau) - 140G_6(\tau)$$

(see, for example, [K] or [La]) and induction, it is easy to see that $\wp_m(z, \tau)$ for $m \geq 2$ can be written as polynomials of $G_4(\tau)$, $G_6(\tau)$, $\wp_2(z;\tau)$ and

$$\frac{\partial}{\partial z}\wp_2(z;\tau) = -2\wp_3(z;\tau).$$

Since $\tilde{G}_4(q)$, $\tilde{G}_6(q)$ and $\tilde{\wp}_m(z;q)$ for $m \geq 2$ are nothing but the q-expansion of $G_4(\tau)$, $G_6(\tau)$ and $\wp_m(z;\tau)$, respectively, $\tilde{\wp}_m(z;q)$ for $m \geq 2$ can be written as polynomials of $\tilde{G}_4(q)$, $\tilde{G}_6(q)$, $\tilde{\wp}_2(z;q)$ and $\tilde{\wp}_3(z;q)$. Consequently for $m \geq 2$ and $i, j = 1, \dots, n$, $i < j$, $\tilde{\wp}_m(z_i - z_j; q)$ can be written as polynomials of $\tilde{G}_4(q)$, $\tilde{G}_6(q)$, $\tilde{\wp}_2(z_i - z_j; q)$ and $\tilde{\wp}_3(z_i - z_j; q)$ and thus are elements of R. Since $\tilde{\wp}_m(z;q)$ for $m \geq 2$ are either even or odd in z, $\tilde{\wp}_m(z_i - z_j; q)$ for $m \geq 2$, $i, j = 1, \dots, n$, $i > j$ are also in R. ∎

As in the preceding two sections, we fix a vertex operator algebra V of central charge c in this section. We say that a V-module W is C_2-*cofinite* or satisfies the C_2-*cofiniteness condition* if $\dim W/C_2(W) < \infty$, where $C_2(W)$ is the subspace of W spanned by elements of the form $u_{-2}w$ for $u \in V$ and $w \in W$. In this section, we assume that all V-modules are $\mathbb{R}$-graded and C_2-cofinite. We also assume that for any W_i and $\tilde{W}_i$, $\mathcal{Y}_i$, $i = 1, \dots, n$, as in Proposition 2.1, any $w_i \in W_i$ $(i = 1, \dots, n)$, $\tilde{w}_n \in \tilde{W}_n$ and $\tilde{w}_n' \in \tilde{W}_n'$,

$$\langle \tilde{w}_n', \mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_n(w_n, z_1)\, \tilde{w}_n \rangle_{\tilde{W}_n}$$

is absolutely convergent when $|z_1| > \cdots > |z_n| > 0$. We also assume that the commutativity and associativity for intertwining operators hold. Note that all the conditions on the vertex operator algebra V assumed in the preceding two sections hold. So we can use the results of these sections.

Let $T = R \otimes W_1 \otimes \cdots \otimes W_n$. Then T and also $\mathbb{G}_{|q_{z_1}| > \cdots > |q_{z_n}| > 0}((q))$ are R-modules.

Let J be the R-submodule of T generated by elements of the form

$$\begin{aligned}
&\mathcal{A}_j(u; w_1, \dots, w_n)\\
&\quad = 1 \otimes w_1 \otimes \cdots \otimes w_{j-1} \otimes u_{-2}w_j \otimes w_{j+1} \otimes \cdots \otimes w_n\\
&\quad + \sum_{k \geq 1} (2k+1) \tilde{G}_{2k+2}(q) \otimes w_1 \otimes \cdots \otimes w_{j-1} \otimes u_{2k}w_j \otimes w_{j+1} \otimes \cdots \otimes w_n\\
&\quad + \sum_{i \neq j} \sum_{m \geq 0} (-1)^m (m+1)\\
&\qquad\quad \tilde{\wp}_{m+2}(z_i - z_j; q) \otimes w_1 \otimes \cdots \otimes w_{i-1} \otimes u_m w_i \otimes w_{i+1} \otimes \cdots \otimes w_n
\end{aligned}$$

for $j = 1, \dots, n$ and $w_i \in W_i$, $i = 1, \dots, n$.

The gradings (by conformal weights) on W_i for $i = 1, \dots, n$ induce a grading on T and this grading on T induces a grading on J. We shall use $T_{(r)}$ and $J_{(r)}$ to denote the homogeneous subspaces of T and J, respectively, of conformal weight $r \in \mathbb{R}$. Let $F_r(T) = \coprod_{s \leq r} T_{(s)}$ and $F_r(J) = \coprod_{s \leq r} J_{(s)}$. We also introduce another grading on R and T. We say that the elements $\tilde{G}_{2k}(q)$ for any $k \geq 1$ have *modular weight* $2k$ and the element $\tilde{\wp}_m(z_i - z_j; q)$ for any $m \geq 2$ and $i < j$ have *modular weight* m. These modular weights of the generators of R give a grading which is called the grading by *modular weights*. For $m \in \mathbb{Z}$, let R_m be the subspace of R consisting of all elements of modular weight m. Then we have $R_m = 0$ if $m < 0$ and $R = \oplus_{m \in \mathbb{N}} R_m$. An element of T is said to have *modular weight* m if it is a linear combination of elements of the form $f \otimes w_1 \otimes \cdots \otimes w_n$ where $w_1 \in W_1, \dots, w_n \in W_n$ are homogeneous and $f \in R$ has modular weight $m - \sum_{i=1}^n \mathrm{wt}\ w_i$. We have:

Proposition 3.2 *Let* $w_1 \in W_1, \ldots, w_n \in W_n$ *be homogeneous. For* $j = 1, \ldots, n$, $\mathcal{A}_j$ *have modular weight* $\sum_{i=1}^n \mathrm{wt}\ w_i + 2$. *In particular, the grading by modular weights on* T *induce a grading, also called the grading by modular weights, on* T/J.

Proof. The conclusion follows directly from the definition of $\mathcal{A}_j$. ∎

Proposition 3.3 *There exists* $N \in \mathbb{R}$ *such that for any* $r \in \mathbb{R}$, $F_r(T) \subset F_r(J) + F_N(T)$. *In particular,* $T = J + F_N(T)$.

Proof. Since $\dim W_i/C_2(W_i) < \infty$ for $i = 1, \ldots, n$, there exists $N \in \mathbb{R}$ such that

$$\coprod_{r>N} T_{(r)} \subset \sum_{j=1}^{n} R \otimes W_1 \otimes \cdots \otimes W_{j-1} \otimes C_2(W_j) \otimes W_{j+1} \otimes W_n. \tag{3.1}$$

We use induction on $r \in \mathbb{R}$. If r is equal to N, $F_N(T) \subset F_N(J) + F_N(T)$. Now we assume that $F_r(T) \subset F_r(J) + F_N(T)$ for $r < s$ where $s > N$. We want to show that any homogeneous element of $T_{(s)}$ can be written as a sum of an element of $F_s(J)$ and an element of $F_N(T)$. Since $s > N$, by (3.1), any element of $T_{(s)}$ is an element of the right-hand side of (3.1). Thus this element of $T_{(s)}$ is a sum of elements of

$$R \otimes W_1 \otimes \cdots \otimes W_{j-1} \otimes C_2(W_j) \otimes W_{j+1} \otimes W_n$$

for $j = 1, \ldots, n$. So we need only discuss elements of the form

$$1 \otimes w_1 \otimes \cdots \otimes w_{j-1} \otimes u_{-2} w_j \otimes w_{j+1} \otimes \cdots \otimes w_n \tag{3.2}$$

where $w_i \in W_i$ for $i = 1, \ldots, n$ and $u \in V$. By assumption, the conformal weight of (3.2) is s and the conformal weights of

$$1 \otimes w_1 \otimes \cdots \otimes w_{j-1} \otimes u_{2k} w_j \otimes w_{j+1} \otimes \cdots \otimes w_n$$

for $k \in \mathbb{Z}_+$ and

$$1 \otimes w_1 \otimes \cdots \otimes w_{i-1} \otimes u_m w_i \otimes w_{i+1} \otimes \cdots \otimes w_n$$

for $i \neq j$ and $m \geq 0$ are all less than the conformal weight s of (3.2). So $\mathcal{A}_j(u; w_1, \ldots, w_n) \in F_s(J)$. Thus (3.2) can be written as a sum of an element of $F_s(J)$ and elements of T of conformal weights less than s. Then by the induction assumption, we know that (3.2) can be written as a sum of an element of $F_s(J)$ and an element of $F_N(T)$.

Now we have

$$\begin{aligned} T &= \cup_{r\in\mathbb{R}} F_r(T) \\ &\subset \cup_{r\in\mathbb{R}} F_r(J) + F_N(T) \\ &= J + F_N(T). \end{aligned}$$

But we know that $J + F_N(T) \subset T$. So we have $T = J + F_N(T)$. ∎

Corollary 3.4 *The quotient R-module T/J is finitely generated.*

Proof. Since $T = J+F_N(T)$ and $F_N(T)$ is finitely-generated, T/J is finitely-generated. ∎

For V-modules W_i and $\tilde{W}_i$, $i = 1, \dots, n$, and intertwining operators $\mathcal{Y}_i$, $i = 1, \dots, n$, as in Proposition 2.1, and for any $w_i \in W_i$, $i = 1, \dots, n$, we shall consider the element

$$\begin{aligned} &F_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; q) \\ &\quad = \mathrm{Tr}_{\tilde{W}_n} \mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1}) \cdots \mathcal{Y}_1(\mathcal{U}(q_{z_n})w_n, q_{z_n}) q^{L(0)-\frac{c}{24}} \end{aligned} \tag{3.3}$$

of $\mathbb{G}_{|q_{z_1}|>\cdots>|q_{z_n}|>0}((q))$. The map from $W_1 \otimes \cdots \otimes W_n$ to N given by

$$w_1 \otimes \cdots \otimes w_n \mapsto F_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; q)$$

can be naturally extended to an R-module map

$$\psi_{\mathcal{Y}_1,\dots,\mathcal{Y}_n} : T \to \mathbb{G}_{|q_{z_1}|>\cdots>|q_{z_n}|>0}((q)).$$

Remark 3.5 We consider (3.3) instead of

$$\mathrm{Tr}_{\tilde{W}_n} \mathcal{Y}_1 \left(q_{z_1}^{L(0)} w_1, q_{z_1}\right) \cdots \mathcal{Y}_n \left(q_{z_n}^{L(0)} w_n, q_{z_1}\right) q^{L(0)-\frac{c}{24}}$$

because (3.3) satisfies simple identities and have simple duality properties. Geometrically, (3.3) corresponds to a torus with punctures and local coordinates whose local coordinates at the punctures are the standard ones in terms of the global coordinates on the torus obtained from the parallelogram defining the torus. In fact, to construct and study chiral genus-one correlation functions, we need to use the description of tori in terms of parallelograms. But geometrically q-traces of products of intertwining operators correspond to annuli, not such parallelograms. Thus we have to use the conformal transformation $w \mapsto \tilde{w} = q_w$ to map parallelograms to annuli. Under this map, the standard local coordinates $\tilde{w} - q_{z_i}$ vanishing at q_{z_i} for $i = 1, \dots, n$ on an annulus are not pulled back to the standard local coordinates $w - z_i$ vanishing at z_i, respectively, on the corresponding parallelogram. The operators $\mathcal{U}(q_{z_i})$ for $i = 1, \dots, n$ correspond exactly to the local coordinates vanishing at q_{z_i}, respectively, on the annulus such that their pull-backs to the parallelogram are the standard local coordinates vanishing at z_i, respectively.

We have:

Proposition 3.6 *The R-submodule J of T is in the kernel of $\psi_{\mathcal{Y}_1, \dots, \mathcal{Y}_n}$.*

Proof. This result follows from the definitions of J and $\psi_{\mathcal{Y}_1, \dots, \mathcal{Y}_n}$ and the case $l = 2$ of (2.22) immediately. ∎

Proposition 3.7 ($L(-1)$-derivative property) *Let W_i and $\tilde{W}_i$ for $i = 1, \dots, n$ be V-modules and $\mathcal{Y}_i$ intertwining operators of types $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$ ($i = 1, \dots, n$, $\tilde{W}_0 = \tilde{W}_n$), respectively. Then*

$$\frac{\partial}{\partial z_j} F_{\mathcal{Y}_1, \dots, \mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; q) \\ = F_{\mathcal{Y}_1, \dots, \mathcal{Y}_n}(w_1, \dots, w_{j-1}, L(-1)w_j, w_{j+1}, \dots, w_n; z_1, \dots, z_n; q)$$

for $1 \leq j \leq n$ and $w_i \in W_i$, $i = 1, \dots, n$.

Proof. The result follows immediately from the definition (3.3) and Proposition 1.3. ∎

Lemma 3.8 *Let W_i and $\tilde{W}_i$ for $i = 1, \dots, n$ be V-modules and $\mathcal{Y}_i$ intertwining operators of types $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$ ($i = 1, \dots, n$, $\tilde{W}_0 = \tilde{W}_n$), respectively. Then for $j = 1, \dots, n$ and any homogeneous elements $w_1 \in W_1, \dots, w_n \in W_n$,*

$$\left((2\pi i)^2 q \frac{\partial}{\partial q} + \tilde{G}_2(q) \sum_{i=1}^{n} \text{wt } w_i + \tilde{G}_2(q) \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i} - \sum_{i \neq j} \tilde{\wp}_1(z_i - z_j; q) \frac{\partial}{\partial z_i} \right) \cdot$$

$$
\begin{aligned}
&\qquad \cdot F_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1,\dots,w_n;z_1,\dots,z_n;q)\\
&= F_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1,\dots,w_{j-1},L(-2)w_j,w_{j+1},\dots,w_n;z_1,\dots,z_n;q)\\
&\quad -\sum_{k\geq 1}\tilde{G}_{2k+2}(q)\cdot\\
&\qquad \cdot F_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1,\dots,w_{j-1},L(2k)w_j,w_{j+1},\dots,w_n;z_1,\dots,z_n;q)\\
&\quad +\sum_{i\neq j}\sum_{m\geq 1}(-1)^m\tilde{\wp}_{m+1}(z_i-z_j;q)\cdot\\
&\qquad \cdot F_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1,\dots,w_{i-1},L(m-1)w_i,w_{i+1},\dots,w_n;z_1,\dots,z_n;q).
\end{aligned}
\tag{3.4}
$$

Proof. In (2.22), we take $u=\omega$ and $l=1$. Using $\omega_m = L(m-1)$ for $m\in\mathbb{Z}$ and Lemma 1.1, we obtain (3.4). ∎

For simplicity, we introduce, for any $\alpha\in\mathbb{C}$, the notation

$$\mathcal{O}_j(\alpha) = (2\pi i)^2 q\frac{\partial}{\partial q} + \tilde{G}_2(q)\alpha + \tilde{G}_2(q)\sum_{i=1}^n z_i\frac{\partial}{\partial z_i} - \sum_{i\neq j}\tilde{\wp}_1(z_i-z_j;q)\frac{\partial}{\partial z_i}$$

for $j=1,\dots,n$. We shall also use the notation

$$\prod_{j=1}^m \mathcal{O}(\alpha_j)$$

to denote

$$\mathcal{O}(\alpha_1)\cdots\mathcal{O}(\alpha_m).$$

Note the order of the product. We have:

Theorem 3.9 *Let V be a vertex operator algebra satisfying the conditions stated in the beginning of this section and let W_i for $i=1,\dots,n$ be V-modules. Then for any homogeneous $w_i\in W_i$ ($i=1,\dots,n$), there exist*

$$a_{p,\,i}(z_1,\dots,z_n;q)\in R_p, b_{p,\,i}(z_1,\dots,z_n;q)\in R_{2p}$$

for $p=1,\dots,m$ and $i=1,\dots,n$ such that for any V-modules $\tilde{W}_i$ ($i=1,\dots,n$) and intertwining operators $\mathcal{Y}_i$ of types $\binom{\tilde{W}_{i-1}}{W_i\tilde{W}_i}$ ($i=1,\dots,n$, $\tilde{W}_0=$

$\tilde{W}_n$), respectively, the series (3.3) satisfies the expansion of the system of differential equations

$$\frac{\partial^m \varphi}{\partial z_i^m} + \sum_{p=1}^{m} a_{p,\,i}(z_1, \dots, z_n; q) \frac{\partial^{m-p} \varphi}{\partial z_i^{m-p}} = 0, \tag{3.5}$$

$$\prod_{k=1}^{m} \mathcal{O}_i \left(\sum_{i=1}^{n} \mathrm{wt}\ w_i + 2(m-k) \right) \varphi + \sum_{p=1}^{m} b_{p,\,i}(z_1, \dots, z_n; q) \prod_{k=1}^{m-p} \mathcal{O}_i \left(\sum_{i=1}^{n} \mathrm{wt}\ w_i + 2(m-p-k) \right) \varphi = 0, \tag{3.6}$$

$i = 1, \dots, n$, in the regions $1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q| > 0$.

Proof. For fixed $w_i \in W_i$, $i = 1, \dots, n$, let Π_i for $i = 1, \dots, n$ be the R-submodules of T/J generated by

$$[1 \otimes w_1 \otimes \cdots \otimes w_{i-1} \otimes L^k(-1) w_i \otimes w_{i+1} \otimes \cdots \otimes w_n] \tag{3.7}$$

for $k \in \mathbb{N}$, respectively. Since R is Noetherian and T/J is a finitely-generated R-module, the R-submodules Π_i for $i = 1, \dots, n$ are all finitely-generated. Thus there exist $a_{p,\,i}(z_1, \dots, z_n; q) \in R$ for $p = 1, \dots, m$ and $i = 1, \dots, n$ such that in Π_i

$$\begin{aligned} &[1 \otimes w_1 \otimes \cdots \otimes w_{i-1} \otimes L^m(-1) w_i \otimes w_{i+1} \otimes \cdots \otimes w_n] \\ &\quad + \sum_{p=1}^{m} a_{p,\,i}(z_1, \dots, z_n; q) \cdot \\ &\qquad \cdot [1 \otimes w_1 \otimes \cdots \otimes w_{i-1} \otimes L^{m-p}(-1) w_i \otimes w_{i+1} \otimes \cdots \otimes w_n] \\ &\quad = 0. \end{aligned} \tag{3.8}$$

(Note that in the argument above, we actually first obtain $a_{p,\,i}(z_1, \dots, z_n; q)$ for $p = 1, \dots, m$ for any fixed i. Thus m in fact depends on i. But since there are only finitely many i, we can always choose a sufficiently large m such that it is independent of i.) Since for any $k \geq 0$, (3.7) has modular weight $\sum_{i=1}^{n} w_i + k$, we see from (3.8) that $a_{p,\,i}(z_1, \dots, z_n; q)$ for $p = 1, \dots, m$ and $i = 1, \dots, n$ can be chosen to have modular weights p. Applying $\psi_{\mathcal{Y}_1, \dots, \mathcal{Y}_n}$ to both sides of (3.8) and then using the $L(-1)$-derivative property (Proposition 3.7) for (3.3), we see that (3.3) satisfies the equations (3.5).

On the other hand, let $\mathcal{Q}_i : T \to T$ for $i = 1, \dots, n$ be the linear map defined by

$$\begin{aligned}
&\mathcal{Q}_i(1 \otimes w_1 \otimes \cdots \otimes w_n)\\
&\quad = 1 \otimes w_1 \otimes \cdots \otimes w_{i-1} \otimes L(-2)w_i \otimes w_{i+1} \otimes \cdots \otimes w_n\\
&\qquad - \sum_{k \geq 1} \tilde{G}_{2k+2}(q) \otimes w_1 \otimes \cdots \otimes w_{i-1} \otimes L(2k)w_i \otimes w_{i+1} \otimes \cdots \otimes w_n\\
&\qquad + \sum_{j \neq i} \sum_{m \geq 1} (-1)^m \tilde{\wp}_{m+1}(z_j - z_i; q)\\
&\qquad\qquad\qquad \otimes w_1 \otimes \cdots \otimes w_{i-1} \otimes L(m-1)w_i \otimes w_{i+1} \otimes \cdots \otimes w_n.
\end{aligned}$$

For the same fixed $w_i \in W_i$, $i = 1, \dots, n$, as above, let Λ_i for $i = 1, \dots, n$ be the R-submodules of T/J generated by $[\mathcal{Q}_i^k(1 \otimes w_1 \otimes \cdots \otimes w_n)]$ for $k \geq 0$. Since R is Noetherian and T/J is a finitely-generated R-module, the R-submodule Λ_i is also finitely generated. Thus there exist $b_{p,\, i}(z_1, \dots, z_n; q) \in R$ for $p = 1, \dots, m$ and for $i = 1, \dots, n$ such that in Λ_i

$$[\mathcal{Q}_i^m(1 \otimes w_1 \otimes \cdots \otimes w_n)] + \sum_{p=1}^{t} b_{p,\, i}(z_1, \dots, z_n; q)[\mathcal{Q}_i^{t-p}(1 \otimes w_1 \otimes \cdots \otimes w_n)] = 0. \tag{3.9}$$

(Note that using the same argument above we can always find m sufficiently large such that it is independent of i. But in general it might be different from the m in (3.8). But we can always take the m in (3.8) and the m obtained here to be sufficiently large so that these two m are equal.) Since $[\mathcal{Q}_i^k(1 \otimes w_1 \otimes \cdots \otimes w_n)]$ for any $k \geq 0$ has modular weight $\sum_{i=1}^n w_i + 2k$, we see from (3.9) that $b_{p,\, i}(z_1, \dots, z_n; q)$ for $p = 1, \dots, m$ and $i = 1, \dots, n$ can be chosen to have modular weights $2p$. Applying $\psi_{\mathcal{Y}_1, \dots, \mathcal{Y}_n}$ to both sides of (3.9) and then using (3.4), we see that (3.3) satisfies (3.6). ∎

4 Chiral genus-one correlation functions and genus-one duality

In this section, using the systems of differential equations obtained in the preceding section, we construct chiral genus-one correlation functions and establish their duality properties.

We still fix a vertex operator algebra V satisfying the same conditions assumed in the preceding section.

Theorem 4.1 *In the region $1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q_\tau| > 0$, the series*

$$F_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1,\dots,w_n;z_1,\dots,z_n;q_\tau) \tag{4.1}$$

is absolutely convergent and can be analytically extended to a (multivalued) analytic function in the region given by $\Im(\tau) > 0$ (here $\Im(\tau)$ is the imaginary part of τ), $z_i \neq z_j + k\tau + l$ for $i,j = 1,\dots,n$, $i \neq j$, $k,l \in \mathbb{Z}$.

Proof. We know that the coefficients of (4.1) as a series in powers of q_τ are absolutely convergent when $1 > |q_{z_1}| > \cdots > |q_{z_n}| > 0$. So for fixed $z_1,\dots,z_n$ satisfying $1 > |q_{z_1}| > \cdots > |q_{z_n}| > 0$, (4.1) is a well-defined series in powers of q_τ. For fixed $z_1,\dots,z_n$ satisfying $1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q_\tau| > 0$, the ordinary differential equation (3.6) with the variable q_τ has a regular singular point at $q_\tau = 0$. Since the series (4.1) satisfies (3.6) with $q = q_\tau$, it is absolutely convergent as a series in powers of q_τ. Since the coefficients of the equation (3.6) are analytic in $z_1,\dots,z_n$, the sum of (4.1) as a series in powers of q_τ is also analytic in $z_1,\dots,z_n$. In particular, (4.1) as the expansion of an analytic function in the region $1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q_\tau| > 0$ must be absolutely convergent as a series of multiple sums. Thus the sum of this series is analytic in $z_1,\dots,z_n$ and q_τ and give a (multivalued) analytic function in the region $1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q_\tau| > 0$. So the first part of the theorem is proved.

Now we know that the coefficients of the system (3.5)–(3.6) with $q = q_\tau$ are analytic in $z_1,\dots,z_n$ and τ with the only possible singularities $z_i \neq z_j + k\tau + l$ for $i,j = 1,\dots,n$, $i \neq j$, $k,l \in \mathbb{Z}$. So (4.1) as a solution of the system in the region $1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q| > 0$ can be analytically extended to the region given by $\Im(\tau) > 0$, $z_i \neq z_j + k\tau + l$ for $i,j = 1,\dots,n$, $i \neq j$, $k,l \in \mathbb{Z}$. ∎

We shall call functions in the region $\Im(\tau) > 0$, $z_i \neq z_j + k\tau + l$ for $i,j = 1,\dots,n$, $i \neq j$, $k,l \in \mathbb{Z}$ obtained by analytically extending (4.1) *(chiral) genus-one correlation functions* or simply *genus-zero correlation functions* and we shall use

$$\overline{F}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1,\dots,w_n;z_1,\dots,z_n;\tau) \tag{4.2}$$

to denote these functions.

Theorem 4.2 (Genus-one commutativity) *Let W_i and $\tilde{W}_i$ be V-modules and $\mathcal{Y}_i$ intertwining operators of types $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$ ($i = 1, \dots, n$, $\tilde{W}_0 = \tilde{W}_n$), respectively. Then for any $1 \le k \le n-1$, there exist V-modules $\hat{W}_k$ and intertwining operators $\hat{\mathcal{Y}}_k$ and $\hat{\mathcal{Y}}_{k+1}$ of types $\binom{\hat{W}_k}{W_k \tilde{W}_{k+1}}$ and $\binom{\tilde{W}_{k-1}}{W_{k+1} \hat{W}_k}$, respectively, such that*

$$F_{\mathcal{Y}_1, \dots, \mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; q_\tau)$$

and

$$F_{\mathcal{Y}_1, \dots, \mathcal{Y}_{k-1}, \hat{\mathcal{Y}}_{k+1}, \hat{\mathcal{Y}}_k, \mathcal{Y}_{k+2} \dots, \mathcal{Y}_n}(w_1, \dots, w_{k-1}, w_{k+1}, w_k, w_{k+2}, \dots, w_n; \\ z_1, \dots, z_{k-1}, z_{k+1}, z_k, z_{k+2}, \dots, z_n; q_\tau)$$

are analytic extensions of each other, or equivalently,

$$\begin{aligned} &\overline{F}_{\mathcal{Y}_1, \dots, \mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; \tau) \\ &\quad = \overline{F}_{\mathcal{Y}_1, \dots, \mathcal{Y}_{k-1}, \hat{\mathcal{Y}}_{k+1}, \hat{\mathcal{Y}}_k, \mathcal{Y}_{k+2} \dots, \mathcal{Y}_n}(w_1, \dots, w_{k-1}, w_{k+1}, w_k, w_{k+2}, \dots, w_n; \\ &\qquad\qquad z_1, \dots, z_{k-1}, z_{k+1}, z_k, z_{k+2}, \dots, z_n; \tau). \end{aligned}$$

More generally, for any $\sigma \in S_n$, there exist V-modules $\hat{W}_i$ ($i = 1, \dots, n$) and intertwining operators $\hat{\mathcal{Y}}_i$ of types $\binom{\hat{W}_{i-1}}{W_{\sigma(i)} \hat{W}_i}$ ($i = 1, \dots, n$, $\hat{W}_0 = \hat{W}_n = \tilde{W}_n$), respectively, such that

$$F_{\mathcal{Y}_1, \dots, \mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; q_\tau)$$

and

$$F_{\hat{\mathcal{Y}}_1, \dots, \hat{\mathcal{Y}}_n}(w_{\sigma(1)}, \dots, w_{\sigma(n)}; z_{\sigma(1)}, \dots, z_{\sigma(n)}; q_\tau)$$

are analytic extensions of each other, or equivalently,

$$\begin{aligned} &\overline{F}_{\mathcal{Y}_1, \dots, \mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; \tau) \\ &\quad = \overline{F}_{\hat{\mathcal{Y}}_1, \dots, \hat{\mathcal{Y}}_n}(w_{\sigma(1)}, \dots, w_{\sigma(n)}; z_{\sigma(1)}, \dots, z_{\sigma(n)}; \tau). \end{aligned}$$

Proof. This follows immediately from commutativity for intertwining operators. ∎

For any V-module W and $r \in \mathbb{R}$, we use P_r to denote the projection from W or $\overline{W}$ to $W_{(r)}$.

Theorem 4.3 (Genus-one associativity) *Let W_i and $\tilde{W}_i$ for $i = 1, \dots, n$ be V-modules and $\mathcal{Y}_i$ intertwining operators of types $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$ ($i = 1, \dots, n$, $\tilde{W}_0 = \tilde{W}_n$), respectively. Then for any $1 \le k \le n-1$, there exist a V-module $\hat{W}_k$ and intertwining operators $\hat{\mathcal{Y}}_k$ and $\hat{\mathcal{Y}}_{k+1}$ of types $\binom{\hat{W}_k}{W_k W_{k+1}}$ and $\binom{\tilde{W}_{k-1}}{\hat{W}_k \tilde{W}_{k+1}}$, respectively, such that*

$$\begin{aligned}\overline{F}_{\mathcal{Y}_1,\dots,\mathcal{Y}_{k-1},\hat{\mathcal{Y}}_{k+1},\mathcal{Y}_{k+2},\dots,\mathcal{Y}_n}(w_1, \dots, w_{k-1}, \hat{\mathcal{Y}}(w_k, z_k - z_{k+1})w_{k+1},\\ w_{k+2}, \dots, w_n; z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n; \tau)\\ = \sum_{r\in\mathbb{R}} \overline{F}_{\mathcal{Y}_1,\dots,\mathcal{Y}_{k-1},\hat{\mathcal{Y}}_{k+1},\mathcal{Y}_{k+2},\dots,\mathcal{Y}_n}(w_1, \dots, w_{k-1}, P_r(\hat{\mathcal{Y}}(w_k, z_k - z_{k+1})w_{k+1}),\\ w_{k+2}, \dots, w_n; z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n; \tau)\end{aligned} \tag{4.3}$$

is absolutely convergent when $1 > |q_{z_1}| > \cdots > |q_{z_{k-1}}| > |q_{z_{k+1}}| > \dots > |q_{z_n}| > |q_\tau| > 0$ and $1 > |q_{(z_k - z_{k+1})} - 1| > 0$ and is convergent to

$$\overline{F}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; \tau)$$

when $1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q_\tau| > 0$ and $|q_{(z_k - z_{k+1})}| > 1 > |q_{(z_k - z_{k+1})} - 1| > 0$.

Proof. Using Proposition 1.4, we know that for any $1 \le k \le n-1$, there exist a V-module $\tilde{W}_k$ and intertwining operators $\hat{\mathcal{Y}}_k$ and $\hat{\mathcal{Y}}_{k+1}$ of $\binom{\hat{W}_k}{W_k W_{k+1}}$ and $\binom{\tilde{W}_{k-1}}{\hat{W}_k \tilde{W}_{k+1}}$, respectively, such that for any $z_1, \dots, z_n \in \mathbb{C}$ satisfying $1 > |q_{z_1}| > \dots > |q_{z_n}| > 0$ and $|q_{z_{k+1}}| > |q_{z_k} - q_{z_{k+1}}| > 0$, we have

$$\begin{aligned}&\langle \tilde{w}'_n, \mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1}) \cdots \mathcal{Y}_{k-1}(\mathcal{U}(q_{z_{k-1}})w_{k-1}, q_{z_{k-1}})\cdot\\ &\quad \cdot\hat{\mathcal{Y}}_{k+1}(\mathcal{U}(q_{z_{k+1}})\hat{\mathcal{Y}}_k(w_k, z_k - z_{k+1})w_{k+1}, q_{z_{k+1}})\cdot\\ &\quad \cdot\mathcal{Y}_{k+2}(\mathcal{U}(q_{z_{k+2}})w_{k+2}, q_{z_{k+2}}) \cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})\tilde{w}_n\rangle\\ &= \langle \tilde{w}'_n, \mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1}) \cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})\tilde{w}_n\rangle\end{aligned} \tag{4.4}$$

for any $\tilde{w}_n \in \tilde{W}_n$ and $\tilde{w}'_n \in \tilde{W}'_n$. Thus as series in q,

$$\begin{aligned}&\mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1}) \cdots \mathcal{Y}_{k-1}(\mathcal{U}(q_{z_{k-1}})w_{k-1}, q_{z_{k-1}})\cdot\\ &\quad \cdot\hat{\mathcal{Y}}_{k+1}(\mathcal{U}(q_{z_{k+1}})\hat{\mathcal{Y}}_k(w_k, z_k - z_{k+1})w_{k+1}, q_{z_{k+1}})\cdot\\ &\quad \cdot\mathcal{Y}_{k+2}(\mathcal{U}(q_{z_{k+2}})w_{k+2}, q_{z_{k+2}}) \cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)-\frac{c}{24}}\\ &= \mathrm{Tr}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}(q_{z_1})w_1, q_{z_1}) \cdots \mathcal{Y}_n(\mathcal{U}(q_{z_n})w_n, q_{z_n})q^{L(0)-\frac{c}{24}}.\end{aligned} \tag{4.5}$$

Since the right-hand side of (4.5) is convergent absolutely when $q = q_\tau$ and $1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q_\tau| > 0$, the left-hand side is also convergent absolutely when $q = q_\tau$, $1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q_\tau| > 0$ and $|q_{z_{k+1}}| > |q_{z_k} - q_{z_{k+1}}| > 0$. Also, since the right-hand side of (4.5) satisfies the system (3.5)–(3.6), so does the left-hand side. Thus the left-hand side of (4.5) with $q = q_\tau$ is convergent absolutely to an analytic function, in the region $1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q_\tau| > 0$ and $|q_{z_{k+1}}| > |q_{z_k} - q_{z_{k+1}}| > 0$, which can be analytically extended to the multivalued function

$$\overline{F}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1,\dots,w_n; z_1,\dots,z_n; \tau)$$

on $\mathbb{C}^n \times \mathbb{H}$. Moreover, in the region $1 > |q_{z_1}| > \cdots > |q_{z_{k-1}}| > |q_{z_{k+1}}| > \dots > |q_{z_n}| > |q_\tau| > 0$ and $1 > |q_{(z_k - z_{k+1})} - 1| > 0$, this function has the expansion (4.3), proving the theorem. ∎

5 The regularity of the singular points for fixed $q = q_\tau$

In this section, we prove that for fixed $\tau \in \mathbb{H}$, we can actually obtain the coefficients in the system (3.5) with $q = q_\tau$ such that (3.5) with $q = q_\tau$ is regular at its singular points of the form $z_i = z_j + \alpha\tau + \beta$ for $1 \leq i < j \leq n$. In particular, at these singular points, all the chiral genus-one correlation functions as functions of $z_1, \dots, z_n$ are regular. The method used here is an adaption of the method used in [H9]. Though it will be important for the further study of chiral genus-one correlation functions, the result in this section will not be needed in the next two sections.

As in the genus-zero case discussed in [H9], we need certain filtrations on R and on the R-module T. For $m \in \mathbb{Z}_+ + 1$, let $F^{\rm sing}_m(R)$ be the vector subspace of R spanned by elements of the form

$$f(q) \prod_{1\leq i<j\leq n} (\tilde{\wp}_2(z_i - z_j; q))^{k_{i,j}} (\tilde{\wp}_3(z_i - z_j; q))^{l_{i,j}}$$

where $k_{i,j}, l_{i,j} \in \mathbb{Z}_+$ satisfying $\sum_{1\leq i<j\leq n} 2k_{i,j} + 3l_{i,j} \leq m$ and $f(q)$ is a polynomial of $\tilde{G}_4(q)$ and $\tilde{G}_6(q)$. Clearly these subspaces of R give a filtration of R. With respect to this filtration, R is a filtered algebra, that is, $F^{\rm sing}_{m_1}(R) \subset F^{\rm sing}_{m_2}(R)$ for $m_1 \leq m_2$, $R = \cup_{m\in\mathbb{Z}} F^{\rm sing}_m(R)$ and $F^{\rm sing}_{m_1}(R) F^{\rm sing}_{m_2}(R) \subset F^{\rm sing}_{m_1+m_2}(R)$ for any $m_1, m_2 \in \mathbb{Z}_+ + 1$.

Lemma 5.1 *For $m \in \mathbb{Z}_+ + 1$ and $i, j = 1, \dots, n$ satisfying $i < j$, we have $\tilde{\wp}_m(z_i - z_j; q) \in F_m^{\text{sing}}(R)$.*

Proof. Using induction, this lemma follows easily from (2.8), the fact that $\tilde{\wp}_m(z_i - z_j; q)$ is a polynomial of $\tilde{\wp}_2(z_i - z_j; q)$ and $\tilde{\wp}_3(z_i - z_j; q)$ and the fact that $\tilde{\wp}_m(z_i - z_j; q)$ is either even or odd in the variable $z_i - z_j$. ∎

For convenience, we shall use σ to denote $\sum_{i=1}^n \text{wt } w_i$ for homogeneous $w_i \in W_i$, $i = 1, \dots, n$, when the dependence on w_i is clear. Let $F_r^{\text{sing}}(T)$ for $r \in \mathbb{R}$ be the subspace of T spanned by elements of the form

$$\left(f(q) \prod_{1 \leq i < j \leq n} (\tilde{\wp}_2(z_i - z_j; q))^{k_{i,j}} (\tilde{\wp}_3(z_i - z_j; q))^{l_{i,j}} \right) \otimes w_1 \otimes \cdots \otimes w_n$$

where $k_{i,j}, l_{i,j} \in \mathbb{Z}_+$ satisfying $\sum_{1 \leq i < j \leq n} 2k_{i,j} + 3l_{i,j} + \sigma \leq r$ and $f(q)$ is a polynomial of $\tilde{G}_4(q)$ and $\tilde{G}_6(q)$. These subspaces give a filtration of T in the following sense: $F_r^{\text{sing}}(T) \subset F_s^{\text{sing}}(T)$ for $r \leq s$; $T = \cup_{r \in \mathbb{R}} F_r^{\text{sing}}(T)$; $F_m^{\text{sing}}(R) F_r^{\text{sing}}(T) \subset F_{r+m}^{\text{sing}}(T)$.

Let $F_r^{\text{sing}}(J) = F_r^{\text{sing}}(T) \cap J$ for $r \in \mathbb{R}$. Then we have the following:

Proposition 5.2 *For any $r \in \mathbb{R}$, there exists $N \in \mathbb{R}$ such that $F_r^{\text{sing}}(T) \subset F_r^{\text{sing}}(J) + F_N(T)$.*

Proof. The proof is a refinement of the proof of Proposition 3.3. The only additional property we need is that the elements $\mathcal{A}_j(u, w_1, \dots, w_n)$, are all in $F_{\text{wt } u + \sigma}^{\text{sing}}(J)$. By Lemma 5.1, this is true. ∎

Let R^{reg} be the commutative associative algebra over $\mathbb{C}$ generated by the series $\tilde{G}_4(q)$, $\tilde{G}_6(q)$, $(z_i - z_j)^2 \tilde{\wp}_2(z_i - z_j; q)$, $(z_i - z_j)^3 \tilde{\wp}_3(z_i - z_j; q)$ and $z_i - z_j$ for $i, j = 1, \dots, n$ satisfying $i < j$. Note that R^{reg} is a subalgebra of the algebra $R[z_i - z_j]_{1 \leq i < j \leq n}$. Since R^{reg} is finitely generated over the field $\mathbb{C}$, it is a Noetherian ring. We also consider the R^{reg}-module

$$T^{\text{reg}} = R^{\text{reg}} \otimes W_1 \otimes \cdots \otimes W_n.$$

Note that T^{reg} is a subspace of

$$R[z_i - z_j]_{1 \leq i < j \leq n} \otimes W_1 \otimes \cdots \otimes W_n.$$

The grading by conformal weights on $W_1 \otimes \cdots \otimes W_n$ induces a grading (by conformal weights) on T^{reg}. Let $T^{\mathrm{reg}}_{(r)}$ for $r \in \mathbb{R}$ be the space of elements of T^{reg} of conformal weight r. Then $T^{\mathrm{reg}} = \coprod_{r\in\mathbb{R}} T^{\mathrm{reg}}_{(r)}$.

Let $w_i \in W_i$ for $i = 1, \dots, n$ be homogeneous. Then by Proposition 5.2,

$$w_1 \otimes \cdots \otimes w_n = \mathcal{W}_1 + \mathcal{W}_2$$

where $\mathcal{W}_1 \in F^{\mathrm{sing}}_{\sigma}(J)$ and $\mathcal{W}_2 \in F_N(T)$ (and $\sigma = \sum_{i=1}^{n} \mathrm{wt}\ w_i$).

Lemma 5.3 *For any $s \in [0, 1)$, there exist $S \in \mathbb{R}$ such that $s + S \in \mathbb{Z}_+$ and for any homogeneous $w_i \in W_i$, $i = 1, \dots, n$, satisfying $\sigma \in s + \mathbb{Z}$, $\prod_{1\le i<j\le n}(z_i - z_j)^{\sigma+S}\mathcal{W}_2 \in T^{\mathrm{reg}}$.*

Proof. Let S be a real number such that $s + S \in \mathbb{Z}_+$ and such that for any $r \in \mathbb{R}$ satisfying $r \le -S$, $T_{(r)} = 0$. By definition, elements of $F^{\mathrm{sing}}_{r}(T)$ for any $r \in \mathbb{R}$ are sums of elements of the form

$$\left(f(q) \prod_{1\le i<j\le n} (\tilde{\wp}_2(z_i - z_j; q))^{k_{i,j}} (\tilde{\wp}_3(z_i - z_j; q))^{l_{i,j}} \right) \otimes \tilde{w}_1 \otimes \cdots \otimes \tilde{w}_n$$

where $k_{i,j}, l_{i,j} \in \mathbb{Z}_+$ satisfying

$$\sum_{1\le i<j\le n} 2k_{i,j} + 3l_{i,j} + \sum_{i=1}^{n} \mathrm{wt}\ \tilde{w}_i \le r \tag{5.1}$$

and $f(q)$ is a polynomial of $\tilde{G}_4(q)$ and $\tilde{G}_6(q)$. Since for nonzero $\tilde{w}_1 \otimes \cdots \otimes \tilde{w}_n$, $\sum_{i=1}^{n} \mathrm{wt}\ \tilde{w}_i > -S$ which together with (5.1) implies

$$r > \sum_{1\le i<j\le n} 2k_{i,j} + 3l_{i,j} - S$$

or

$$r + S - \sum_{1\le i<j\le n} 2k_{i,j} + 3l_{i,j} > 0.$$

Consequently,

$$r + S - (2k_{i,j} + 3l_{i,j}) > 0, \quad 1 \le i < j \le n.$$

Thus $\prod_{1\le i<j\le n}(z_i - z_j)^{r+S} F^{\mathrm{sing}}_{r}(T) \in T^{\mathrm{reg}}$.

By definition,

$$\mathcal{W}_2 = w_1 \otimes \cdots \otimes w_n - \mathcal{W}_1,$$

where

$$\mathcal{W}_1 \in F_\sigma^{\rm sing}(J) \subset F_\sigma^{\rm sing}(T).$$

By the discussion above, $\prod_{1\le i<j\le n}(z_i - z_j)^{\sigma+S}\mathcal{W}_1 \in T^{\rm reg}$ and by definition,

$$w_0 \otimes w_1 \otimes w_2 \otimes w_3 \in T^{\rm reg}.$$

Thus $\prod_{1\le i<j\le n}(z_i - z_j)^{\sigma+S}\mathcal{W}_2 \in T^{\rm reg}$. ∎

Theorem 5.4 *Let W_i and $w_i \in W_i$ for $i = 1, \dots, n$ be the same as in Theorem 3.9 and let $\tau \in \mathbb{H}$. Then there exist*

$$a_{p,\,j}(z_1, \dots, z_n; q) \in R_p$$

for $p = 1, \dots, m$ and $j = 1, \dots, n$ such that the (possible) singular points of the form $z_i = z_j + \alpha\tau + \beta$ for $1 \le i < j \le n$ and $\alpha, \beta \in \mathbb{Z}$ of the system (3.5) with $q = q_\tau$ satisfied by

$$F_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; q_\tau)$$

are regular.

Proof. We need only prove that for any fixed integers i, j satisfying $1 \le i < j \le n\}$, the singular point $z_i = z_j$ is regular because the coefficients of the system (3.5) are periodic with periods 1 and τ.

By Proposition 5.2, for $k = 1, \dots, n$,

$$1 \otimes w_1 \otimes \cdots \otimes w_{k-1} \otimes L^p(-1)w_k \otimes w_{k+1} \otimes \cdots \otimes w_n = \mathcal{W}_1^{(p)} + \mathcal{W}_2^{(p)}$$

for $p \ge 0$, where $\mathcal{W}_1^{(p)} \in F_{\sigma+p}^{\rm sing}(J)$ and $\mathcal{W}_2^{(p)} \in F_N(T)$.

By Lemma 5.3, there exists $S \in \mathbb{R}$ such that $\sigma + S \in \mathbb{Z}_+$ and

$$\prod_{1\le i<j\le n}(z_i - z_j)^{\sigma+p+S}\mathcal{W}_2^{(p)} \in T^{\rm reg}$$

and thus

$$\prod_{1\le i<j\le n}(z_i - z_j)^{\sigma+p+S}\mathcal{W}_2^{(p)} \in \coprod_{r\le N} T_{(r)}^{\rm reg}$$

for $p \geq 0$ and $1 \leq i < j \leq n$. Since $R^{\rm reg}$ is a Noetherian ring and $\coprod_{r \leq N} T_{(r)}^{\rm reg}$ is a finitely-generated $R^{\rm reg}$-module, the submodule of $\coprod_{r \leq N} T_{(r)}^{\rm reg}$ generated by $\prod_{1\leq i<j\leq n}(z_i - z_j)^{\sigma+p+S}\mathcal{W}_2^{(p)}$ for $p \geq 0$ is also finitely generated. Let

$$\prod_{1\leq i<j\leq n}(z_i - z_j)^{\sigma+m-p+S}\mathcal{W}_2^{(m-p)}$$

for $p = 1, \dots, m$ be a set of generators of this submodule. (Note that as in the proof of Theorem 3.9, we can always choose m to be independent of i.) Then there exist $c_{p,\,l}(z_1, \dots, z_n; q) \in R^{\rm reg}$ for $p = 1, \dots, m$ such that

$$\begin{aligned}\prod_{1\leq i<j\leq n}&(z_i - z_j)^{\sigma+m+S}\mathcal{W}_2^{(m)}\\ &= -\sum_{p=1}^{m} c_{p,\,i}(z_1, \dots, z_n; q) \prod_{1\leq i<j\leq n}(z_i - z_j)^{\sigma+m-p+S}\mathcal{W}_2^{(m-p)}. \qquad (5.2)\end{aligned}$$

Since $\mathcal{W}_2^{(m-p)} \in T$ for $p = 1, \dots, m$,

$$\prod_{1\leq i<j\leq n}(z_i - z_j)^{\sigma+m-p+S}\mathcal{W}_2^{(m-p)} \in \prod_{1\leq i<j\leq n}(z_i - z_j)^{\sigma+m-p+S}T.$$

Projecting both sides of (5.2) to $\prod_{1\leq i<j\leq n}(z_i - z_j)^{\sigma+m+S}T$, we obtain

$$\begin{aligned}\prod_{1\leq i<j\leq n}&(z_i - z_j)^{\sigma+m+S}\mathcal{W}_2^{(m)}\\ &= -\sum_{p=1}^{m} d_{p,\,i}(z_1, \dots, z_n; q) \prod_{1\leq i<j\leq n}(z_i - z_j)^{\sigma+m-p+S}\mathcal{W}_2^{(m-p)}, \qquad (5.3)\end{aligned}$$

where $d_{p,\,i}(z_1, \dots, z_n; q)$ for $p = 1, \dots, m$ are the projection images in

$$\prod_{1\leq i<j\leq n}(z_i - z_j)^{p} R \cap R^{\rm reg}$$

of $c_{p,\,i}(z_1, \dots, z_n; q)$. The equality (5.2) is equivalent to

$$\mathcal{W}_2^{(m)} + \sum_{p=1}^{m} d_{p,\,i}(z_1, \dots, z_n; q) \prod_{1\leq i<j\leq n}(z_i - z_j)^{-p}\mathcal{W}_2^{(m-p)} = 0. \qquad (5.4)$$

Let

$$a_{p,\,i}(z_1, \dots, z_n; q) = d_{p,\,i}(z_1, \dots, z_n; q) \prod_{1\le i<j\le n} (z_i - z_j)^{-p} \in R \tag{5.5}$$

for $p = 1, \dots, m$. Then (5.4) gives

$$\begin{aligned}
1 \otimes w_1 \otimes \cdots \otimes w_{i-1} L^m(-1) w_i \otimes w_{i+1} \otimes \cdots \otimes w_n \\
+ \sum_{p=1}^{m} a_{p,\,i}(z_1, \dots, z_n; q) \cdot \\
\cdot (1 \otimes w_1 \otimes \cdots \otimes w_{i-1} \otimes L^{m-p}(-1) w_i \otimes w_{i+1} \otimes \cdots \otimes w_n) \\
= \mathcal{W}_1^{(m)} + \sum_{p=1}^{m} a_{p,\,i}(z_1, \dots, z_n; q) \mathcal{W}_1^{(m-p)}.
\end{aligned} \tag{5.6}$$

Since $\mathcal{W}_1^{(m-p)} \in F^{\mathrm{sing}}_{\sigma+p}(J) \subset J$ for $p = 0, \dots, m$, the right-hand side of (5.6) is in J. Thus we obtain

$$\begin{aligned}
[1 \otimes w_1 \otimes \cdots \otimes w_{i-1} L^m(-1) w_i \otimes w_{i+1} \otimes \cdots \otimes w_n] \\
+ \sum_{p=1}^{m} a_{p,\,i}(z_1, \dots, z_n; q) \cdot \\
\cdot [1 \otimes w_1 \otimes \cdots \otimes w_{i-1} \otimes L^{m-p}(-1) w_i \otimes w_{i+1} \otimes \cdots \otimes w_n] = 0.
\end{aligned}$$

Now using the fact that $d_{p,\,i}(z_1, \dots, z_n; q) \in R^{\mathrm{reg}}$ for $p = 1, \dots, m$, (5.5), we see that the singular point $z_i = z_j$ for $1 \le i < j \le n$ of the system (3.5) is regular. As in the proof of Theorem 3.9, $a_{p,\,i}(z_1, \dots, z_n; q)$ can be further chosen to be also in R_p for $p = 1, \dots, m$. ∎

6 Associative algebras and vertex operator algebras

In this section, we introduce a new product in a vertex operator algebra and using this new product, we construct an associative algebra. We study this associative algebra and its representations. These results are needed in the next section. The new product we introduce is very different from the one introduced by Zhu in [Z] and thus our algebra looks very different from Zhu's algebra. It turns out that our algebra is in fact isomorphic to Zhu's algebra and the results for our algebra are parallel to those results in [Z], [FZ] and [Li] for Zhu's algebra. Thus many of the results in this section can also be proved using the isomorphism between these two algebras and the results for Zhu's algebra.

On the other hand, the product introduced in this section is conceptual and geometric (see Remark 6.2 below) and many of the formulas are simpler than those in [Z]. Also in the present paper, we assume that the reader is familiar only with some basic notions and results on vertex operator algebras, their representations and intertwining operator algebras. Because of these reasons, we shall give direct proofs of all results.

We define a product $\bullet$ in V as follows: For $u, v \in V$,

$$\begin{aligned} u \bullet v &= \mathrm{Res}_y y^{-1} Y\left(u, \frac{1}{2\pi i}\log(1+y)\right) v \\ &= \mathrm{Res}_x \frac{2\pi i e^{2\pi i x}}{e^{2\pi i x}-1} Y(u, x)v. \end{aligned}$$

Let $\tilde{O}(V)$ be the subspace of V spanned by elements of the form

$$\mathrm{Res}_y y^{-n} Y\left(u, \frac{1}{2\pi i}\log(1+y)\right) v = \mathrm{Res}_x \frac{2\pi i e^{2\pi i x}}{(e^{2\pi i x}-1)^n} Y(u, x)v$$

for $n > 1$ and $u, v \in V$. Let $\tilde{A}(V) = V/\tilde{O}(V)$.

Proposition 6.1 *The product $\bullet$ in V induces a product (denoted still by $\bullet$) in $\tilde{A}(V)$ such that $\tilde{A}(V)$ together with this product, the equivalence class of the vacuum $\mathbf{1} \in V$ is an associative algebra with identity. Moreover, for any $u, v \in V$, $(L(-1)u) \bullet v \in \tilde{O}(V)$ and $u \bullet v \equiv v \bullet u - 2\pi i u_0 v \mod \tilde{O}(V)$. In particular, $\omega + \tilde{O}(V)$ is in the center of $\tilde{A}(V)$.*

Proof. Let $u_1, u_2, u_3 \in V$ and $n \in \mathbb{Z}_+ + 1$. Using the commutator formula for vertex operators, we have

$$\begin{aligned} &\mathrm{Res}_{x_1}\mathrm{Res}_{x_2} \frac{2\pi i e^{2\pi i x_1}}{e^{2\pi i x_1}-1} \frac{2\pi i e^{2\pi i x_2}}{(e^{2\pi i x_2}-1)^n} Y(u_1, x_1)Y(u_2, x_2)u_3 \\ &\quad = \mathrm{Res}_{x_1}\mathrm{Res}_{x_2} \frac{2\pi i e^{2\pi i x_1}}{e^{2\pi i x_1}-1} \frac{2\pi i e^{2\pi i x_2}}{(e^{2\pi i x_2}-1)^n} Y(u_2, x_2)Y(u_1, x_1)u_3 \end{aligned}$$

$$+\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\mathrm{Res}_{x_0}\frac{2\pi i e^{2\pi i x_1}}{e^{2\pi i x_1}-1}\frac{2\pi i e^{2\pi i x_2}}{(e^{2\pi i x_2}-1)^n}\cdot$$
$$\cdot x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right)Y(Y(u_1,x_0)u_2,x_2)u_3. \tag{6.7}$$

Since the left-hand side of (6.7) spans $V\bullet\tilde{O}(V)$ and the right-hand side of (6.7) is in $\tilde{O}(V)$, we conclude that $V\bullet\tilde{O}(V)\subset\tilde{O}(V)$.

Now let $u_1,u_2,u_3\in V$ and $n\in\mathbb{Z}_+$. Using the Jacobi identity, we have

$$\begin{aligned}
&\mathrm{Res}_{x_0}\mathrm{Res}_{x_2}\frac{2\pi i e^{2\pi i x_0}}{(e^{2\pi i x_0}-1)^n}\frac{2\pi i e^{2\pi i x_2}}{e^{2\pi i x_2}-1}Y(Y(u_1,x_0)u_2,x_2)u_3\\
&\quad=\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\mathrm{Res}_{x_0}\frac{2\pi i e^{2\pi i x_0}}{(e^{2\pi i x_0}-1)^n}\frac{2\pi i e^{2\pi i x_2}}{e^{2\pi i x_2}-1}\cdot\\
&\qquad\qquad\cdot x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u_1,x_1)Y(u_2,x_2)u_3\\
&\qquad-\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\mathrm{Res}_{x_0}\frac{2\pi i e^{2\pi i x_0}}{(e^{2\pi i x_0}-1)^n}\frac{2\pi i e^{2\pi i x_2}}{e^{2\pi i x_2}-1}\cdot\\
&\qquad\qquad\cdot x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(u_2,x_2)Y(u_1,x_1)u_3\\
&\quad=\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\mathrm{Res}_{x_0}\frac{2\pi i e^{2\pi i(x_1-x_2)}}{(e^{2\pi i(x_1-x_2)}-1)^n}\frac{2\pi i e^{2\pi i x_2}}{e^{2\pi i x_2}-1}\cdot\\
&\qquad\qquad\cdot x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y(u_1,x_1)Y(u_2,x_2)u_3\\
&\qquad-\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\mathrm{Res}_{x_0}\frac{2\pi i e^{-2\pi i(x_2-x_1)}}{(e^{-2\pi i(x_2-x_1)}-1)^n}\frac{2\pi i e^{2\pi i x_2}}{e^{2\pi i x_2}-1}\cdot\\
&\qquad\qquad\cdot x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y(u_2,x_2)Y(u_1,x_1)u_3\\
&\quad=\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\frac{2\pi i e^{2\pi i(x_1-x_2)}}{((e^{2\pi i x_1}-1)e^{-2\pi i x_2}+(e^{-2\pi i x_2}-1))^n}\frac{2\pi i e^{2\pi i x_2}}{e^{2\pi i x_2}-1}\cdot\\
&\qquad\qquad\cdot Y(u_1,x_1)Y(u_2,x_2)u_3\\
&\qquad-\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}\frac{2\pi i e^{-2\pi i(x_2-x_1)}}{(e^{-2\pi i x_2}-1)e^{2\pi i x_1}-(e^{2\pi i x_1}-1))^n}\frac{2\pi i e^{2\pi i x_2}}{e^{2\pi i x_2}-1}\cdot\\
&\qquad\qquad\cdot Y(u_2,x_2)Y(u_1,x_1)u_3\\
&\quad=\sum_{k\in\mathbb{N}}\binom{-n}{k}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2}2\pi i e^{2\pi i(x_1-x_2)}(e^{2\pi i x_1}-1)^{-n-k}e^{-2\pi i(-n-k)x_2}\cdot
\end{aligned}$$

$$
\begin{aligned}
&\cdot (e^{-2\pi i x_2}-1)^k \frac{2\pi i e^{2\pi i x_2}}{e^{2\pi i x_2}-1} Y(u_1,x_1)Y(u_2,x_2)u_3\\
&-\sum_{k\in\mathbb{N}}\binom{-n}{k}\mathrm{Res}_{x_1}\mathrm{Res}_{x_2} 2\pi i e^{-2\pi i(x_2-x_1)}(e^{-2\pi i x_2}-1)^{-n-k}e^{2\pi i(-n-k)x_1}\cdot\\
&\cdot (e^{2\pi i x_1}-1)^k \frac{2\pi i e^{2\pi i x_2}}{e^{2\pi i x_2}-1} Y(u_2,x_2)Y(u_1,x_1)u_3. \qquad (6.8)
\end{aligned}
$$

When $n > 1$, the left-hand side of (6.8) spans $\tilde{O}(V)\bullet V$ and the right-hand side of (6.8) is in $\tilde{O}(V)$. So $\tilde{O}(V)\bullet V\subset \tilde{O}(V)$. Thus $\tilde{O}(V)$ is a two-side ideal for the product $\bullet$ so that $\bullet$ induces a product, denoted still by $\bullet$, in $\tilde{A}(V)$.

When $n = 1$, the left-hand side of (6.8) is $(u_1\bullet u_2)\bullet u_3$ and the right-hand side of (6.8) is a sum of $u_1\bullet(u_2\bullet u_3)$ and elements of $\tilde{O}(V)$. Thus the product $\bullet$ in $\tilde{A}(V)$ is associative.

It is obvious that $\mathbf{1}+\tilde{O}(V)$ is the identity of $\tilde{A}(V)$.

For $u, v\in V$, using the $L(-1)$-derivative property and the property of Res_x, we have

$$
\begin{aligned}
(L(-1)u)\bullet v &= \mathrm{Res}_x \frac{2\pi i e^{2\pi i x}}{e^{2\pi i x}-1} Y(L(-1)u,x)v\\
&= \mathrm{Res}_x \frac{2\pi i e^{2\pi i x}}{e^{2\pi i x}-1}\frac{d}{dx} Y(u,x)v\\
&= -\mathrm{Res}_x \frac{d}{dx}\frac{2\pi i e^{2\pi i x}}{e^{2\pi i x}-1} Y(u,x)v\\
&= \mathrm{Res}_x \frac{(2\pi i)^2 e^{2\pi i x}}{(e^{2\pi i x}-1)^2} Y(u,x)v\\
&\in \tilde{O}(V).
\end{aligned}
$$

In particular, we have $L(-1)u = (L(-1)u)\bullet\mathbf{1}\in\tilde{O}(V)$.

For $u, v\in V$, using this fact and skew-symmetry, we have

$$
\begin{aligned}
u\bullet v &= \mathrm{Res}_x \frac{2\pi i e^{2\pi i x}}{e^{2\pi i x}-1} Y(u,x)v\\
&= \mathrm{Res}_x \frac{2\pi i e^{2\pi i x}}{e^{2\pi i x}-1} e^{xL(-1)}Y(v,-x)u\\
&\equiv \mathrm{Res}_x \frac{2\pi i e^{2\pi i x}}{e^{2\pi i x}-1} Y(v,-x)u \mod \tilde{O}(V)\\
&= \mathrm{Res}_y \frac{-2\pi i e^{-2\pi i y}}{e^{-2\pi i y}-1} Y(v,y)u
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{Res}_y \left(\frac{2\pi i e^{2\pi i y}}{e^{2\pi i y}-1} - 2\pi i \right) Y(v,y)u \\
&= v \bullet u - 2\pi i u_0 v.
\end{aligned}
$$

Since $\omega_0 = L(-1)$, we have

$$
\begin{aligned}
\omega \bullet u &\equiv u \bullet \omega - 2\pi i \omega_0 u \mod \tilde{O}(V) \\
&\equiv u \bullet \omega - 2\pi i L(-1) u \mod \tilde{O}(V) \\
&\equiv u \bullet \omega \mod \tilde{O}(V).
\end{aligned}
$$

∎

Remark 6.2 The product $\bullet$ has a very clear geometric meaning. We know that vertex operators correspond to the sphere $\mathbb{C} \cup \{\infty\}$ with the negatively oriented puncture ∞ and the positively oriented and ordered puncture z and 0 and with the standard locally coordinates vanishing at these punctures (see [H3]). The variable in the vertex operators corresponds to the position of the first oriented puncture z. We can restrict z to be in some subsets of $\mathbb{C} \setminus \{0\}$ to get vertex operators defined only locally. Since Riemann surfaces are constructed locally using subsets of $\mathbb{C}$, we have vertex operators locally on any Riemann surfaces (see, for example, [FB]). The map $y \mapsto \frac{1}{2\pi i}\log(1+y)$ in fact maps an annulus in the sphere $\mathbb{C} \cup \{\infty\}$ to a parallelogram in the universal covering of the torus corresponding to the parallelogram. So the product $\bullet$ can be understood as the constant term of the pull-back of the local vertex operator map on the torus by this map to the annulus. Such pull-backs of local vertex operators are needed because we want to construct (global) correlation functions on one Riemann surface (a torus) from the (global) correlation functions on another Riemann surface (the sphere), not just want to study local vertex operators on a single Riemann surface or sheaves of vertex operators obtained from local vertex operators.

Proposition 6.3 *The associative algebra $\tilde{A}(V)$ is isomorphic to the associative algebra $A(V)$ constructed by Zhu in [Z].*

Proof. For $u, v \in V$ $n \in \mathbb{Z}_+$, by (1.5), we have

$$
\begin{aligned}
&\mathcal{U}(1)\mathrm{Res}_y y^{-n} Y\left(u, \frac{1}{2\pi i}\log(1+y)\right) v \\
&\quad = \mathrm{Res}_y y^{-n} Y\left((1+y)^{L(0)}\mathcal{U}(1)u, y\right) \mathcal{U}(1)v.
\end{aligned}
$$

By this formula in the cases of $n \geq 2$ and the fact that $\mathcal{U}(1)$ is invertible, we have

$$\mathcal{U}(1)(\tilde{O}(V)) = O(V).$$

Thus $\mathcal{U}(1)$ induces a linear isomorphism from $\tilde{A}(V)$ to $A(V)$.

For $u, v \in V$, using the formula above in the case of $n = 1$, we have

$$\begin{aligned}
\mathcal{U}(1)(u \bullet v) &= \mathrm{Res}_y y^{-1} \mathcal{U}(1) Y\left(u, \frac{1}{2\pi i}\log(1+y)\right) v \\
&= \mathrm{Res}_y y^{-1} Y\left((1+y)^{L(0)}\mathcal{U}(1)u, y\right)\mathcal{U}(1)v \\
&= (\mathcal{U}(1)u) * (\mathcal{U}(1)v),
\end{aligned}$$

which shows that the linear isomorphism induced by $\mathcal{U}(1)$ is in fact an isomorphism of algebras. ∎

We need the notion of $\mathbb{N}$-gradable weak V-module (which was called V-module in [Z]). A *weak V-module* is a vector space W equipped with a vertex operator map

$$\begin{aligned}
Y : V \otimes W &\to W[[x, x^{-1}]], \\
u \otimes w &\mapsto Y(u, x)w = \sum_{n \in \mathbb{Z}} u_n w x^{-n-1}
\end{aligned}$$

satisfying all the axioms for V-modules except for those involving the grading. An *$\mathbb{N}$-gradable weak V-module* is a weak V-module such that there exists an $\mathbb{N}$-grading $W = \coprod_{n \in \mathbb{N}} W_{(n)}$ satisfying the condition that $u_n w \in W_{(\mathrm{wt}\ u - n - 1 + \mathrm{wt}\ w)}$ for homogeneous $u \in V$, $w \in W$ and $n \in \mathbb{Z}$.

Let W be an $\mathbb{N}$-gradable weak V-module and

$$T(W) = \{w \in W \mid u_n w = 0, u \in V, \mathrm{wt}\ u - n - 1 < 0\}.$$

Let $P_{T(W)} : W \to T(W)$ be the projection from W to $T(W)$. For any $u \in V$, we define $\rho_W(u) : T(W) \to T(W)$ by

$$\begin{aligned}
\rho_W(u)w &= P_{T(W)}(o(\mathcal{U}(1)u)w) \\
&= \mathrm{Res}_x x^{-1} P_{T(W)}(Y(\mathcal{U}(x)u, x)w)
\end{aligned}$$

for $w \in T(W)$.

In remaining part of this section, we assume for simplicity that $V_{(n)} = 0$ for $n < 0$.

Proposition 6.4 *For any $\mathbb{N}$-gradable weak V-module W, $\rho_W(u) = 0$ for $u \in \tilde{O}(V)$ and the map given by $u + \tilde{O}(V) \mapsto \rho_W(u)$ for $u \in V$ gives $T(W)$ an $\tilde{A}(V)$-module structure. The functor T from the category of $\mathbb{N}$-gradable weak V-modules to the category of $\tilde{A}(V)$-modules given by $W \mapsto T(W)$ has a right inverse, that is, there exists a functor S from the category of $\tilde{A}(V)$-modules to the category of $\mathbb{N}$-gradable weak V-modules such that $TS = I$, where I is the identity functor on the category of $\tilde{A}(V)$-modules. In particular, for any $\tilde{A}(V)$-module M, $T(S(M)) = M$. Moreover we can find such an S such that for any $\mathbb{N}$-gradable weak V-module W, there exists a natural surjective homomorphism of V-modules from $S(T(W))$ to the $\mathbb{N}$-gradable weak V-submodule of W generated by $T(W)$.*

Proof. For $n \in \mathbb{Z}_+$, $u, v \in V$ and $w \in T(W)$, using the definitions, (1.5), the $L(0)$-conjugation formula, the Jacobi identity and the assumption that the weight of a nonzero element of V is nonnegative, we have

$$
\begin{aligned}
&\rho\left(\mathrm{Res}_{x_0}\frac{2\pi i e^{2\pi i x_0}}{(e^{2\pi i x_0}-1)^n}Y(u,x_0)v\right)w\\
&\quad= \mathrm{Res}_{x_2}\frac{1}{x_2}\mathrm{Res}_{x_0}\frac{2\pi i e^{2\pi i x_0}}{(e^{2\pi i x_0}-1)^n}P_{T(W)}(Y(\mathcal{U}(x_2)Y(u,x_0)v,x_2)w)\\
&\quad= \mathrm{Res}_{x_2}\mathrm{Res}_{x_0}\frac{2\pi i e^{2\pi i x_0}}{x_2(e^{2\pi i x_0}-1)^n}\cdot\\
&\qquad\quad \cdot P_{T(W)}(Y(x_2^{L(0)}Y(\mathcal{U}(e^{2\pi i x_0})u, e^{2\pi i x_0}-1)\mathcal{U}(1)v,x_2)w)\\
&\quad= \mathrm{Res}_{x_2}\mathrm{Res}_{x_0}\frac{2\pi i e^{2\pi i x_0}}{x_2(e^{2\pi i x_0}-1)^n}\cdot\\
&\qquad\quad \cdot P_{T(W)}(Y(Y(x_2^{L(0)}\mathcal{U}(e^{2\pi i x_0})u, x_2(e^{2\pi i x_0}-1))x_2^{L(0)}\mathcal{U}(1)v,x_2)w)\\
&\quad= \mathrm{Res}_{x_2}\mathrm{Res}_{y_0}\frac{1}{x_2y_0^n}P_{T(W)}(Y(Y(\mathcal{U}(x_2+y_0)u,y_0)\mathcal{U}(x_2)v,x_2)w)\\
&\quad= \mathrm{Res}_{x_2}\mathrm{Res}_{y_0}\mathrm{Res}_{x_1}\frac{1}{x_2y_0^n}y_0^{-1}\delta\left(\frac{x_1-x_2}{y_0}\right)\cdot\\
&\qquad\quad \cdot P_{T(W)}(Y(\mathcal{U}(x_2+y_0)u,x_1)Y(\mathcal{U}(x_2)v,x_2)w)\\
&\qquad -\mathrm{Res}_{x_2}\mathrm{Res}_{y_0}\mathrm{Res}_{x_1}\frac{1}{x_2y_0^n}y_0^{-1}\delta\left(\frac{x_2-x_1}{-y_0}\right)\cdot\\
&\qquad\quad \cdot P_{T(W)}(Y(\mathcal{U}(x_2)v,x_2)Y(\mathcal{U}(x_2+y_0)u,x_1)w)\\
&\quad= \mathrm{Res}_{x_2}\mathrm{Res}_{x_1}\frac{1}{x_2(x_1-x_2)^n}P_{T(W)}(Y(\mathcal{U}(x_1)u,x_1)Y(\mathcal{U}(x_2)v,x_2)w)
\end{aligned}
$$

$$-\mathrm{Res}_{x_2}\mathrm{Res}_{x_1}\frac{1}{x_2(-x_2+x_1)^n}P_{T(W)}(Y(\mathcal{U}(x_2)v,x_2)Y(\mathcal{U}(x_1)u,x_1)w)$$
$$= \mathrm{Res}_{x_2}\mathrm{Res}_{x_1}\frac{1}{x_2(x_1-x_2)^n}P_{T(W)}(Y(\mathcal{U}(x_1)u,x_1)Y(\mathcal{U}(x_2)v,x_2)w). \tag{6.9}$$

When $n \geq 2$, the left-hand side of (6.9) spans $\rho(\tilde{O}(V))$ and the right-side of (6.9) is equal to 0. When $n = 1$, the left-hand and right-hand sides of (6.9) are equal to $\rho(u \bullet v)w$ and $\rho(u)\rho(v)w$, respectively. Thus the first conclusion is proved.

To prove the second conclusion, we need to construct an $\mathbb{N}$-gradable weak V-module from an $\tilde{A}(V)$-module. Consider the affinization

$$V[t,t^{-1}] = V \otimes \mathbb{C}[t,t^{-1}] \subset V \otimes \mathbb{C}((t)) \subset V \otimes \mathbb{C}[[t,t^{-1}]] \subset V[[t,t^{-1}]]$$

of V and the tensor algebra $\mathcal{T}(V[t,t^{-1}])$ generated by $V[t,t^{-1}]$. For simplicity, we shall denote $u \otimes t^m$ for $u \in V$ and $m \in \mathbb{Z}$ by $u(m)$ and we shall omit the tensor product sign $\otimes$ when we write an element of $\mathcal{T}(V[t,t^{-1}])$. Thus $\mathcal{T}(V[t,t^{-1}])$ is spanned by elements of the form $u_1(m_1)\cdots u_k(m_k)$ for $u_i \in V$ and $m_i \in \mathbb{Z}$, $i = 1,\dots,k$.

Let M be an $\tilde{A}(V)$-module and let $\rho : \tilde{A}(V) \to \mathrm{End}\ M$ be the map giving the representation of $\tilde{A}(V)$ on M. Consider $\mathcal{T}(V[t,t^{-1}]) \otimes M$. Again for simplicity we shall omit the tensor product sign. So $\mathcal{T}(V[t,t^{-1}]) \otimes M$ is spanned by elements of the form $u_1(m_1)\cdots u_k(m_k)w$ for $u_i \in V$, $m_i \in \mathbb{Z}$, $i = 1,\dots,k$, and $w \in M$ and for any $u \in V$, $m \in \mathbb{Z}$, $u(m)$ acts from the left on $\mathcal{T}(V[t,t^{-1}]) \otimes M$. For homogeneous $u_i \in V$, $m_i \in \mathbb{Z}$, $i = 1,\dots,k$, and $w \in M$, we define the *degree* of $u_1(m_1)\cdots u_k(m_k)w$ to be $(\mathrm{wt}\ u_1 - m_1 - 1) + \cdots + (\mathrm{wt}\ u_k - m_k - 1)$. For any $u \in V$, let

$$Y_t(u,x) : \mathcal{T}(V[t,t^{-1}]) \otimes M \to (\mathcal{T}(V[t,t^{-1}]) \otimes M)[[x,x^{-1}]]$$

be defined by

$$Y_t(u,x) = \sum_{m\in\mathbb{Z}} u(m)x^{-m-1}.$$

For a homogeneous element $u \in V$, let $o_t(u) = u(\mathrm{wt}\ u - 1)$. Using linearity, we extend $o_t(u)$ to nonhomogeneous u.

Let $\mathcal{I}$ be the $\mathbb{Z}$-graded $\mathcal{T}(V[t,t^{-1}])$-submodule of $\mathcal{T}(V[t,t^{-1}]) \otimes M$ generated by elements of the forms $u(m)w$ ($u \in V$, $\mathrm{wt}\ u - m - 1 < 0$, $w \in M$),

$o_t(\mathcal{U}(1)u)w - \rho(u + \tilde{O}(V))w$ ($u \in V$, $w \in M$) and the coefficients in x_1 and x_2 of

$$Y_t(u, x_1)Y_t(v, x_2)w - Y_t(v, x_2)Y_t(u, x_1)w$$
$$-\mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y_t(Y(u, x_0)v, x_2)w$$

($u, v \in V$ and $w \in \mathcal{T}(V[t, t^{-1}]) \otimes M$). (Note that the coefficients of the formal expression above are indeed in $\mathcal{T}(V[t, t^{-1}]) \otimes M$.) Let $S_1(M) = (\mathcal{T}(V[t, t^{-1}]) \otimes M)/\mathcal{I}$. Then $S_1(M)$ is also a $\mathbb{Z}$-graded $\mathcal{T}(V[t, t^{-1}])$-module. In fact, by definition of $\mathcal{I}$, we see that $S_1(M)$ is spanned by elements of the form $u_1(m_1) \cdots u_k(m_k)w + \mathcal{I}$ for homogeneous $u_i \in V$, $m_i < \mathrm{wt}\ u_i - 1$, $i = 1, \dots, k$ and $w \in M$. In particular, we see that $S_1(M)$ has an $\mathbb{N}$-grading. Thus for $u \in V$ and $w \in S_1(M)$, $u(m)w = 0$ when m is sufficiently large.

Let $\mathcal{J}$ be the $\mathbb{N}$-graded $\mathcal{T}(V[t, t^{-1}])$-submodule of $S_1(M)$ generated by the coefficients in x of

$$Y_t(L(-1)u, x)w - \frac{d}{dx} Y_t(u, x)w$$

and the coefficients in x_0 and x_2 of

$$Y_t(Y(u, x_0)v, x_2)w - \mathrm{Res}_{x_1} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_t(u, x_1)Y_t(v, x_2)w$$
$$+x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_t(v, x_2)Y_t(u, x_1)w$$

(which are indeed in $S_1(M)$) for $u, v \in V$, $w \in S_1(M)$.

Let $S(M) = S_1(M)/\mathcal{J}$. Then $S(M)$ is also an $\mathbb{N}$-graded $\mathcal{T}(V[t, t^{-1}])$-module. We can still use elements of $\mathcal{T}(V[t, t^{-1}]) \otimes M$ to represent elements of $S(M)$. But note that these elements now satisfy relations. We equip $S(M)$ with the vertex operator map $Y : V \otimes S(M) \to S(M)[[x, x^{-1}]]$ given by $u \otimes w \mapsto Y(u, x)w = Y_t(u, x)w$. As in $S_1(M)$, for $u \in V$ and $w \in S(M)$, we also have $u(m)w = 0$ when $m > \mathrm{wt}\ u - 1$. Clearly $Y(\mathbf{1}, x) = I_{S(M)}$ (where $I_{S(M)}$ is the identity operator on $S(M)$). By definition, we know that the commutator formula, the associator formula and the $L(-1)$-derivative property all hold. Thus $S(M)$ is an $\mathbb{N}$-gradable weak V-module such that $T(S(M)) = M$.

Let W be an $\mathbb{N}$-gradable weak V-module. We define a linear map from $S(T(W))$ to W by mapping $u_1(m_1) \cdots u_k(m_k)w$ of $S(M)$ to $(u_1)_{m_1} \cdots (u_k)_{m_k} w$

of W for $u_i \in V$, $m_i \in \mathbb{Z}$ $(i = 1, \dots, k)$ and $w \in T(W)$. Note that the relations among $u_1(m_1) \cdots u_k(m_k)w$ for $u_i \in V$, $m_i \in \mathbb{Z}$ $(i = 1, \dots, k)$ and $w \in T(W)$ are given by just the action of V on $T(W)$, the commutator formula, the associator formula and the $L(-1)$-derivative property for vertex operators. These relations also hold in W. Thus this map is well-defined. Clearly, this is a surjective homomorphism of V-modules from $S(T(W))$ to the $\mathbb{N}$-gradable weak V-submodule of W generated by $T(W)$. ∎

Proposition 6.5 *Assume that every $\mathbb{N}$-gradable weak V-module is completely reducible. Then T and S are equivalences of categories. In particular, $\tilde{A}(V)$ is semi-simple and M is an irreducible $\tilde{A}(V)$-module if and only if $S(M)$ is an irreducible V-module.*

Proof. We need only prove that for any $\mathbb{N}$-gradable weak V-module W, $S(T(W))$ is naturally isomorphic to W. Since every $\mathbb{N}$-gradable weak V-module is completely reducible, we need only to consider the case that W is irreducible.

Let W be an irreducible $\mathbb{N}$-gradable weak V-module. Then by assumption, $S(T(W))$ is completely reducible. If $S(T(W))$ is not irreducible, then there exist nonzero V-modules W_1 and W_2 such that $S(T(W)) = W_1 \oplus W_2$. In particular, we have $T(W) = T(S(T(W))) = T(W_1) \oplus T(W_2)$. Here $T(W_1)$ and $T(W_2)$ are both nonzero because both W_1 and W_2 are nonzero. we have a V-submodule $\tilde{W}_1$ of W generated by $T(W_1)$. This V-submodule $\tilde{W}_1$ is obviously nonzero because $T(W_1)$ is nonzero. It is also not W since $T(W_2)$ is not in $\tilde{W}_1$. Thus W is not irreducible. Contradiction. So $S(T(W))$ is irreducible. Since W is irreducible, the natural homomorphism in Proposition 6.4 from $S(T(W))$ to W is surjective. This homomorphism must also be injective because $S(T(W))$ is irreducible. Thus $S(T(W))$ is naturally isomorphic to W. ∎

Corollary 6.6 *If every $\mathbb{N}$-gradable weak V-module is completely reducible, then there are only finitely many inequivalent irreducible V-modules.*

Proof. By Proposition 6.5, $\tilde{A}(V)$ is semisimple. Thus there are only finitely many inequivalent irreducible $\tilde{A}(V)$-modules. By Proposition 6.5 again, these finitely many irreducible $\tilde{A}(V)$-modules are mapped under S to a complete set of inequivalent V-modules. Thus there are only finitely many inequivalent irreducible V-modules. ∎

Given an $\mathbb{N}$-gradable weak V-module W, we can also construct an $\tilde{A}(V)$-bimodule: For $u \in V$ and $w \in W$, we define

$$\begin{aligned} u \bullet w &= \mathrm{Res}_y y^{-1} Y\left(u, \frac{1}{2\pi i}\log(1+y)\right) w \\ &= \mathrm{Res}_x \frac{2\pi i e^{2\pi i x}}{e^{2\pi i x}-1} Y(u,x)w, \\ w \bullet u &= \mathrm{Res}_y y^{-1} e^{\frac{1}{2\pi i}\log(1+y)L(-1)} Y\left(u, -\frac{1}{2\pi i}\log(1+y)\right) w \\ &= \mathrm{Res}_x \frac{2\pi i e^{2\pi i x}}{e^{2\pi i x}-1} e^{xL(-1)} Y(u,-x)w. \end{aligned}$$

Let $\tilde{O}(W)$ be the subspace of V spanned by elements of the form

$$\mathrm{Res}_y y^{-n} Y\left(u, \frac{1}{2\pi i}\log(1+y)\right) w = \mathrm{Res}_x \frac{2\pi i e^{2\pi i x}}{(e^{2\pi i x}-1)^n} Y(u,x)w$$

for $n \in \mathbb{Z}_+ + 1$ and $u \in V$ and $w \in W$. Let $\tilde{A}(W) = W/\tilde{O}(W)$.

Proposition 6.7 *The left and right actions of V on W induce an $\tilde{A}(V)$-bimodule structure on $\tilde{A}(W)$.*

Proof. The proof is the same as the proof that $\tilde{A}(V)$ is an associative algebra above. ∎

We now assume that every $\mathbb{N}$-gradable weak V-modules is completely reducible. It is easy to show that an irreducible $\mathbb{N}$-gradable weak V-module must be a V-module (see [Z]). Thus we can reduce the study of $\mathbb{N}$-gradable weak V-modules to the study of V-modules.

Let W_1, W_2 and W_3 be V-modules and $\mathcal{Y}$ an intertwining operator of type $\binom{W_3}{W_1 W_2}$. Then $\tilde{A}(W_1) \otimes_{\tilde{A}(V)} T(W_2)$ and $T(W_3)$ are both left $\tilde{A}(V)$-modules. For homogeneous $w_1 \in W_1$, let

$$o_{\mathcal{Y}}(w_1) = \mathcal{Y}_{\mathrm{wt}\ w_1 - 1}(w_1),$$

where for $n \in \mathbb{C}$ and $w_1 \in W_1$, $\mathcal{Y}_n(w_1) \in \mathrm{Hom}(W_2, W_3)$ is given by

$$\mathcal{Y}(w_1, x) = \sum_{n \in \mathbb{C}} \mathcal{Y}_n(w_1) x^{-n-1}.$$

We define $o_{\mathcal{Y}}(w_1)$ for general $w_1 \in W_1$ by linearity.

Lemma 6.8 *For $w_1 \in \tilde{O}(W_1)$, $o_{\mathcal{Y}}(\mathcal{U}(1)w_1) = 0$.*

Proof. The proof is the same as the one for the first conclusion in Proposition 6.4. ∎

Let

$$\rho(\mathcal{Y}) : \tilde{A}(W_1) \otimes_{\tilde{A}(V)} T(W_2) \to W_3$$

be defined by

$$\begin{aligned}\rho(\mathcal{Y})((w_1 + \tilde{O}(W_1)) \otimes w_2) &= o_{\mathcal{Y}}(\mathcal{U}(1)w_1)w_2 \\ &= \mathrm{Res}_x x^{-1}\mathcal{Y}(\mathcal{U}(x)w_1, x)w_2\end{aligned}$$

for $w_1 \in W_1, w_2 \in T(W_2)$. By Lemma 6.8, $\rho(\mathcal{Y})$ is indeed well defined. We have:

Proposition 6.9 *The image of $\rho(\mathcal{Y})$ is in fact in $T(W_3)$ and $\rho(\mathcal{Y})$ is in fact in*

$$\mathrm{Hom}_{\tilde{A}(V)}(\tilde{A}(W_1) \otimes_{\tilde{A}(V)} T(W_2), T(W_3)).$$

The map

$$\begin{aligned}\rho : \mathcal{V}_{W_1W_2}^{W_3} &\to \mathrm{Hom}_{\tilde{A}(V)}(\tilde{A}(W_1) \otimes_{\tilde{A}(V)} T(W_2), T(W_3)) \\ \mathcal{Y} &\mapsto \rho(\mathcal{Y})\end{aligned}$$

is a linear isomorphism.

Proof. For any $w_1 \in W_1$, since the weight of $o_{\mathcal{Y}}(\mathcal{U}(1)w_1)$ is 0, it is clear that the image of $\rho(\mathcal{Y})$ is in fact in $T(W_3)$. Thus ρ is a linear map from $\mathcal{V}_{W_1W_2}^{W_3}$ to $\mathrm{Hom}_{\tilde{A}(V)}(\tilde{A}(W_1) \otimes_{\tilde{A}(V)} T(W_2), T(W_3))$. We still need to show that it is in fact an isomorphism.

To show that ρ is injective, we need the following obvious fact: For any $\tilde{w}_1 \in W_1$, $\tilde{w}_2 \in W_2$ and $\tilde{w}_3'$, using the Jacobi identity and (1.5), we can always write $\langle \tilde{w}_3', \mathcal{Y}(\tilde{w}_1, x)\tilde{w}_2\rangle$ as a linear combination of series of the form $\langle w_3', \mathcal{Y}(w_1, x)w_2\rangle$ for $w_1 \in W_1$, $w_2 \in T(W_2)$ and $w_3' \in T(W_3')$ with Laurent polynomials of x as coefficients. If $\rho(\mathcal{Y}) = 0$, $\langle w_3', \mathcal{Y}(w_1, x)w_2\rangle = 0$ for $w_1 \in W_1$, $w_2 \in T(W_2)$ and $w_3' \in T(W_3')$. Thus the fact above shows that $\mathcal{Y} = 0$. So ρ is injective.

We now prove that ρ is surjective. Given any element f of

$$\mathrm{Hom}_{\tilde{A}(V)}(\tilde{A}(W_1) \otimes_{\tilde{A}(V)} T(W_2), T(W_3)),$$

we want to construct an element $\mathcal{Y}^f$ of $\mathcal{V}_{W_1 W_2}^{W_3}$ such that $\rho(\mathcal{Y}^f) = f$. We assume that W_1, W_2, W_3 and thus W_3' are all irreducible V-modules. The general case follows from this case by using the assumption that every $\mathbb{N}$-gradable weak V-module is completely reducible. Since W_1, W_2 and W_3 are irreducible V-modules, there exists $h_1, h_2, h_3 \in \mathbb{C}$ such that the weights of nonzero homogeneous elements of W_1, W_2 and W_3 are in $h_1 + \mathbb{N}$, $h_2 + \mathbb{N}$ and $h_3 + \mathbb{N}$, respectively, and $(W_1)_{(h_1)}$, $(W_3)_{(h_3)}$ and $(W_1)_{(h_3)}$ are not 0. Let $h = h_3 - h_1 - h_2$. Then we know that for any intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$, $\mathcal{Y}(w_1, x)w_2 \in x^h W_3[[x, x^{-1}]]$.

We consider the affinization $V[t, t^{-1}]$ and also

$$t^{-h}W_1[t, t^{-1}] = W_1 \otimes t^{-h}\mathbb{C}[t, t^{-1}].$$

For simplicity, we shall use $u(m)$ and $w_1(n)$ to denote $u \otimes t^m$ and $w_1 \otimes t^n$, respectively, for $u \in V$, $w_1 \in W_1$, $m \in \mathbb{Z}$ and $n \in h + \mathbb{Z}$. We consider the tensor algebra $\mathcal{T}(V[t, t^{-1}] \oplus t^{-h}W_1[t, t^{-1}])$ generated by $V[t, t^{-1}]$ and $t^{-h}W_1[t, t^{-1}]$. The tensor algebra $\mathcal{T}(V[t, t^{-1}])$ is a subalgebra of $\mathcal{T}(V[t, t^{-1}] \oplus t^{-h}W_1[t, t^{-1}])$ and $t^{-h}W_1[t, t^{-1}]$ is a subspace. Let $\mathcal{T}_{V;W_1}$ be the $\mathcal{T}(V[t, t^{-1}])$-sub-bimodule of $\mathcal{T}(V[t, t^{-1}] \oplus t^{-h}W_1[t, t^{-1}])$ generated by $t^{-h}W_1[t, t^{-1}]$. For simplicity we shall omit the tensor product sign for elements.

Consider $\mathcal{T}_{V;W_1} \otimes T(W_2)$. For simplicity again we shall omit the tensor product sign for elements. So $\mathcal{T}_{V;W_1} \otimes T(W_2)$ is spanned by elements of the form

$$u_1(m_1) \cdots u_k(m_k) w_1(n) u_{k+1}(m_{k+1}) \cdots u_{k+l}(m_{k+l}) w_2$$

for $u_i \in V$, $m_i \in \mathbb{Z}$, $i = 1, \ldots, k+l$, $w_1 \in W_1$ and $w_2 \in T(W_2)$. For any $u \in V$, $m \in \mathbb{Z}$, $u(m)$ acts from the left on $\mathcal{T}_{V;W_1} \otimes T(W_2)$. For homogeneous $u_i \in V$, $m_i \in \mathbb{Z}$, $i = 1, \ldots, k+l$, and homogeneous $w_1 \in W_1$ and $w_2 \in T(W_2)$, we define the weight of

$$u_1(m_1) \cdots u_k(m_k) w_1(n) u_{k+1}(m_{k+1}) \cdots u_{k+l}(m_{k+l}) w_2$$

to be

$$(\mathrm{wt}\ u_1 - m_1 - 1) + \cdots + (\mathrm{wt}\ u_k - m_{k+l} - 1) + (\mathrm{wt}\ w_1 - n - 1) + \mathrm{wt}\ w_2.$$

For any $u \in V$ and $w_1 \in W_1$, let

$$\begin{aligned} Y_t(u,x) : \mathcal{T}_{V;W_1} \otimes T(W_2) &\to (\mathcal{T}_{V;W_1} \otimes T(W_2))[[x,x^{-1}]], \\ \mathcal{Y}_t(w_1,x) : \mathcal{T}_{V;W_1} \otimes T(W_2) &\to x^h(\mathcal{T}_{V;W_1} \otimes T(W_2))[[x,x^{-1}]] \end{aligned}$$

be defined by

$$\begin{aligned} Y_t(u,x) &= \sum_{m\in\mathbb{Z}} u(m)x^{-m-1}, \\ Y_t(w_1,x) &= \sum_{n\in h+\mathbb{Z}} w_1(n)x^{-n-1}, \end{aligned}$$

respectively. For homogeneous elements $u \in V$ and $w_1 \in W_1$, let $o_t(u) = u(\mathrm{wt}\ u - 1)$ and $o_t(w_1) = w_1(\mathrm{wt}\ w_1 - 1)$. Using linearity, we extend $o_t(u)$ and $o_t(w_1)$ to nonhomogeneous u and w_1.

Let $\mathcal{I}_{V;W_1,W_2}$ be the $h_1+h_2+\mathbb{Z}$-graded $\mathcal{T}(V[t,t^{-1}])$-submodule of $\mathcal{T}_{V;W_1} \otimes T(W_2)$ generated by elements of the forms $u(m)w_2$ ($u \in V$, $m \in \mathbb{Z}$, $\mathrm{wt}\ u - m - 1 < 0$, $w_2 \in T(W_2)$), $w_1(n)w_2$ ($w_1 \in W_1$, $n \in \mathbb{Z} - h$, $\mathrm{wt}\ w_1 - n - 1 + \mathrm{wt}\ w_2 < h_3$, $w_2 \in T(W_2)$), $o_t(\mathcal{U}(1)u)w_2 - \rho(u + \tilde{O}(V))w_2$ ($u \in V$, $w_2 \in T(W_2)$), and the coefficients in x_1 and x_2 of

$$\begin{aligned} &Y_t(u,x_1)Y_t(v,x_2)w - Y_t(v,x_2)Y_t(u,x_1)w \\ &\quad -\mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) Y_t(Y(u,x_0)v,x_2)w, \\ &Y_t(u,x_1)\mathcal{Y}_t(w_1,x_2)w_2 - \mathcal{Y}_t(w_1,x_2)Y_t(u,x_1)w \\ &\quad -\mathrm{Res}_{x_0} x_2^{-1}\delta\left(\frac{x_1-x_0}{x_2}\right) \mathcal{Y}_t(Y(u,x_0)w_1,x_2)w_2 \end{aligned}$$

($u, v \in V$, $w_1 \in W_1$ and $w_2 \in \mathcal{T}_{V;W_1} \otimes T(W_2)$). Let

$$S_1(V;W_1,W_2) = (\mathcal{T}_{V;W_1} \otimes T(W_2))/\mathcal{I}_{V;W_1,W_2}.$$

Then $S_1(V;W_1,W_2)$ is also an $h_1+h_2+\mathbb{Z}$-graded $\mathcal{T}(V[t,t^{-1}])$-module. In fact, by definition of $\mathcal{I}_{V;W_1,W_2}$, $S_1(V;W_1,W_2)$ is spanned by elements of the form

$$u_1(m_1)\cdots u_k(m_k)w_1(n)w_2 + \mathcal{I}_{V;W_1,W_2}$$

for homogeneous $u_i \in V$, $m_i < \mathrm{wt}\ u_i - 1$, $i = 1, \ldots, k$, homogeneous $w_1 \in W_1$, $n \leq \mathrm{wt}\ w_1 - 1 + \mathrm{wt}\ w_2 - h_3$ and $w_2 \in T(W_2)$. In particular, we see that

$S_1(V;W_1,W_2)$ has an $\mathbb{N}$-grading. Thus for $u \in V$, $w_1 \in W_1$ and $w_2 \in S_1(V;W_1,W_2)$, $u(m)w_2 = 0$ and $w_1(n)w_2 = 0$ when m and n are sufficiently large.

Let $\mathcal{J}_{V;W_1,W_2}$ be the $h_1+h_2+\mathbb{Z}$-graded $\mathcal{T}(V[t,t^{-1}])$-submodule of $S_1(V;W_1,W_2)$ generated by the coefficients in x of

$$Y_t(L(-1)u,x)w_2 - \frac{d}{dx}Y_t(u,x)w_2,$$
$$\mathcal{Y}_t(L(-1)w_1,x)w_2 - \frac{d}{dx}\mathcal{Y}_t(w_1,x)w_2$$

and the coefficients in x_0 and x_2 of

$$Y_t(Y(u,x_0)v,x_2)w_2 - \mathrm{Res}_{x_1}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_t(u,x_1)Y_t(v,x_2)w_2$$
$$+x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_t(v,x_2)Y_t(u,x_1)w_2,$$
$$\mathcal{Y}_t(Y(u,x_0)w_1,x_2)w_2 - \mathrm{Res}_{x_1}x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_t(u,x_1)\mathcal{Y}_t(w_1,x_2)w_2$$
$$+x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\mathcal{Y}_t(w_1,x_2)Y_t(u,x_1)w_2,$$

for $u,v \in V$, $w_1 \in W_1$ and $w_2 \in S_1(V;W_1,W_2)$.

Let $S(V;W_1,W_2) = S_1(V;W_1,W_2)/\mathcal{J}_{V;W_1,W_2}$. Then $S(V;W_1,W_2)$ is also a $\mathcal{T}(V[t,t^{-1}])$-module with an $\mathbb{N}$-grading. We can still use elements of $\mathcal{T}_{V;W_1} \otimes T(W_2)$ to represent elements of $S(M)$. But note that these elements now satisfy relations. Now we have operators $Y_t(u,x)$ and $\mathcal{Y}_t(w_1,x)$ for $u \in V$ and $w_1 \in W_1$ acting on $S(V;W_1,W_2)$. By construction, these operators satisfy the lower truncation property, the identity property (for Y_t), the commutator formula (for Y_t and for Y_t and $\mathcal{Y}_t$), the associator formula (for Y_t and $\mathcal{Y}_t$) and the $L(-1)$-derivative property (for Y_t and $\mathcal{Y}_t$).

We also have a linear map $\mu : S(V;W_1,W_2) \to W_3$ defined as follows: For $u_i \in V$, $m_i < \mathrm{wt}\ u_i - 1$, $i = 1,\ldots,k$, $w_1 \in W_1$, $n \leq \mathrm{wt}\ w_1 - 1$ and $w_2 \in T(W_2)$, we define

$$\mu(u_1(m_1)\cdots u_k(m_k)w_1(n)w_2) = (u_1)_{m_1}\cdots(u_k)_{m_k}f((w_1+\tilde{O}(W_1))\otimes w_2).$$

Since all the relations in $S(V;W_1,W_2)$ for representatives of the form

$$u_1(m_1)\cdots u_k(m_k)w_1(n)w_2$$

above are also satisfied by their images in W_3, μ is well defined.

Now we construct an intertwining operator $\mathcal{Y}^f$ of type $\binom{W_3}{W_1 W_2}$ as follows: By Proposition 6.5, W_2 is isomorphic to $S(T(W_2))$. So we can work with $S(T(W_2))$ instead of W_2. We know that $S(T(W_2))$ is spanned by elements of the form $u_1(m_1)\cdots u_k(m_k)w_2$ for $u_i \in V$, $m_i < \mathrm{wt}\ u_i - 1$ for $i = 1, \ldots, k$ and $w_2 \in T(W_2)$. Let $w_1 \in W_1$. We define

$$\mathcal{Y}^f(w_1, x)u_1(m_1)\cdots u_k(m_k)w_2 = \mu(\mathcal{Y}_t(w_1, x)u_1(m_1)\cdots u_k(m_k)w_2).$$

Since $\mathcal{Y}_t$ satisfies the commutator formula, the associator formula and the $L(-1)$-derivative property, so does $\mathcal{Y}^f$. Thus $\mathcal{Y}^f$ satisfies the Jacobi identity and the $L(-1)$-derivative property. So it is an intertwining operator of the desired type. It is clear from the construction that $\rho(\mathcal{Y}^f) = f$. ∎

Lemma 6.10 *Let A be a semisimple associative algebra over $\mathbb{C}$, ω an element in the center of A, M an A-bimodule and $F : M \to \mathbb{C}$ a linear functional on M satisfying the following property:*

1. *For $u \in A$ and $w \in M$, $F(uw) = F(wu)$.*
2. *There exist $h \in \mathbb{C}$ and $s \in \mathbb{Z}_+$ such that for $w \in M$, $F((\omega - h)^s w) = 0$.*

Then there exist irreducible left A-modules M_i and left module maps

$$\begin{aligned} f_i : M \otimes_A M_i &\to M_i \\ w \otimes_A w_i &\mapsto f_i(w)w_i \end{aligned}$$

for $i = 1, \ldots, m$ such that

$$F(w) = \sum_{i=1}^{m} \mathrm{Tr}_{M_i} f_i(w)$$

for $w \in M$.

Proof. This is a result in the classical theory of semisimple associative algebras. We just give the idea of the proof. It is easy for the reader to fill in all the details using the theory presented in, for example, [J] or [FD]. One can also find a proof in [M1].

Since A is semisimple, it must be a direct sum of simple ideals. Because of this fact, we can reduce our lemma to the case that A is simple. But when A is simple, we know that A is isomorphic to a matrix algebra. The lemma now can be verified directly for simple matrix algebras. ∎

7 Modular invariance

In this section, we prove the modular invariance of the space of chiral genus-one correlation functions using the results we have obtained in the preceding sections.

We first discuss the modular invariance of the system (3.5)–(3.6). We need the following modular transformation formulas (see, for example, [K]): For any

$$\left(\begin{array}{cc}\alpha & \beta\\ \gamma & \delta\end{array}\right)\in SL(2,\mathbb{Z}),$$

we have

$$G_2\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = (\gamma\tau+\delta)^2 G_2(\tau)-2\pi i\gamma(\gamma\tau+\delta), \tag{7.1}$$

$$G_{2k}\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = (\gamma\tau+\delta)^{2k} G_{2k}(\tau), \tag{7.2}$$

$$\wp_m\left(\frac{z}{\gamma\tau+\delta};\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = (\gamma\tau+\delta)^{m}\wp_m(z,\tau), \tag{7.3}$$

for $k\geq 2$ and $m\geq 1$.

We have:

Proposition 7.1 *Let $\varphi(z_1,\dots,z_n;\tau)$ be a solution of the system (3.5)–(3.6) with $q=q_\tau$. Then for any*

$$\left(\begin{array}{cc}\alpha & \beta\\ \gamma & \delta\end{array}\right)\in SL(2,\mathbb{Z}),$$

$$\left(\frac{1}{\gamma\tau+\delta}\right)^{\mathrm{wt}\ w_1+\cdots \mathrm{wt}\ w_n}\varphi\left(\frac{z_1}{c\tau+d},\dots,\frac{z_n}{c\tau+d};\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$$

is also a solution of the system (3.5)–(3.6) with $q=q_\tau$.

Proof. The proof is a straightforward calculation using (7.1)–(7.3). ∎

From this result, we know that the space of solutions of the system (3.5)–(3.6) with $q=q_\tau$ is invariant under the action of the modular group $SL(2,\mathbb{Z})$. But this is not enough for the modular invariance we would like to prove because we want to prove that the space of those solutions obtained from the q_τ-traces of products of geometrically modified intertwining operators are invariant under this action of $SL(2, \mathbb{Z})$.

We need the following:

Theorem 7.2 *Let V be a vertex operator algebra satisfying the following conditions:*

1. *For $n < 0$, $V_{(n)} = 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$.*
2. *Every $\mathbb{N}$-gradable weak V-module is completely reducible.*
3. *V is C_2-cofinite.*

Then all the conclusions of the results in Sections 1–6 hold.

Proof. By Corollary 6.6, there are only finitely many inequivalent irreducible V-modules. By a result of Anderson-Moore [AM] and Dong-Li-Mason [DLM], every irreducible V-module is in fact $\mathbb{Q}$-graded. Also in this case, it is clear that every finitely-generated lower-truncated generalized V-module is a V-module. In [H9], the author proved that for such a vertex operator algebra, the direct sum of a complete set of inequivalent irreducible V-modules has a natural structure of intertwining operator algebra. By a result of Abe, Buhl and Dong [ABD], we also know that for such a vertex operator algebra V, every V-module is C_2-cofinite. Thus the conditions for V needed in Sections 1–6 are all satisfied. ∎

Let V be a vertex operator algebra satisfying the conditions in Theorem 7.2. Let W_i be V-modules and $w_i \in W_i$ for $i = 1, \ldots, n$. For any V-modules $\tilde{W}_i$ and any intertwining operators $\mathcal{Y}_i$, $i = 1, \ldots, n$, of types $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$, respectively, we have a genus-one correlation function

$$\overline{F}_{\mathcal{Y}_1, \ldots, \mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; \tau).$$

Note that these multivalued functions actually have preferred branches in the region $1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q_\tau| > 0$ given by the intertwining operators $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$. Thus linear combinations of these functions make sense. For fixed V-modules W_i and $w_i \in W_i$ for $i = 1, \ldots, n$, we now denote the vector space spanned by all such functions by $\mathcal{F}_{w_1, \ldots, w_n}$. The following theorem is the main result of this section:

Theorem 7.3 *Let V be a vertex operator algebra satisfying the conditions in Theorem 7.2. Then for any V-modules $\tilde{W}_i$ and any intertwining operators $\mathcal{Y}_i$ $(i = 1, \dots, n)$ of types $\binom{\tilde{W}_{i-1}}{W_i \tilde{W}_i}$, respectively, and any*

$$\left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right) \in SL(2, \mathbb{Z}),$$

$$\overline{F}_{\mathcal{Y}_1, \dots, \mathcal{Y}_n}\Bigg(\left(\frac{1}{\gamma\tau + \delta}\right)^{L(0)} w_1, \dots, \left(\frac{1}{\gamma\tau + \delta}\right)^{L(0)} w_n;$$
$$\frac{z_1}{\gamma\tau + \delta}, \dots, \frac{z_n}{\gamma\tau + \delta}; \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\Bigg)$$

is in $\mathcal{F}_{w_1, \dots, w_n}$.

Proof. By Theorem 7.2, all the results in Sections 1–6 can be used.

The case of $n = 1$, as we have mentioned in the introduction, was proved by Miyamoto in [M1] using the method of Zhu in [Z]. Since the differential equations we obtained in this paper are explicitly modular invariant, we have a simpler proof than the one given in [M1]. Here we give this proof. In this case, the identity (3.4) becomes

$$\left(2\pi i \frac{\partial}{\partial \tau} + G_2(\tau) \mathrm{wt}\ w_1 + G_2(\tau) z_1 \frac{\partial}{\partial z_1}\right) F_{\mathcal{Y}_1}(w_1; z_1; q_\tau)$$
$$= F_{\mathcal{Y}_1}(L(-2)w_1; z_1; q_\tau) - \sum_{k \in \mathbb{Z}_+} G_{2k+2}(\tau) F_{\mathcal{Y}_1}(L(2k)w_1; z_1; q_\tau). \quad (7.4)$$

But in this case,

$$\frac{\partial}{\partial z_1} F_{\mathcal{Y}_1}(w_1; z_1; q_\tau)$$
$$= \mathrm{Tr}_{\tilde{W}_1} \mathcal{Y}_1((2\pi i L(0) + 2\pi i q_{z_1} L(-1)) \mathcal{U}(q_{z_1}) w_1, q_{z_1}) q^{L(0) - \frac{c}{24}}$$
$$= 2\pi i \mathrm{Tr}_{\tilde{W}_1} [L(0), \mathcal{Y}_1(\mathcal{U}(q_{z_1}) w_1, q_{z_1})] q^{L(0) - \frac{c}{24}}$$
$$= 0.$$

In other words, $F_{\mathcal{Y}_1}(w_1; z_1; q_\tau)$ is in fact independent of z_1. So from (7.4), we obtain

$$\left(2\pi i \frac{\partial}{\partial \tau} + G_2(\tau) \mathrm{wt}\ w_1\right) F_{\mathcal{Y}_1}(w_1; z_1; q_\tau)$$

$$= F_{\mathcal{Y}_1}\left(L(-2)w_1 - \sum_{k\in\mathbb{Z}_+} G_{2k+2}(\tau)L(2k)w_1; z_1; q_\tau\right). \tag{7.5}$$

We use τ' to denote $\frac{\alpha\tau+\beta}{\gamma\tau+\delta}$ and $z_1' = \frac{z_1}{\gamma\tau+\delta}$. Then (7.5) also holds with τ replaced by τ' and z_1 replaced by z_1'. Using the modular transformation property (7.1) and (7.2) for $G_{2k}(\tau)$ for $k \in \mathbb{Z}_+$ and the fact that $F_{\mathcal{Y}_1}(w_1; z_1; q_\tau)$ is independent of z_1, we see that by a straightforward calculation, (7.5) with τ, z_1 and w_1 replaced by τ', z_1' and $(\gamma\tau+\delta)^{-L(0)}w_1$, respectively, is equivalent to

$$\begin{aligned}
&\left(2\pi i\frac{\partial}{\partial\tau} + G_2(\tau)\mathrm{wt}\ w_1\right) F_{\mathcal{Y}_1}((\gamma\tau+\delta)^{-L(0)}w_1; z_1'; q_{\tau'})\\
&\quad = F_{\mathcal{Y}_1}((\gamma\tau+\delta)^{-L(0)}L(-2)w_1; z_1'; q_{\tau'})\\
&\qquad - \sum_{k\in\mathbb{Z}_+} G_{2k+2}(\tau)F_{\mathcal{Y}_1}((\gamma\tau+\delta)^{-L(0)}L(2k)w_1; z_1'; q_{\tau'})
\end{aligned}$$

or equivalently

$$\begin{aligned}
&2\pi i\frac{\partial}{\partial\tau}F_{\mathcal{Y}_1}((\gamma\tau+\delta)^{-L(0)}w_1; z_1'; q_{\tau'})\\
&\quad = F_{\mathcal{Y}_1}((\gamma\tau+\delta)^{-L(0)}L(-2)w_1; z_1'; q_{\tau'})\\
&\qquad - \sum_{k\in\mathbb{N}} G_{2k+2}(\tau)F_{\mathcal{Y}_1}((\gamma\tau+\delta)^{-L(0)}L(2k)w_1; z_1'; q_{\tau'})
\end{aligned} \tag{7.6}$$

The $n = 1$ cases of the identities (2.2) and (2.22) give

$$F_{\mathcal{Y}_1}((\gamma\tau+\delta)^{-L(0)}u_0w_1; z_1'; q_{\tau'}) = 0 \tag{7.7}$$

and

$$\begin{aligned}
&F_{\mathcal{Y}_1}\Bigg((\gamma\tau+\delta)^{-L(0)}u_{-2}w_1\\
&\qquad\qquad + \sum_{k\in\mathbb{Z}_+}(2k+1)G_{2k+2}(\tau)(\gamma\tau+\delta)^{-L(0)}u_{2k}w_1; z_1'; q_{\tau'}\Bigg)\\
&\quad = 0,
\end{aligned} \tag{7.8}$$

respectively, where in (7.8), we have used (7.2).

Using (2.9), we have

$$L(-2)w_1 - \sum_{k\in\mathbb{N}} G_{2k+2}(\tau)L(2k)w_1$$
$$= \mathrm{Res}_x(\wp_1(x;\tau) - G_2(\tau)x)Y(\omega, x)w_1 \tag{7.9}$$

and

$$u_{-2}w_1 + \sum_{k\in\mathbb{Z}_+}(2k+1)G_{2k+2}(\tau)u_{2k}w_1$$
$$= \mathrm{Res}_x\wp_2(x;\tau)Y(u,x)w_1. \tag{7.10}$$

Using (2.10) and (1.5), we see that the constant terms of the expansions of (7.9) and (7.10) as power series in q_τ are

$$\begin{aligned}&\mathrm{Res}_x\pi i\frac{e^{2\pi ix}+1}{e^{2\pi ix}-1}Y(omega, x)w_1\\ &= \mathrm{Res}_x\pi i\frac{2\pi ie^{2\pi ix}}{e^{2\pi ix}-1}Y(\omega, x)w_1 - \pi i\mathrm{Res}_xY(u,x)w_1\\ &= \omega \bullet w_1 - \pi i\omega_0 w_1\end{aligned} \tag{7.11}$$

and

$$\begin{aligned}&\mathrm{Res}_x\wp_2(x;\tau)Y(u,x)w_1\\ &= \mathrm{Res}_x\left(-\frac{\pi^2}{3} + \frac{(2\pi i)^2e^{2\pi ix}}{(e^{2\pi ix}-1)^2}\right)Y(u,x)w_1\\ &= -\frac{\pi^2}{3}u_0w_1 + 2\pi i\mathrm{Res}_x\frac{2\pi ie^{2\pi ix}}{(e^{2\pi ix}-1)^2}Y(u,x)w_1,\end{aligned} \tag{7.12}$$

respectively.

By Proposition 7.1, we know that $F_{\mathcal{Y}_1}((\gamma\tau+\delta)^{-L(0)}w_1; z_1'; q_{\tau'})$ satisfies the same equation of regular singular points as the one for $F_{\mathcal{Y}_1}(w_1; z_1; q_\tau)$. Thus we have

$$F_{\mathcal{Y}_1}((\gamma\tau+\delta)^{-L(0)}w_1; z_1'; q_{\tau'}) = \sum_{k=0}^{K}\sum_{l=1}^{N}\sum_{m\in\mathbb{N}} C_{k,l,m}(w_1)\tau^k q_\tau^{r_l+m}$$

where r_l for $l = 1, \dots, N$ are real numbers such that $r_{l_1} - r_{l_2} \notin \mathbb{Z}$ when $l_1 \neq l_2$. From (7.6)–(7.12), we obtain

$$\begin{aligned} C_{K,l,0}(u \bullet w_1) &= C_{K,l,0}(w_1 \bullet u), & (7.13)\\ C_{K,l,0}(\tilde{O}(W_1)) &= 0, & (7.14)\\ C_{K,l,0}\left(\left(\omega - \frac{c}{24} - r_l\right) \bullet w_1\right) &= 0. & (7.15)\end{aligned}$$

Thus we see that $C_{K,l,0}$ gives a linear functional on the $\tilde{A}(V)$-module $\tilde{A}(W_1)$ satisfying the conditions in Lemma 6.10. By Lemma 6.10, we can find irreducible left $\tilde{A}(V)$-modules M_i and left $\tilde{A}(V)$-module maps

$$f_i : \tilde{A}(W_1) \otimes_{\tilde{A}(V)} M_i \to M_i$$

such that for $i = 1, \dots, m$ such that

$$C_{K,l,0}(w_1) = \sum_{i=1}^{m} \mathrm{Tr}_{M_i} f_i(w_1)$$

for $w_1 \in W_1$. By Propositions 6.4, 6.5 and 6.9, there exist irreducible V-modules $W_i^{(1)}$ and intertwining operators $\mathcal{Y}_i^{(1)}$ of types $\binom{W_i^{(1)}}{W_1 W_i^{(1)}}$ for $i = 1, \dots, p$ such that $M_i = T(W_i^{(1)})$ and $\mathcal{Y}_i^{(1)}$ correspond to f_i for $i = 1, \dots, p$.

It is clear that

$$\sum_{l=1}^{N} \sum_{m \in \mathbb{N}} C_{K,l,m}(w_1) q_\tau^{r_l+m} - \sum_{i=1}^{p} F_{\mathcal{Y}_i^{(1)}}(w_1; z_1; q_\tau)$$

must be of the form

$$\sum_{l=1}^{N^{(1)}} \sum_{m \in \mathbb{N}} C^{(1)}_{K,l,m}(w_1) q_\tau^{r_l^{(1)}+m}$$

where for $l = 1, \dots, N^{(1)}$, there must be k satisfying $1 \leq k \leq N$ such that $r_l^{(1)} - r_k$ is a positive integer. In addition, this series also satisfies (7.6)–(7.8) and thus $C^{(1)}_{K,l,0}(\cdot)$ also satisfy (7.13)–(7.15). Repeating the argument above again and again, say s times and noticing that there are only finitely many inequivalent irreducible V-modules, finally we can find V-modules $W_j^{(s)}$ and intertwining operators $\mathcal{Y}_j^{(s)}$ of type $\binom{W_j^{(s)}}{W_1 W_j^{(s)}}$ for $j = 1, \dots, p^{(s)}$ such that

$$\sum_{l=1}^{N} \sum_{m \in \mathbb{N}} C_{K,l,m}(w_1) q_\tau^{r_l+m} - \sum_{j=1}^{p^{(s)}} F_{\mathcal{Y}_j^{(s)}}(w_1; z_1; q_\tau)$$

is of the form

$$\sum_{1=1}^{N^{(s)}}\sum_{m\in\mathbb{N}} C^{(s)}_{K,l,m}(w_1)q_\tau^{r_l^{(s)}+m},$$

where for $l=1,\dots,N^{(s)}$, $r_l^{(s)}$ are larger than the lowest weights of all irreducible V-modules. But this happens only when this series is 0. So we obtain

$$\sum_{l=1}^{N}\sum_{m\in\mathbb{N}} C_{K,l,m}(w_1)q_\tau^{r_l+m}=\sum_{j=1}^{p^{(s)}} F_{\mathcal{Y}_j^{(s)}}(w_1;z_1;q_\tau).$$

We still need to show that $K=0$. Assume $K\neq 0$. Then $K-1\geq 0$. Let

$$S_k(w_i;\tau)=\sum_{l=1}^{N}\sum_{m\in\mathbb{N}} C_{k,l,m}(w_1)q_\tau^{r_l+m}$$

for $k=1,\dots,K$. Then

$$F_{\mathcal{Y}_1}((\gamma\tau+\delta)^{-L(0)}w_1;z_1';q_{\tau'})=\sum_{k=1}^{K} S_k(w_i;\tau)\tau^k.$$

Using (7.6)–(7.8), we obtain

$$S_{K-1}(u_0w_1;\tau)=0, \tag{7.16}$$

$$S_{K-1}\left(u_{-2}w_1+\sum_{k\in\mathbb{Z}_+}(2k+1)G_{2k+2}(\tau)u_{2k}w_1;\tau\right)=0, \tag{7.17}$$

$$\begin{aligned}(2\pi i)^2KS_K(w_i;\tau)&+2\pi i\frac{d}{d\tau}S_{K-1}(w_1;\tau)\\ &=S_{K-1}\left(\left(L(-2)-\sum_{k\in\mathbb{N}}G_{2k+2}(\tau)L(2k)\right)w_1;\tau\right),\end{aligned} \tag{7.18}$$

$$2\pi i\frac{d}{d\tau}S_K(w_1;\tau)=S_K\left(\left(L(-2)-\sum_{k\in\mathbb{N}}G_{2k+2}(\tau)L(2k)\right)w_1;\tau\right). \tag{7.19}$$

From (7.18) and (7.19), we obtain

$$\begin{aligned}4\pi iS_{K-1}(w_i;\tau)&+(2\pi i)^2\frac{d^2}{d\tau^2}S_{K-1}(w_1;\tau)\\ &=S_{K-1}\left(\left(L(-2)-\sum_{k\in\mathbb{N}}G_{2k+2}(\tau)L(2k)\right)^2w_1;\tau\right).\end{aligned} \tag{7.20}$$

From (7.16), (7.17) and (7.20), we obtain

$$\begin{aligned} C_{K-1,l,0}(u \bullet w_1) &= C_{K,l,0}(w_1 \bullet u), \\ C_{K-1,l,0}(\tilde{O}(W_1)) &= 0, \\ C_{K-1,l,0}\left(\left(\omega - \frac{c}{24} - r_l\right)^2 \bullet w_1\right) &= 0. \end{aligned}$$

The same argument as we have used above shows that

$$S_{K-1}(w_1;\tau) = \sum_{j=1}^{p} F_{\hat{\mathcal{Y}}_j}(w_1; z_1; q_\tau)$$

for some irreducible modules $\hat{W}_j$ and intertwining operators of type $\binom{\hat{W}_j}{W_1 \hat{W}_j}$ for $j = 1, \dots, p$. Since $F_{\hat{\mathcal{Y}}_j}(w_1; z_1; q_\tau)$, $j = 1, \dots, p$, satisfy (7.6), we must have

$$2\pi i \frac{d}{d\tau} S_{K-1}(w_1;\tau) = S_{K-1}\left(\left(L(-2) - \sum_{k\in\mathbb{N}} G_{2k+2}(\tau) L(2k)\right) w_1; \tau\right)$$

which together with (7.18) gives

$$(2\pi i)^2 K S_K(w_1;\tau) = 0,$$

a contradiction.

Now we prove the case for any $n \geq 1$. This is the case where the method of Zhu [Z], further developed by Dong-Li-Mason [DLM] and Miyamoto [M1], cannot be used since there is no recurrence formula. We use induction. When $n = 1$, the theorem is just proved. Assume that when $n = k$, the theorem is proved. We now prove the case $n = k + 1$ using the genus-one associativity (Theorem 4.3). By Theorem 4.3, we have

$$\begin{aligned} &\overline{F}_{\mathcal{Y}_1,\dots,\mathcal{Y}_{n+1}}(w_1, \dots, w_{n+1}; z_1, \dots, z_{n+1}; \tau) \\ &\quad = \sum_{r\in\mathbb{R}} \overline{F}_{\mathcal{Y}_1,\dots,\mathcal{Y}_{n-1},\hat{\mathcal{Y}}_{n+1}}(w_1, \dots, w_{n-1}, P_r(\hat{\mathcal{Y}}_n(w_n, z_n - z_{n+1})w_{n+1}); \\ &\qquad\qquad z_1, \dots, z_{n-1}, z_{n+1}; \tau). \qquad (7.21) \end{aligned}$$

Using the induction assumption, we know that for any $r \in \mathbb{R}$,

$$\begin{aligned} &\overline{F}_{\mathcal{Y}_1,\dots,\mathcal{Y}_{n-1},\hat{\mathcal{Y}}_{n+1}}((\gamma\tau+\delta)^{-L(0)}w_1, \dots, (\gamma\tau+\delta)^{-L(0)}w_{n-1}, \\ &\qquad (\gamma\tau+\delta)^{-L(0)} P_r(\hat{\mathcal{Y}}(w_n, z_n - z_{n+1})w_{n+1}); z_1', \dots, z_{n-1}', z_{n+1}'; \tau') \end{aligned}$$

is in

$$\mathcal{F}_{w_1,\dots,w_{n-1},P_r(\hat{\mathcal{Y}}_n(w_n,z_n-z_{n+1})w_{n+1})}.$$

Thus by Theorem 4.3 again,

$$\sum_{r\in\mathbb{R}}\overline{F}_{\mathcal{Y}_1,\dots,\mathcal{Y}_{n-1},\hat{\mathcal{Y}}_{n+1}}((\gamma\tau+\delta)^{-L(0)}w_1,\dots,(\gamma\tau+\delta)^{-L(0)}w_{n-1},$$
$$(\gamma\tau+\delta)^{-L(0)}P_r(\hat{\mathcal{Y}}(w_n,z_n-z_{n+1})w_{n+1});z_1',\dots,z_{n-1}',z_{n+1}';\tau'). \tag{7.22}$$

is absolutely convergent and is a linear combination of absolutely convergent series of the form

$$\sum_{r\in\mathbb{R}}\overline{F}_{\check{\mathcal{Y}}_1,\dots,\check{\mathcal{Y}}_{n-1},\bar{\mathcal{Y}}_{n+1}}(w_1,\dots,w_{n-1},P_r(\bar{\mathcal{Y}}_n(w_n,z_n-z_{n+1})w_{n+1});$$
$$z_1,\dots,z_{n-1},z_{n+1};\tau) \tag{7.23}$$

for suitable intertwining operators $\check{\mathcal{Y}}_1,\dots,\check{\mathcal{Y}}_{n-1}$, $\bar{\mathcal{Y}}_n$ and $\bar{\mathcal{Y}}_{n+1}$. Moreover, there exist suitable intertwining operators $\check{\mathcal{Y}}_n$ and $\check{\mathcal{Y}}_{n+1}$ such that (7.23) is equal to

$$\overline{F}_{\check{\mathcal{Y}}_1,\dots,\check{\mathcal{Y}}_{n+1}}(w_1,\dots,w_{n+1};z_1,\dots,z_{n+1};\tau) \tag{7.24}$$

which is in $\mathcal{F}_{w_1,\dots,w_{n+1}}$. Thus, (7.22) as a linear combination of elements of the form (7.24) is also in $\mathcal{F}_{w_1,\dots,w_{n+1}}$. By (7.21), we see that

$$\overline{F}_{\mathcal{Y}_1,\dots,\mathcal{Y}_{n+1}}(w_1,\dots,w_{n+1};z_1,\dots,z_{n+1};\tau)$$

is also in $\mathcal{F}_{w_1,\dots,w_{n+1}}$. ∎

Remark 7.4 Let $W_1=\cdots=W_n=V$. Assume that $\tilde{W}_i$ for $i=1,\dots,n$ are irreducible. Then any intertwining operator of type $\binom{\tilde{W}_i}{V\tilde{W}_{i+1}}$ is 0 when $\tilde{W}_i$ is not isomorphic to $\tilde{W}_{i+1}$ and is a multiple of the vertex operator $Y_{\tilde{W}_i}$ for the V-module $\tilde{W}_i$ when $\tilde{W}_i$ is isomorphic (and then is identified with) $\tilde{W}_{i+1}$. In this case, if we take $w_i=(\mathcal{U}(1))^{-1}v_i\in V$ for $i=1,\dots,n$, Theorem 7.3 gives the modular invariance result proved by Zhu in [Z]. Similarly, if we take all but one of the V-modules $W_1,\dots,W_n$ to be V, Theorem 7.3 gives the generalization of Zhu's result by Miyamoto in [M1].

Remark 7.5 If we replace Conditions 1 and 3 by the conditions that for $n < 0$, $V_{(n)} = 0$ and every V-module is C_2-cofinite, then the conclusions of Theorems 7.2 and 7.3 are also true.

Remark 7.6 Geometrically, Theorems 7.1 and 7.3 and the fact that the coefficients of the system (3.5) is doubly-periodic in 1 and τ actually say that the space of the solutions of the system (3.5)–(3.6) and the space $\mathcal{F}_{w_1,\dots,w_{n+1}}$ are the spaces of (multivalued) global sections of some vector bundles with flat connections over the moduli space of genus-one Riemann surfaces with n punctures and standard local coordinates vanishing at these punctures.

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