$N = 2$ Superconformal Field Theory and Operator Algebras

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Operator algebraic approach to quantum field theory, particularly to chiral superconformal field theory.

Connecting the Jones subfactor theory and the Connes noncommutative geometry through superconformal field theory.

Outline of the talk:

1. Vertex operator algebras and local conformal nets
2. Representation theory and Jones theory of subfactors
3. $N = 2$ superconformal field theory and classification theory
4. Supersymmetry and the Connes noncommutative geometry

(with S. Carpi, R. Hillier, R. Longo, F. Xu)

Vertex operator algebras (with unitarity)

Spacetime and its symmetry group: S^1 and $\mathrm{Diff}(S^1)$

Quantum fields on S^1 have Fourier expansions (vertex operators):

$Y(v,z)=\sum_{n\in\mathbb{Z}} v_n z^{-n-1}$ with $v\in V$, $v_n\in \mathrm{End}(V)$, where V is a pre-Hilbert space with postive definite inner product. (Originally z is a complex number with modulus 1.)

Axioms: locality, vacuum vector, etc.

Virasoro element ω: $Y(\omega,z)=\sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, where L_n's satisfy the Virasoro relations. (conformal symmetry)

Representations: v_n's act on another space $\rightarrow$ a module of a vertex operator algebra.

We have a notion of a tensor product of modules.

Operator algebraic axioms of conformal field theory:

Motivation: Start with vertex operators $\{Y(v,z)\}$ on S^1.

Fix an interval $I \subset S^1$, consider $\langle Y(v,z), f(z)\rangle$ with $\mathrm{supp}\, f \subset I$. (Smeared fields)

$A(I)$: the operator algebra generated by these (possibly unbounded) operators

1. $I_1 \subset I_2 \Rightarrow A(I_1) \subset A(I_2)$.
2. $I_1 \cap I_2 = \varnothing \Rightarrow [A(I_1), A(I_2)] = 0$. (the commutator)
3. $\mathrm{Diff}(S^1)$-covariance (conformal covariance)
4. Positive energy
5. Vacuum vector

Such a family $\{A(I)\}$ is called a local conformal net.

In all the explicitly known examples, each algebra $A(I)$ is always mutually isomorphic. $\Rightarrow$ A single algebra $A(I)$ carries no information, but it is the relative relations of $A(I)$'s that contain information on quantum field theory.

In order to construct a family $\{A(I)\}$ actually from vertex operators, one needs certain Sobolev norm estimates for operators.

Basic examples of local conformal nets arise from Kac-Moody (and Virasoro) algebras and lattices.
We also have simple current extensions, the orbifold construction and the coset construction (and more) for local conformal nets. The Moonshine vertex operator algebra has a corresponding local conformal net.

Representation theory of a local conformal net $\{A(I)\}$.

Each $A(I)$, called a factor, acts on the initial Hilbert space from the beginning, but consider a representation on another Hilbert space (without a vacuum vector).

Each representation is given with an endomorphism of one factor $A(I_0)$. The image of the endomorphism is a subfactor of the factor $A(I_0)$, and it has the Jones index. Its square root is defined to be the (quantum) dimension of the representation π, whose value is in $[1, \infty]$.

We compose the two endomorphisms. This gives a notion of a tensor product of representations. We have a braided tensor category. (Doplicher-Haag-Roberts + Fredenhagen-Rehren-Schroer)

[K-Longo-Müger] Operator algebraic characterization of complete rationality: We have only finitely many irreducible representations and all have finite dimensions. ($\to$ a modular tensor category)

Complete rationality passes to an irreducible extension, a finite index subnet including an orbifold with a finite group action and a coset with a cofinite inclusion.

Representations of a local conformal net are supposed to correspond to modules of a vertex operator algebra, but proving the correspondence is often technically difficult and indirect.

Computations of fusion rules are often difficult, and we use techniques of subfactor theory. We also have various applications to the Jones theory of subfactors.

We have a classical notion of an induced representation for a group and its subgroup. Now introduce a similar construction for an inclusion of local conformal nets.

Let $\{A(I) \subset B(I)\}$ be an inclusion of local conformal nets. We extend an endomorphism of $A(I_0)$, giving a representation, to a larger operator algebra $B(I_0)$, using a braiding, for some fixed interval I_0. ($\alpha^{\pm}$-induction: Longo-Rehren, Xu, Ocneanu, Böckenhauer-Evans-K)

The intersection of irreducible endomorphisms arising from α^+-induction and those from α^--induction are exactly the genuine representations of $\{B(I)\}$. Somewhat similar to twisted modules. This produces a matrix called a modular invariant.

Apply the above machinery to classify local conformal nets.

Classification of local conformal nets with $c < 1$.

(K-Longo, Ann. Math. 2004):

We now apply the above theory to classify local conformal nets with $c < 1$, since they are extensions of the Virasoro nets with $c < 1$. Here is the classification list.

(1) Virasoro nets $\{\mathrm{Vir}_c(I)\}$ with $c < 1$.

(2) Their simple current extensions with index 2.

(3) Four exceptionals at $c = 21/22$, $25/26$, $144/145$, $154/155$.

Three exceptionals in the above (3) are identified with coset constructions, but the other one does not seem to be related to any other known constructions. ($\rightarrow$ Xu's mirror extensions)

Geometric aspects of local conformal nets

Classical geometry: Consider the Laplacian Δ on an n-dimensional compact oriented Riemannian manifold. The classical Weyl formula gives an asymptotic expansion

$$\mathrm{Tr}(e^{-t\Delta}) \sim \frac{1}{(4\pi t)^{n/2}}(a_0 + a_1 t + \cdots),$$

as $t \to 0+$, where a_0 is the volume of the manifold, and if $n = 2$, then a_1 is (constant times) the Euler characteristic of the manifold.

So the coefficients in the asymptotic expansion have a geometric meaning and the dimension is recovered. We look for their analogues in the setting of local conformal nets.

The conformal Hamiltonian L_0 of a local conformal net is the generator of the rotation group of S^1.

For a nice local conformal net, we have an expansion

$$\log \mathrm{Tr}(e^{-tL_0}) \sim \frac{1}{t}(a_0 + a_1 t + \cdots),$$

where a_0, a_1, a_2 are explicitly given. (K-Longo $\leftarrow$ modular invariance)

This gives an analogy of the Laplacian Δ of a manifold and the conformal Hamiltonian L_0 of a local conformal net.

A “square root” of the Laplacian gives a classical Dirac operator. The Connes approach in noncommutative geometry uses its abstract axiomatization.

Noncommutative geometry:

Noncommutative operator algebras are regarded as function algebras on noncommutative spaces.

In geometry, we need manifolds rather than compact Hausdorff spaces or measure spaces.

The Connes axiomatization of a noncommutative compact Riemannian spin manifold: a spectral triple $(\mathcal{A}, H, D)$.

1. $\mathcal{A}$: $*$-subalgebra of $B(H)$, the smooth algebra $C^\infty(M)$.
2. H: a Hilbert space, the space of L^2-spinors.
3. D: an (unbounded) self-adjoint operator with compact resolvents, the Dirac operator.
4. We require $[D, x] \in B(H)$ for all $x \in \mathcal{A}$.

$N = 1$ super Virasoro algebras: (Adding a square root of L_0)

The infinite dimensional super Lie algebras generated by central element c, even elements L_n, $n \in \mathbb{Z}$, and odd elements G_r, $r \in \mathbb{Z}$ or $r \in \mathbb{Z} + 1/2$, with the following relations:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c, \\ [L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}, \\ [G_r, G_s] &= 2L_{r+s} + \frac{1}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}c. \end{aligned}$$

Ramond ⟦Neveu-Schwarz⟧ algebra, if $r \in \mathbb{Z}$ ⟦$r \in \mathbb{Z} + 1/2$⟧.
Note $G_0^2 = L_0 - c/24$ for $r = s = 0$ in a representation.

We again consider a unitary representation of (one of) the $N = 1$ super Virasoro algebras. Consider $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ and $G(z) = \sum_r G_r z^{-r-3/2}$ as operator-valued distributions on S^1.

Using test functions supported in an interval I, they produce a family $\{A(I)\}$ of operator algebras parametrized by $I \subset S^1$. This gives a superconformal net, for which now the bracket in the axioms means a graded commutator. They have been classified for the discrete series of the values of c, that is, $c < 3/2$.

To make a further study in connection to noncommutative geometry, we work on $N = 2$ super Virasoro algebra and its unitary representations. Instead of one series $\{G_r\}$, we next have two series $\{G_r^\pm\}$ for the $N = 2$ case.

$N = 2$ super Virasoro algebra: Generated by c, L_n, J_n and $G^{\pm}_{n\pm a}$, $n \in \mathbb{Z}$, with the following relations. (a: a real parameter)

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}c, \\
[J_m, J_n] &= \frac{m}{3}\delta_{m+n,0}c \\
[L_n, J_m] &= -mJ_{m+n}, \\
[G^{+}_{n+a}, G^{+}_{m+a}] &= [G^{-}_{n-a}, G^{-}_{m-a}] = 0, \\
[L_n, G^{\pm}_{m\pm a}] &= \left(\frac{n}{2} - (m \pm a)\right) G^{\pm}_{m+n\pm a}, \\
[J_n, G^{\pm}_{m\pm a}] &= \pm G^{\pm}_{m+n\pm a}, \\
[G^{+}_{n+a}, G^{-}_{m-a}] &= 2L_{m+n} + (n-m+2a)J_{n+m} + \\
&\quad \frac{1}{3}\left((n+a)^2 - \frac{1}{4}\right)\delta_{m+n,0}c.
\end{aligned}
$$

It is known that an irreducible unitary representation maps c to a scalar in the set

$$\left\{ \frac{3m}{m+2} \middle| m = 1, 2, 3, \dots \right\} \cup [3, \infty).$$

We consider only the case $c = 3m/(m+2)$ in the discrete series now.

We use $G_n^1 = (G_n^+ + G_n^-)/\sqrt{2}$ and $G_n^2 = -i(G_n^+ - G_n^-)/\sqrt{2}$. We fix a unitary representation and write L_n, G_n^1, G_n^2, J_n for their images in the representation. They are closed unbounded operators. We then use the four operator-valued distributions $L(z) = \sum_n L_n z^{-n-2}$, $G^j(z) = \sum_n G_n^j z^{-n-3/2}$ $(j = 1, 2)$ and $J(z) = \sum_n J_n z^{-n-1}$, where $z \in \mathbb{C}$ with $|z| = 1$.

As before, using these four operator-valued distributions and test functions supported in $I \subset S^1$, we obtain a family of operator algebras $\{A(I)\}$ parametrized by the intervals I.

For the vacuum representation of the $N = 2$ super Virasoro algebra with $c = 3m/(m+2)$ in the discrete series, we have a vacuum representation with the coset construction arising from $U(1)_{2m+4} \subset SU(2)_m \otimes U(1)_4$ due to Di Vecchia-Petersen-Yu-Zheng.

However, while it is clear that the coset contains a representation of the $N = 2$ super Virasoro algebra, it is not clear whether this representation gives all of the coset or not. The equality here has been often taken as a mathematically established result, but we have been unable to find a complete proof in literature.

Operator algebraic methods give a proof of this equality for the coset.

The extensions of the cosets are $N = 2$ superconformal nets by definition. They are classified and listed completely. Typical methods to give such an extension are the coset construction and the mirror extension in the sense of Xu, which copies one extension to give another through a coset.

Now these two are mixed together, and we have a simple current extension with a finite cyclic group of an arbitrary order. This was not the case in the previous classification results for (super) conformal nets.

This is partly based on Gannon's classification and a kind of A-D-E classification.

We now connect these to noncommutative geometry by constructing a family of spectral triples parameterized by the intervals I. We need the Dirac operator, and have two candidates G_0^1 and G_0^2 in the Ramond representation, but they are unitarily equivalent, so we just choose G_0^1, and put $\delta(x) = [G_0^1, x]$ for a bounded linear operator x on the representation space.

We put

$$\mathcal{A}(I) = A(I) \cap \bigcap_{n=1}^{\infty} \mathrm{dom}(\delta^n).$$

Each $\mathcal{A}(I)$ is dense in $A(I)$ and satisfies $\delta(\mathcal{A}(I)) \subset \mathcal{A}(I)$. That is, for each I, our spectral triple $(\mathcal{A}(I), H, G_0^1)$ gives a quantum algebra in the sense of Jaffe-Lesniewski-Osterwalder.

We now deal with entire cyclic cohomology introduced by Connes, a nice cohomology theory here. Our Dirac operator $D = G_0^1$ satisfies the condition $\mathrm{Tr}(e^{-tD^2}) < \infty$ for all $t > 0$, which is called the θ-summability condition.

A JLO cocycle for a θ-summable spectral triple is defined by a sequence of multilinear functionals defined in terms of traces and integrals involving e^{-tD^2} and $[D, \cdot\,]$ on the $*$-algebra. This gives an element in the entire cyclic cohomology.

We are interested in spectral triples arising from the Ramond representations with the lowest conformal weight $h = c/24$. They produce subfactors and then projections in the universal operator algebra for a local conformal net, and in turn elements in the K_0-group.

Within the universal operator algebra, we define a $*$-subalgebra with different representations, and each image gives a spectral triple with an appropriate Dirac operator in each representation. We thus have different JLO-cocycles for the same $*$-algebra.

In general, we have the index pairing between the K_0-group and the entire cyclic cohomology, producing a number.

In the above, we have the K_0-elements depending on the Ramond representations, and also the JLO-cocycles in the entire cyclic cohomology given by the same Ramond representations.

Our result then says that the pairing between them give the Kronecker δ of the representations. In this way, subfactor theory and noncommutative geometry are connected.