Eigenfunctions on a Riemannian manifold and representations of a vertex operator algebra

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Outline

The vertex operator algebra approach to two-dimensional conformal field theory

Nonlinear sigma models

- Classical nonlinear sigma models: harmonic maps
- Quantization

Heisenberg algebras and smooth functions

- A homomorphism from the symmetric algebra $S(T_pM)$ to the algebra of linear operators on $C^\infty(B(p, r))$
- Modules for the Heisenberg algebra $\widehat{T_pM}$ generated by germs of smooth functions at p

$L(0)$-semisimple lower-bounded generalized modules generated by eigenfucntions

- Sheaves of vertex operator algebras and weak modules
- Eigenfunctions and $L(0)$-semisimple lower-bounded generalized modules for a vertex operator algebra

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Vertex operator algebras

Belavin-Polyakov-Zamolodchikov, Borcherds, Frenkel-Lepowsky-Meurman.

A vertex operator algebra consists the following data:

- A $\mathbb{Z}$-graded vector space $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$.
- A vertex operator map

$$\begin{aligned} Y_V : V \otimes V &\to V[[z, z^{-1}]], \\ u \otimes v &\mapsto Y(u, z)v. \end{aligned}$$

- A vacuum $\mathbf{1} \in V_{(0)}$.
- A conformal vector $\omega \in V_{(2)}$.

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Vertex operator algebras

These data satisfy the following axioms:

- Grading-restriction property: $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$ and $V_{(n)} = 0$ when n is sufficiently negative.
- Lower-truncation property: For $u, v \in V$, $Y(u, z)v$ contains only finitely many negative power terms.
- Axioms for the vacuum: For $u \in V$, $Y(\mathbf{1}, z)u = u$ and $\lim_{z \to 0} Y(u, z)\mathbf{1} = u$.
- Axioms for the conformal element: Let $L(n) : V \to V$ be defined by $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$, then $[L(m), L(n)] = (m - n)L(m + n) + \frac{c}{12}(m^3 - m)\delta_{m+b,0}$, $\frac{d}{dz} Y(u, z) = Y(L(-1)u, z)$ ($L(-1)$-derivative property) for $u \in V$ and $L(0)u = nu$ for $u \in V_{(n)}$ ($L(0)$-grading property).

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Vertex operator algebras

- Duality property: For $u_1, u_2, v \in V$, $v' \in V' = \coprod_{n\in\mathbb{Z}} V^*_{(n)}$, the series

$$\langle v', Y(u_1, z_1)Y(u_2, z_2)v\rangle$$
$$\langle v', Y(u_2, z_2)Y(u_1, z_1)v\rangle$$
$$\langle v', Y(Y(u_1, z_1 - z_2)u_2, z_2)v\rangle$$

  are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$ and $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

Modules

- Let V be a vertex operator algebra. A V-module is a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{(n)}$ equipped with a vertex operator map $Y_W : V \otimes W \to W[[z, z^{-1}]]$ satisfying all those axioms for V which still make sense.
- A lower-bounded generalized V-module is a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{(n)}$ equipped with a vertex operator map $Y_W : V \otimes W \to W[[z, z^{-1}]]$ satisfying all the axioms for a V-module except that $W_{(n)}$ are generalized eigenspaces instead of eigenspaces of $L(0)$ and are not necessarily finite dimensional. A weak V-module is a vector space W equipped with a vertex operator map $Y_W : V \otimes W \to W[[z, z^{-1}]]$ satisfying all those axioms for V-modules which still make sense.

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Intertwining operators

Let W_1, W_2 and W_3 be V-modules. An intertwining operator of type $\binom{W_3}{W_1\,W_2}$ is a linear map $\mathcal{Y} : W_1 \otimes W_2 \to W_3\{z\}$, where $W_3\{z\}$ is the space of all series in complex powers of z with coefficients in W_3, satisfying all those axioms for V which still make sense, that is, a lower-truncation property, an $L(-1)$-derivative property and a duality property.

Examples

- Free bosons: Representations of infinite-dimensional Heisenberg algebras.
- Free bosons on tori: Vertex operator algebras, modules and classical vertex operators (intertwining operators) associated to lattices.
- Wess-Zumino-Novikov-Witten models: Representations of affine Lie algebras.
- Minimal models: Representations of the Virasoro algebra.
- Fermion theories: Representations of infinite-dimensional Clifford algebras, affine Lie superalgebras and superconformal algebras.
- Orbifolds, cosets and $\mathcal{W}$-algebras, including in particular the moonshine module.

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Rational conformal field theories

Theorem (H. 2002–2005)

Let V be a simple vertex operator algebra satisfying the following conditions:

1. $V_{(n)} = 0$ *for* $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ *and* V' *is isomorphic to* V *as a* V*-module.*
2. *Every lower-bounded generalized* V*-module is a direct sum of irreducible* V*-modules.*
3. V *is* C_2*-cofinite, that is,* $\dim V/C_2(V) < \infty$ *where* $C_2(V)$ *is the subspace of* V *spanned by elements of the form* $\mathrm{Res}_z z^{-2} Y(u, z)v$ *for* $u, v \in V$.

Then we have the following conclusion:

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Then we have the following conclusion:

1. *The Moore-Seiberg polynomial equations hold. In particular, the Verlinde conjecture holds.*
2. *The category of V-modules has a natural structure of a modular tensor category. In partciular, we have a modular functor (in all genera) and a 3-dimensional topological field theory.*
3. *All chiral and full correlation functions in genus-zero and genus-one can be constructed from intertwining operators (the part on full correlation functions being done jointly with Kong).*
4. *There exist locally convex topological completions of the spaces involved such that Segal's axioms in genus-zero and genus-one are satisfied.*

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Higher-genus and non-rational theories

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- Many results for rational theories can be generalized to non-rational theories, including unitary non-rational theories and logarithmic theories. But there are still conjectures and open problems to be solved. The construction and study of non-rational theories is important for the study of the moduli space of conformal field theories. Nonlinear sigma models with Calabi-Yau manifolds as targets are important examples to be constructed and studied.

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Harmonic maps from a Riemann surface to a Riemnannian manifold

- Σ: 2-diemnsional Riemannian manifold.
  M: n-dimensional Riemannian manifold.
  φ: smooth map from Σ to M.
- Action: $\int_{\Sigma} \parallel d\varphi \parallel dS$.
  Locally: $\parallel d\varphi \parallel = \eta^{ij} g_{\mu\nu} \frac{\partial \varphi^{\mu}}{\partial x^i} \frac{\partial \varphi^{\nu}}{\partial x^j}$.
- Harmonic maps: Critical points of the action above.
- The action is conformally invariant. Only the Riemann surface structure on Σ is needed.

Harmonic maps from a Riemann surface to a Riemnannian manifold

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The vertex operator algebra approach to two-dimensional conformal field theory

Nonlinear sigma models

- Classical nonlinear sigma models: harmonic maps
- Quantization

Heisenberg algebras and smooth functions

- A homomorphism from the symmetric algebra $S(T_pM)$ to the algebra of linear operators on $C^\infty(B(p, r))$
- Modules for the Heisenberg algebra $\widehat{T_pM}$ generated by germs of smooth functions at p

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- Sheaves of vertex operator algebras and weak modules
- Eigenfunctions and $L(0)$-semisimple lower-bounded generalized modules for a vertex operator algebra

Free bosons: Quantization of linear sigma models

- Linear sigma models: M flat.
- $M = \mathbb{R}^n$ and $p \in \mathbb{R}^n$.
  $T_pM = T_p\mathbb{R}^n$: the tangent space of $M = \mathbb{R}^n$ at p.
- Heisenberg algebra: $\widehat{T_pM} = \widehat{T_pM}_- \oplus \widehat{T_pM}_0 \oplus \widehat{T_pM}_+$, where

$$\begin{aligned}
\widehat{T_pM}_- &= T_pM \otimes t^{-1}\mathbb{C}[t^{-1}], \\
\widehat{T_pM}_0 &= T_pM \otimes \mathbb{C}t^0 \oplus \mathbb{C}\mathbf{I} \simeq T_pM \oplus \mathbb{C}\mathbf{I}, \\
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Vertex operator algebra and modules associated to free bosons

- Let $\widehat{T_pM}_+$ act on $|\mathbf{k}\rangle$ as 0, $\mathbf{I}$ on $|\mathbf{k}\rangle$ as 1 and elements of T_pM on $|\mathbf{k}\rangle$ as i times the vector fields obtained by parallel transporting the elements of T_pM. Then $\mathbb{C}|\mathbf{k}\rangle$ becomes a module for $\widehat{T_pM}_0 \oplus \widehat{T_pM}_+$.
- Induced module: $W_{|\mathbf{k}\rangle} = U(\widehat{T_pM}) \otimes_{U(\widehat{T_pM}_0 \oplus \widehat{T_pM}_+)} \mathbb{C}|\mathbf{k}\rangle$ (Fock space generated by $|\mathbf{k}\rangle$).
- $W_{|\mathbf{0}\rangle}$ is a vertex operator algebra and $W_{|\mathbf{k}\rangle}$ for any $\mathbf{k}$ is a module for the vertex operator algebra $W_{|\mathbf{0}\rangle}$.
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Sigma models on tori

- Given a torus, the vertex operator algebra is still $W_{|\mathbf{0}\rangle}$. But consider only modules $W_{|\mathbf{k}\rangle}$ for those **k** such that $e^{ik_\mu x^\mu}$ are well defined on the torus. Such **k** form a lattice.
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First difficulty in generalizing this construction to the curved case

- M: a Riemannian manifold. $p \in M$. T_pM: the tangent space of $\mathbb{R}^n$ at p.
- Heisenberg algebra $\widehat{T_pM}$.
- Problem: How to define an action of the abelian Lie subalgebra T_pM on smooth functions, in particular on eigenfunctions of the Laplacian, at least near p.
- If we choose a local coordinate patch near p and define the action using derivatives with respect to the coordinates, the results depend on the local coordinate patch.
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Notations

- U: an open subset of M.
  $\Gamma_U(S(TM))$: the space of sections on U of the vector bundle $S(TM)$ whose fibers are the symmetric algebras of tangent spaces of M.
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Definition of η_U

- $\sigma \in \Gamma_U(S^k(TM))$ and $q \in U$.
  $\psi : N \to \mathbb{R}^n$: a chart of normal coordinates at q.
- Identify the symmetric algebra $S(T_qM)$ with the space of symmetric tensors at q. Then there exist $a^{\mu_1\cdots\mu_k}(q) \in \mathbb{C}$, symmetric with respect to $\mu_1, \ldots, \mu_k$, such that

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A homomorphism of associative algebras

- $p \in M$ and $r \leq \mathrm{inj}(p)$ (injectivity radius at p). $B(p, r)$: open ball of radius r centered at p.
- Given an element of $S(T_pM)$, using parallel transport along the unique geodesics from p to points in $B(p, r)$, we obtain an element of $\Gamma_{B(p,r)}S(TM)$. Apply $\eta_{B(p,r)}$ to this element, we obtain an operator on $C^\infty(B(p, r))$. Thus we obtain a linear map $\xi_{B(p,r)} : S(T_pM) \to L(C^\infty(B(p, r))$.

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$\xi_{B(p,r)}$ *is a homomorphism of associative algebras.*

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  σ: parallel section on U of $U(\widehat{TM})$ (the vector bundle whose fibers are universal enveloping algebras of Heisenberg algebras).
- Given such σ and $f \in C^{\infty}(U)$, we obtain a map from U to the disjoint union of
  $W_{C^{\infty}(p)} = U(\widehat{T_pM}) \otimes_{U(\widehat{T_pM}_0 \oplus \widehat{T_pM}_+)} C^{\infty}(p)$ for $p \in U$.
- $W^0(U)$: space spanned by all such maps from U to the disjoint union of $W_{C^{\infty}(p)}$ for $p \in U$.
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A sheaf of vertex operator algebra and a sheaf of weak moduels

- Let $\mathcal{W}$ be the sheafication of $\mathcal{W}^0$.
- For an open subset U, the subspace $W_1(U)$ of $W^0(U)$ obtained from parallel sections on U of $U(\widehat{TM})$ and the constant function 1 has a structure of a vertex operator algebra. $U \mapsto W_1(U)$ gives a sheaf $\mathcal{W}_1$ of vertex operator algebras.

Theorem

The space $W(U)$ of sections of $\mathcal{W}$ on U is a weak module for $W_1(U)$ and $\mathcal{W}$ is a sheaf of weak modules for the sheaf $\mathcal{W}_1$ of vertex operator algebras.

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Outline

- The vertex operator algebra approach to two-dimensional conformal field theory
- Nonlinear sigma models
  - Classical nonlinear sigma models: harmonic maps
  - Quantization
- Heisenberg algebras and smooth functions
  - A homomorphism from the symmetric algebra $S(T_pM)$ to the algebra of linear operators on $C^\infty(B(p, r))$
  - Modules for the Heisenberg algebra $\widehat{T_pM}$ generated by germs of smooth functions at p
- $L(0)$-semisimple lower-bounded generalized modules generated by eigenfucntions
  - Sheaves of vertex operator algebras and weak modules
  - Eigenfunctions and $L(0)$-semisimple lower-bounded generalized modules for a vertex operator algebra

$L(0)$-semisimple lower-bounded generalized modules for the vertex operator algebra $W_1(M)$ generated by eigenfunctions

Take $U = M$ and let $W_f(M)$ be the weak submodule of $W(M)$ for $W_1(M)$ generated by an eigenfunction f. Then we have:

Theorem

$W_f(M)$ is an $L(0)$-semisimple lower-bounded generalized mdoule for $W_1(M)$.

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The correspondence $M \mapsto (\mathcal{W}_1, \{\mathcal{W}_f\}_{f \text{ are eigenfunctions}})$ is a functor from the category of Riemannian manifold to the category of of pairs of vertex operator algebras and sequences of $L(0)$-semisimple lower-bounded generalized V-modules.

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- We can now define correlation functions for elements of these modules, in partciular, correlation functions for eigenfunctions of the Laplacian.
- The construction generalizes without difficulties to fermionic theories using eigenforms and spinors.
- We expect that when M is Calabi-Yau, our construction gives a sheaf of vertex operator superalgebras with $N = 2$ superconformal algebras and sheaves of modules for the sheaf of vertex operator superalgebras.
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